Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/110607

Relative Booster Ideals of Distributive p-algebras

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Received: 12 Mar. 2022, Revised: 22 May 2022, Accepted: 2 Jun 2022
Published online: 1 Nov. 2022

Abstract: In this article, the definition and characterization of relative booster ideals in distributive p-algebras are given. The relationship between disjunctive relative booster ideals and normal relative booster ideals is established in the distributive p-algebras. A lattice congruence relation defined via the relative boosters is given and its quotient lattice structure is obtained.

Keywords: Booster ideals, Congruence Relations, Disjunctive lattices, Distributive p-algebras, Ideals, Normal lattices, Stone algebras.

1 Introduction

O. Frink introduced the notion of pseudo-complementation of meet semi-lattices to generalize the class of Boolean algebras. Many authors introduced this concept of pseudo-complementation in different classes for example, the class of bounded lattices to get the class of p-algebras, consequently distributive p-algebras, Stone algebras, modular p-algebras, quasi-modular p-algebras. Ideals are investigated in the class of pseudo-complemented semi-lattice by T.S. Blyth ,refer to [1]

A. Badawy and M. S. Raw [2] introduced the definition of booster ideals in distributive p-algebras in the sense of annihilators with pseudo-complemented elements and discussed some properties. W.H. Cornish studied a normal lattice [3] and announced that disjunctive normal distributive lattices are important in the compactification theory. Y. S. Pawar and S. S. Khopade studied the characterization of disjunctive ideals in 0-distributive lattices, see ([4],Theorem 3.11). Recently, many articles study properties of types of boosters in different classes of algebras, e.g., MS-algebra in [5] and [6], Stone almost distributive lattice in [7], [8] and so on.

In the present article, we generalize the concept of booster which was defined by A.Badawy et al.in [2]. The relative booster ideals in distributive p-algebras is defined, and its important properties are shown. The behavior of disjunctive relative booster and normal relative booster ideals of distributive p-algebras is discussed, and hence the disjunctive booster and normal booster distributive p-algebras are given. Furthermore, the lattice congruence relations defined via the relative boosters is established.

After preliminaries in section 2, the definition and characterization of relative booster ideals of distributive p-algebra L are introduced. Some related properties are proved and the structure of relative booster ideals of L are given in section 3. In section 4, the relationship between disjunctive relative booster ideals and normal relative booster ideals in distributive p-algebra is established. Consequently, the disjunctive and normal booster distributive p-algebras are given, and every Stone algebra is a normal booster. Concluding by section 5, a relationship between the lattice congruence relation Θ and booster ideals on L is given. Moreover, the structure $(L/\Theta;\wedge,\vee,[0]\Theta,D(L)))$ is investigated.

2 Preliminaries

In the present section, we recall some of the basic definitions and theorems that will be needed.

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Definition 1.[1] *Let $(L;\wedge,\vee,0,1)$ be a bounded lattice. An element $a^\star \in L$ is called a pseudo-complement of a if for all $x \in L$*

$x \wedge a = 0$ iff $x \leq a^\star$

Therefore, in a bounded lattice the pseudo-complement of a is the largest element $x \in L$ with the property that $a \wedge x = 0$.

The bounded lattice L is said to be pseudo-complemented lattice (a p-algebra, for short) if every element in it have at most one pseudo-complement.
It is clear that $0^\star = 1$ and $1^\star = 0$. A p-algebra is called distributive (modular) if the underlying lattice is distributive (modular).

A distributive p-algebra is called a Stone algebra if it satisfies Stone identity

$x^\star \vee x^{\star\star} = 1,$ for all $x \in L$

Theorem 1.[1] *Let L be a p-algebra. For all $a,b \in L$,*

(1) $a \leq b$ implies $b^\star \leq a^\star$,
(2) $a \leq a^{\star\star}$,
(3) $a^\star = a^{\star\star\star}$,
(4) $(a \vee b)^\star = a^\star \wedge b^\star$,
(5) $(a \wedge b)^{\star\star} = a^{\star\star} \wedge b^{\star\star}$,
(6) $(a \wedge b)^\star \geq a^\star \vee b^\star$,
(7) $(a \vee b)^{\star\star} = (a^{\star\star} \vee b^{\star\star})^{\star\star} \geq a^{\star\star} \wedge b^{\star\star}$.

Moreover, if L be a Stone algebra, then

(8) $(a \wedge b)^\star = a^\star \vee b^\star$,
(9) $(a \vee b)^{\star\star} = (a^{\star\star} \vee b^{\star\star})^{\star\star} = a^{\star\star} \wedge b^{\star\star}$.

Let L be a p-algebra, an element $a \in L$ is called a closed element if it satisfies the condition $a^{\star\star} = a$. The set of all closed elements of L denoted by $B(L)$. It is known that, $(B(L);\wedge,\triangledown,{}^\star,0,1)$ forms a Boolean algebra where

$a \triangledown b = (a^\star \wedge b^\star)^\star$

An element $d \in L$ is said to be a dense element if it satisfies the condition $d^\star = 0$. The set of all dense elements of L denoted by $D(L)$.

Let $I(L)$ be the set of all ideals of L. Then $(I(L);\subseteq)$ forms a complete lattice, and the lattice operations defined on $I(L)$ are given by, for $I,J \in I(L)$,

$$I \wedge J = I \cap J \text{ and } I \vee J = \{x \in L : x \geq a \vee b, \text{ for some } a \in I \text{ and } b \in J\}$$

It is known that, $(I(L);\wedge,\vee,(0],L)$ forms a complete lattice. In the special case, $I = \{a\}$, $a \in L$, the ideal $(a] = \{x \in L : x \leq a\}$ is called the principal ideal generated by $a \in L$. One can see that

$(a] \wedge (b] = (a \wedge b]$ and $(a] \vee (b] = (a \vee b]$,

refer to [1].

Definition 2.[2] *Let a be an element in a distributive p-algebra L. Then the booster $(a)^\triangle$ of a is the set:*

$$(a)^\triangle = \{x \in L : x \wedge a^\star = 0\}.$$

Clearly, $(0)^\triangle = (0]$ and $(1)^\triangle = L$. In addition to boosters have the following properties:

Proposition 1.[2] *Let L be a distributive p-algebra. Then for any $a,b \in L$:*

(1) $(a)^\triangle$ *is an ideal of L containing a,*
(2) $(a)^\triangle = (a^{\star\star})^\triangle = (a^{\star\star}]$,
(3) $(a)^\triangle = (a]$ *iff* $a \in B(L)$,
(4) $(a)^\triangle = L$ *iff* $a \in D(L)$,
(5) *If* $a \in (b)^\triangle$, *then* $(a)^\triangle \subseteq (b)^\triangle$,
(6) *If* $a \leq b$, *then* $(a)^\triangle \subseteq (b)^\triangle$,
(7) $a^\star = b^\star$ *iff* $(a)^\triangle = (b)^\triangle$,
(8) $(a)^\triangle = (0)^\triangle$ *iff* $a = 0$.

Let $Con(L)$ be the set of all congruence relations on L, it is complete distributive lattice under set inclusion. For any $a \in L$, the congruence class of a with respect to the congruence Θ on L is given by $[a]\Theta = \{x \in L : x \equiv a(\Theta)\}$. The set of all congruence classes is denoted by L/Θ. The operations are defined on L/Θ by $[a]\Theta \wedge [b]\Theta = [a \wedge b]\Theta$ and $[a]\Theta \vee [b]\Theta = [a \vee b]\Theta$ for all $[a]\Theta, [b]\Theta \in L/\Theta$. Then the algebraic system $(L/\Theta;\wedge,\vee,[0]\Theta,[1]\Theta))$ forms a lattice called a quotient lattice of L modulo Θ, refer to [1].

Throughout the present article, L stands for a distributive p-algebra unless otherwise mentioned

3 Relative Boosters

In the present section, the concept of relative booster and related properties are given.

Definition 3. *Let A be a non-empty subset of L and I be a fixed ideal in L. A relative booster of A in an ideal I is the set A^{bo} defined as:*

$A^{bo} = \{x \in I : x \wedge a^\star = 0 \text{ for all } a \in A\}$

where $a^\star$ represents the pseudo-complemented element of a in L.

In the special case, if $A = \{a\}$, we write $(a)^{bo} = \{x \in I : x \wedge a^\star = 0\}$ is called the relative booster of a in I. If $I = (a]$ is a principal ideal generated by a in L, then $(0)^{bo} = (0], (a)^{bo} = (a]$. If $I = L$, then $(a)^{bo}$coincide with the booster $(a)^\triangle$ as in [2].

Lemma 1. *If A is a non-empty subset of L and I is a fixed ideal of L. Then A^{bo} is an ideal of I.*

Proof. Suppose that $x,y \in A^{bo}$, then

$$(x \vee y) \wedge a^\star = (x \wedge a^\star) \vee (y \wedge a^\star) = 0, \text{ for all } a \in A.$$

So, $x \vee y \in A^{bo}$. Let $x \in A^{bo}$ and $z \in I, z \leq x$, meeting both sides with $a^\star$, then

$$z \wedge a^\star \leq x \wedge a^\star \text{ implies that } z \wedge a^\star = 0, \text{ for all } a \in A.$$

Thus $z \in A^{bo}$ and A^{bo} is an ideal of I.

Corollary 1. *The relative booster $(a)^{bo}$ of $a \in L$ is an ideal of I containing a.*

Proposition 2. *For any two elements $a,b \in L$ and a fixed ideal I in L. The following properties are true:*

(1) $(a)^{bo} \subseteq (a^{\star\star}]$,
(2) If $a \le b$ then $(a)^{bo} \subseteq (b)^{bo}$,
(3) $(a)^{bo} \cap (b)^{bo} = (a \wedge b)^{bo}$,
(4) If $b \in (a)^{bo}$ then $(b)^{bo} \subseteq (a)^{bo}$,
(5) $I = \cup_{a \in I} (a)^{bo}$,
(6) If $a \in B(L)$, then $(a)^{bo} \subseteq (a]$,
(7) If $a \in I \cap D(L)$, then $(a)^{bo} = I$,
(8) $(a)^{bo} = (0)^{bo}$ iff $a = 0$,
(9) If $a,b \in L$ and $(a)^{bo} = (b)^{bo}$, then:

$(a \wedge c)^{bo} = (b \wedge c)^{bo}$ and $(a \vee c)^{bo} = (b \vee c)^{bo}$ for all $c \in L$.

Proof.
(1) we have that, $(a)^{bo} = (a)^{\triangle} \cap I$. Then, $(a)^{bo} \subseteq (a)^{\triangle} = (a^{\star\star}]$.
(2) Let $a \le b$ and $x \in (a)^{bo} \subseteq I$. Then $x \wedge b^{\star} \le x \wedge a^{\star} = 0$. Accordingly $x \in (b)^{bo}$ and $(a)^{bo} \subseteq (b)^{bo}$.
(3) Clearly, we have $(a \wedge b)^{bo} \subseteq (a)^{bo}, (b)^{bo}$. Now, let $x \in (a)^{bo} \cap (b)^{bo} \subseteq (a^{\star\star}] \cap b^{\star\star}]$, i.e., $x \le a^{\star\star}$, $x \le b^{\star\star}$. Then $x \le a^{\star\star} \wedge b^{\star\star} = (a \wedge b)^{\star\star}$. Hence, $x \wedge (a \wedge b)^{\star} = 0$, and then $x \in (a \wedge b)^{bo}$. Therefore, $(a)^{bo} \cap (b)^{bo} = (a \wedge b)^{bo}$.
(4) If $b \in (a)^{bo}$, then $b \wedge a^{\star} = 0$ which implies $a^{\star} \le b^{\star}$. Now, let $x \in (b)^{bo}$. Then $x \wedge a^{\star} \le x \wedge b^{\star} = 0$. As a result, $x \wedge a^{\star} = 0$ and so $x \in (a)^{bo}$.
(5) Since $a \in (a)^{bo}$, it means that $I = \cup_{a \in I} \{a\} \subseteq \cup_{a \in I} (a)^{bo} \subseteq I$.
(6) If $a \in B(L)$, then $a = a^{\star\star}$. By using (1), $(a)^{bo} \subseteq (a^{\star\star}] = (a]$.
(7) If $a \in I \cap D(L)$, then $a^{\star} = 0$. Thus, for all $x \in I$ we get $x \wedge a^{\star} = 0$. Therefore $I = (a)^{bo}$.
(8) $a \in (a)^{bo} = (0)^{bo} = (0]$ iff $a = 0$.
(9) Let $(a)^{bo} = (b)^{bo}$, and $x \in (a \wedge c)^{bo}$. Then $(x \wedge a^{\star}) \vee (x \wedge c^{\star}) = x \wedge (a^{\star} \vee c^{\star}) \le x \wedge (a \wedge c)^{\star} = 0$. Hence $(x \wedge a^{\star}) = (x \wedge c^{\star}) = 0$, it implies $x \in (a)^{bo} = (b)^{bo}$ and $x \in (c)^{bo}$. Accordingly $x \in (b)^{bo} \cap (c)^{bo} = (b \wedge c)^{bo}$ (from (3)). As a result, $(a \wedge c)^{bo} \subseteq (b \wedge c)^{bo}$. Similarly, $(b \wedge c)^{bo} \subseteq (a \wedge c)^{bo}$. If $x \in (a \vee c)^{bo}$, then $x \wedge (a \vee c)^{\star} = x \wedge a^{\star} \wedge c^{\star} = 0$. Thus, $x \wedge c^{\star} \in (a)^{bo} = (b)^{bo}$ and $x \wedge b^{\star} \wedge c^{\star} = x \wedge (b \vee c)^{\star} = 0$. Therefore, $x \in (b \vee c)^{bo}$ and $(a \vee c)^{bo} \subseteq (b \vee c)^{bo}$. Similarly, $(b \vee c)^{bo} \subseteq (a \vee c)^{bo}$.

Let the set $B^{bo}(I) = \{(a)^{bo} : a \in L\}$ be the set of all relative boosters of a fixed ideal I in L, and the operations on $B^{bo}(I)$ are defined as the following:

$(a)^{bo} \cap (b)^{bo} = (a \wedge b)^{bo}$ and $(a)^{bo} \sqcup (b)^{bo} = (a \vee b)^{bo}$ for all $(a)^{bo}, (b)^{bo} \in B^{bo}(I)$.

Theorem 2. *The structure $(B^{bo}(I); \wedge, \sqcup, ^c, (0], I)$ forms a Boolean algebra, where $((a)^{bo})^c = (a^{\star})^{bo}$ represents the complemented element of $(a)^{bo}$ in $B^{bo}(I)$.*

Proof. Since for any $a,b \in L$, $(0)^{bo} \subseteq (a)^{bo}, (b)^{bo} \subseteq I$, and by Proposition 2(9), the meet operation in $B^{bo}(I)$ is well defined.

Now, for any $a,b \in L$, $a,b \le a \vee b$, by using Proposition 2 (2), $(a)^{bo}, (b)^{bo} \subseteq (a \vee b)^{bo}$. That is $(a \vee b)^{bo}$ is an upper bound of both $(a)^{bo}$ and $(b)^{bo}$. Let $(c)^{bo}$ be another an upper bound of both $(a)^{bo}$ and $(b)^{bo}$. Then $(a)^{bo}, (b)^{bo} \subseteq (c)^{bo}$. Suppose that $x \in (a \vee b)^{bo}$, i.e., $x \wedge (a \vee b)^{\star} = x \wedge (a^{\star} \wedge b^{\star}) = 0$. So, $x \wedge a^{\star} \in (b)^{bo} \subseteq (c)^{bo}$. It implies that $x \wedge a^{\star} \wedge c^{\star} = 0$ and $x \wedge c^{\star} \in (a)^{bo} \subseteq (c)^{bo}$. Hence $x \in (c)^{bo}$ and $(a \vee b)^{bo} \subseteq (c)^{bo}$. As a result, $(a \vee b)^{bo} = (a)^{bo} \sqcup (b)^{bo}$.

The distributive condition can be proved as the following, let $(a)^{bo}, (b)^{bo}, (c)^{bo} \in B^{bo}(I)$. Then

$$\begin{aligned}(a)^{bo} \cap [(b)^{bo} \sqcup (c)^{bo}] &= (a)^{bo} \cap (b \vee c)^{bo} = (a \wedge (b \vee c))^{bo} \\ &= ((a \wedge b) \vee (a \wedge c))^{bo} \\ &= (a \wedge b)^{bo} \sqcup (a \wedge c)^{bo} \\ &= ((a)^{bo} \cap (b)^{bo}) \sqcup ((a)^{bo} \cap (c)^{bo}).\end{aligned}$$

Hence, $B^{bo}(I)$ is a bounded distributive lattice.

Since for any $(a)^{bo} \in B^{bo}(I)$ there exists $(a^{\star})^{bo} \in B^{bo}(I)$ such that $(a)^{bo} \wedge (a^{*})^{bo} = (a \wedge a^{\star})^{bo} = (0)^{bo} = (0]$, and if $(x)^{bo} \in B^{bo}(I)$ satisfies the property $(x)^{bo} \wedge (a)^{bo} = (0]$, we get $(x \wedge a)^{bo} = (0]$, and so $x \wedge a = 0$, that means $x \le a^{\star}$, by Proposition 2(2), $(x)^{bo} \subseteq (a^{\star})^{bo}$. Hence, $(a^{\star})^{bo}$ represents the pseudo-complementation of $(a)^{bo} \in B^{bo}(I)$, and $(a)^{bo} \sqcup (a^{*})^{bo} = (a \vee a^{\star})^{bo} = I$. Consequently, $((a)^{bo})^c = (a^{\star})^{bo}$ is the complement of $(a)^{bo} \in B^{bo}(I)$. Therefore, $B^{bo}(I)$ is a Boolean algebra.

Example 1. Consider L in Figure 1. The relative booster of the set $A = \{a,c\}$ in the ideal $I = (\zeta]$ is $A^{bo} = \{0,a\}$ and its booster is $A^{\triangle} = \{0,a,c\}$. Figure 2 represent the boolean algebra of the set of all relative doosters in the ideal I.

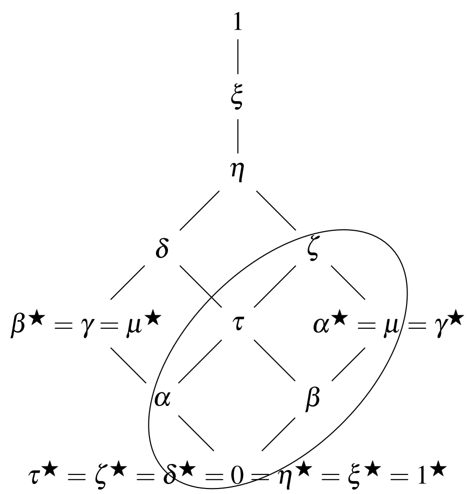

Figure 1: distributive p-algebra L

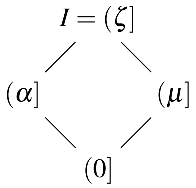

Figure 2: The Boolean algebra $(B^{bo}(I);\wedge,\sqcup,^{c},(0],I)$

Lemma 2. *Let I be a fixed ideal of L and J be an ideal of $B^{bo}(I)$. Then:*

(1)For any ideal K of L, the set $\{(a)^{bo}:a\in K\}$ is an ideal of $B^{bo}(I)$,
(2)The set $\{a\in L:(a)^{bo}\in J\}$ is an ideal of L.

Proof. (1)For all $a,b\in K$ we get $a\vee b\in K$. So, if $(a)^{bo},(b)^{bo}\in\{(a)^{bo}:a\in K\}$, then $(a)^{bo}\sqcup(b)^{bo}=(a\vee b)^{bo}\in\{(a)^{bo}:a\in K\}$. Now let $(a)^{bo}\in\{(a)^{bo}:a\in K\}$ and $(c)^{bo}\in B^{bo}(I)$ such that $(c)^{bo}\subseteq(a)^{bo}$. Then $(c)^{bo}\cap(a)^{bo}=(c)^{bo}=(c\wedge a)^{bo}\in\{(a)^{bo}:a\in K\}$, since $a\wedge c\in K$. Thus $\{(a)^{bo}:a\in K\}$ is an ideal of $B^{bo}(I)$.
(2)If $(a)^{bo},(b)^{bo}\in J$, implies $(a)^{bo}\sqcup(b)^{bo}=(a\vee b)^{bo}\in J$. Accordingly, if $a,b\in\{a\in L:(a)^{bo}\in J\}$, then $a\vee b\in\{a\in L:(a)^{bo}\in J\}$. Assume $a\in\{a\in L:(a)^{bo}\in J\}$ and $c\in L$ such that $c\leq a$. Hence $(c)^{bo}\cap(a)^{bo}=(c)^{bo}=(c\wedge a)^{bo}\in J$ and $c\in\{a\in L:(a)^{bo}\in J\}$. Therefore $\{a\in L:(a)^{bo}\in J\}$ is an ideal of L.

Definition 4. *Let $I(L)$ be a lattice of ideals of L and $I(B^{bo}(I))$ be a lattice of the ideals of $B^{bo}(I)$, for a fixed ideal I of L. Then define the maps:*
$\phi:I(L)\to I(B^{bo}(I))$ as: $\phi(K)=\{(a)^{bo}:a\in K\}$, for any ideal K of L and,
$\psi:I(B^{bo}(I))\to I(L)$ as: $\psi(J)=\{a\in L:(a)^{bo}\in J\}$, for any ideal J of $B^{bo}(I)$.

Theorem 3. *The maps ϕ and ψ are satisfy the following conditions:*

(1)Maps ϕ and ψ are isotones,
(2)The map $\phi(\psi)$ is an identity map,
(3)The map $\psi(\phi)$ is a clousure operartor,
(4)$\psi(\phi(K\cap G))=\psi(\phi(K))\cap\psi(\phi(G))$, for any two ideals $K,G\in I(L)$.

Proof. (1)If $K,G\in I(L)$ and $K\subseteq G$, then $\phi(K)\subseteq\phi(G)$.On the other side, suppose $J,H\in I(B^{bo}(I))$ such that $J\subseteq H$. Then $\{a\in L:(a)^{bo}\in J\}\subseteq\{a\in L:(a)^{bo}\in H\}$. I.e., $\psi(J)\subseteq\psi(H)$.
(2)Assume $J\in I(B^{bo}(I))$ implies $\psi(J)\in I(L)$. Accordingly, $(a)^{bo}\in J$ iff $a\in\psi(J)$ iff $(a)^{bo}\in\phi(\psi(J))$. Therefore $\phi(\psi(J))=J$.
(3)Now we show that $\psi(\phi)$ is extensive, isotone and idempotent.

(i)Let $a\in K\in I(L)$. Then $(a)^{bo}\in\phi(K)$. It means $a\in\psi(\phi(K))$ and $K\subseteq\psi(\phi(K))$. Hence $\psi(\phi)$ is extensive.
(ii)Let $K,G\in I(L)$ such that $K\subseteq G$ and $a\in\psi(\phi(K))$. Then $(a)^{bo}\in\phi(K)$ and there exist $b\in K\subseteq G$ such that $(a)^{bo}=(b)^{bo}\in\phi(G)$. Since $\phi(G)$ is an ideal of $B^{bo}(I)$,hence $a\in\psi(\phi(G))$. As a result, $\psi(\phi(K))\subseteq\psi(\phi(G))$ and $\psi(\phi)$ is isotone.
(iii)We get $\psi(\phi(K))\subseteq\psi(\phi(\psi(\phi(K))))$, Since $\phi(\psi(\phi(K)))\in I(B^{bo}(I))$. Conversly, suppose $a\in\psi(\phi(\psi(\phi(K))))$ implies $(a)^{bo}\in\phi(\psi(\phi(K)))$. Let $b\in\psi(\phi(K))$ and $(b)^{bo}=(a)^{bo}\in\phi(K)$. Therefore $a\in\psi(\phi(K))$ and it is idempotent.

(4)Suppose $a\in\psi(\phi(K))\cap\psi(\phi(G))$.It means $(a)^{bo}\in\phi(K)\cap\phi(G)=\phi(K\cap G)$. So, $a\in\psi(\phi(K\cap G))$ and $\psi(\phi(K\cap G))\supseteq\psi(\phi(K))\cap\psi(\phi(G))$. The convers is clear.

4 Disjunctive and Normal Relative Boosters

In the present section, the behavior of a disjunctive relative booster ideal and a normal relative booster ideal in distributive p-algebra are established.

Definition 5. *An ideal I of L is a disjunctive relative booster if it satisfies*

$$(a)^{bo}=(b)^{bo}\ \text{ implies that }\ a=b\ \text{ for all } a,b\in L.$$

Definition 6. *An ideal I of L is a normal relative booster ideal if it satisfies*

$$(a)^{bo}\vee(b)^{bo}=(a\vee b)^{bo}\ \text{ for all }\ a,b\in L$$

In particular case $I=L$, we can say that L is called disjunctive (or normal) booster.

Example 2. Consider L in Figure (1). The ideal $(\tau]$ is disjunctive and normal relative booster while the ideal $(\eta]$ is normal relative booster but not disjunctive. The ideal $(\xi]$ is neither disjunctive nor normal relative booster in L.

Theorem 4. *Let I be a disjunctive relative booster ideal of L. Then I can be embedded into $B^{bo}(I)$.*

Proof. Suppose that $\varphi:I\to B^{bo}(I)$ is defined by: $\varphi(a)=(a)^{bo}$, for all $a\in I$. It is easy to see that φ is well defined. Moreover,

$$\varphi(a\wedge b)=\varphi(a\wedge b)^{bo}=(a)^{bo}\cap(b)^{bo}=\varphi(a)\cap\varphi(b),\text{and}$$
$$\varphi(a\vee b)=(a\vee b)^{bo}=(a)^{bo}\sqcup(b)^{bo}=\varphi(a)\sqcup\varphi(b).$$

Also, φ is one-to-one follows from the disjunctive of I.

Corollary 2. *L is disjunctive booster iff L is a Boolean algebra.*

A characterization of a normal relative booster ideal is given in the following results:

Theorem 5. *An ideal I of L is normal relative booster iff $B^{bo}(I)$ is a Boolean sub-lattice of $I(L)$.*

*Proof.*The necessary condition follows from Theorem 2. Conversely, suppose that $B^{+}(I)$ is a Boolean sub-lattice of $I(L)$ and for all $(a)^{bo},(b)^{bo}\in B^{bo}(I)$

$$(a)^{bo}\vee(b)^{bo}=(c)^{bo},\quad \text{for some}\quad c\in L.$$

So, $(a)^{bo},(b)^{bo}\subseteq(c)^{bo}$, and $a,b\le a\vee b$, by Proposition 2(2),$(a)^{bo},(b)^{bo}\subseteq(a\vee b)^{bo}$. Thus, $(c)^{bo}$ and $(a\vee b)^{bo}$ are upper bounds of both $(a)^{bo}$ and $(b)^{bo}$. Since L is distributive, then $I(L)$ so, and

$$\begin{aligned}(c)^{bo}\wedge(a\vee b)^{bo}&=(c)^{bo}\wedge[(a)^{bo}\sqcup(b)^{bo}]\\&=[(c)^{bo}\wedge(a)^{bo}]\sqcup[(c)^{bo}\wedge(b)^{bo}]\\&=(a)^{bo}\sqcup(b)^{bo}=(a\vee b)^{bo}.\end{aligned}$$

Therefore $(a\vee b)^{bo}\subseteq(c)^{bo}$ and $(a)^{bo}\vee(b)^{bo}=(a\vee b)^{bo}$.

Corollary 3.*L is normal booster iff $B^{bo}(I)$ is a Boolean sub-lattice of $I(L)$.*

Theorem 6.*If L is a Stone algebra, then L is a normal booster.*

*Proof.*Since, $a,b\le a\vee b$, then by Proposition 2(2), $(a)^{\triangle},(b)^{\triangle}\subseteq(a\vee b)^{\triangle}$ and so $(a)^{\triangle}\vee(b)^{\triangle}\subseteq(a\vee b)^{\triangle}$. Now, suppose that $x\in(a\vee b)^{\triangle}$,i.e., $x\wedge(a\vee b)^{\star}=x\wedge a^{\star}\wedge b^{\star}=0$. It implies that $(x\wedge a^{\star}\wedge b^{\star}]=(x\wedge a^{\star}]\wedge(b^{\star}]=(0]$. Hence $(x\wedge a^{\star}]=(x]\wedge(a^{\star}]\subseteq(b^{\star}]^{\star}=(b^{\star\star}]=(b)^{\triangle}$. But $(x]\wedge(a^{\star\star}]\subseteq(a^{\star\star}]=(a)^{\triangle}$. Accordingly, $((x]\wedge(a^{\star\star}])\vee((x]\wedge(a^{\star}])=(x]\wedge((a^{\star}]\vee(a^{\star\star}])=(x]\wedge(a^{\star}\vee a^{\star\star}]=(x]\subseteq(a)^{\triangle}\vee(b)^{\triangle}$. Therefore, $x\in(a)^{\triangle}\vee(b)^{\triangle}$ and $(a)^{\triangle}\vee(b)^{\triangle}=(a\vee b)^{\triangle}$

Corollary 4.*A disjunctive booster distributive p-algebra is a normal booster*

The convers of the above corollary is not true. For example, the Stone algebra of three elements chain is normal booster but not disjunctive booster.

5 Congruence Relations Via The Relative Boosters

In the present section a lattice congruence defined via the relative boosters is studied and some properties are derived.

Definition 7.*[6] A Gilvenko congruence relation Φ on L is defined as*

$$a\equiv b(\Phi)\ \ iff\ \ a^{\star\star}=b^{\star\star}.$$

Theorem 7.*Let Θ be a binary relation defined on L by the rule, for a fixed element $z\in L$ and fixed ideal I of L,*

$$a\equiv b(\Theta)\ \ \textit{iff}\ \ (a)^{bo}\cap(z)^{bo}=(b)^{bo}\cap(z)^{bo}\ \ \textit{for all}\ \ a,b\in L.$$

Then Θ is a lattice congruence on L.

*Proof.*It is clear that,Θ is an equivalence relation on L. Now, let $a\equiv b(\Theta)$ and $c\equiv d(\Theta)$ for all $a,b,c,d\in L$ by the definition, we get

$$(a)^{bo}\cap(z)^{bo}=(b)^{bo}\cap(z)^{bo}\ \ and\ \ (c)^{bo}\cap(z)^{bo}=(d)^{bo}\cap(z)^{bo},$$

thus

$$\begin{aligned}(a\wedge c)^{bo}\cap(z)^{bo}&=(a)^{bo}\cap(c)^{bo}\cap(z)^{bo}=(b)^{bo}\cap(d)^{bo}\cap(z)^{bo}\\&=(b\wedge d)^{bo}\cap(z)^{bo},\ \text{and}\\(a\vee c)^{bo}\cap(z)^{bo}&=((a\vee c)\wedge z)^{bo}=((a\wedge z)\vee(c\wedge z))^{bo}\\&=(a\wedge z)^{bo}\sqcup(c\wedge z)^{bo}=(b\wedge z)^{bo}\sqcup(d\wedge z)^{bo}\\&=((b\wedge z)\vee(d\wedge z))^{bo}=((b\vee d)\wedge z))^{bo}\\&=(b\vee d)^{bo}\cap(z)^{bo}\end{aligned}$$

Therefore, $a\wedge c\equiv b\wedge d(\Theta)$ and $a\vee c\equiv b\vee d(\Theta))$. That is, Θ is a lattice congruence relation on L.

*Example 3.*Consider relative boosters of elements of lattice L in ideal $I=(\zeta]$. Closed curves in Figure 3 represent the congruence classes of relation Θ which is defined as $a\equiv b(\Theta)$ iff $(a)^{bo}\cap(\alpha)^{bo}=(b)^{bo}\cap(\alpha)^{bo}$ for a fixed element $\alpha\in L$ and for all $a,b\in L$.

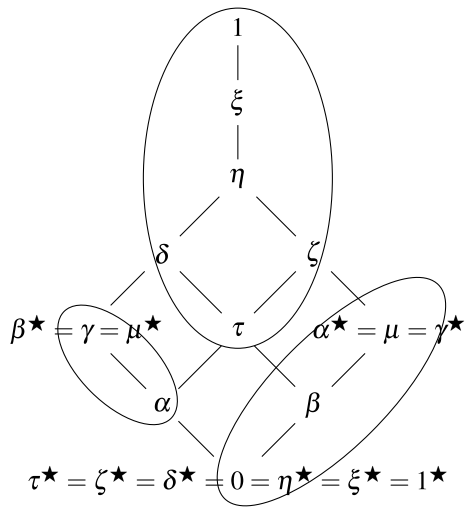

Figure 3: Congruence classes of relation Θ

Corollary 5.*If the fixed element z in L, $z\in I\cap D(L)$ is dense element, then the congruence relation Θ on L defined as*

$$a\equiv b(\Theta)\ \ \textit{iff}\ \ (a)^{bo}=(b)^{bo}\ \ \textit{for all}\ \ a,b\in L$$

In particular case, whenever $I=L$, we have

$$a\equiv b(\Theta)\ \ \textit{iff}\ \ (a)^{\triangle}=(b)^{\triangle}\ \ \textit{for all}\ \ a,b\in L$$

A Gilvinko congruence is an example of this congruence.

Theorem 8.*Let Θ be a congruence relation on L defined as*

$$a\equiv b(\Theta)\ \ \textit{iff}\ \ (a)^{bo}=(b)^{bo}\ \ \textit{for all}\ \ a,b\in L.$$

Then, the structure $(L/\Theta;\wedge,\vee,^{c},(0],D(L))$ forms a Boolean algebra.

Proof. From the definition of the class $[a]\Theta = \{x \in L : (x)^{bo} = (a)^{bo}\}$, we have

$[0]\Theta = \{x \in L : (x)^{bo} = (0)^{bo}\} = \{0\} = (0]$.

So, $[0]\Theta$ is the smallest congruence class of L. Also, let $d \in D(L)$,

$$[d]\Theta = \{x \in L : (x)^{bo} = (d)^{bo}\} = \{x \in L : (x)^{bo} = (1]\} = \{x \in L : x^{\bigstar} = (0]\} = D(L).$$

$D(L)$ is a class of L/Θ, since the set of dense elements $D(L)$ forms a filter of L, then for any $x \in L$ and $d \in D(L)$, we have $x \vee d \in D(L)$. Thus $[x]\Theta \vee [d]\Theta = [x \vee d]\Theta = D(L)$. Therefore, $D(L)$ is the largest element of L/Θ. Moreover, $[a]\Theta \wedge [a^{\bigstar}]\Theta = [a \wedge a^{\bigstar}]\Theta = [0]\Theta$ and $[a]\Theta \vee [a^{\bigstar}]\Theta = [a \vee a^{\bigstar}]\Theta = D(L)$.

It means that the complement $([a]\Theta)^c$ of $[a]\Theta$ is $[a^{\bigstar}]\Theta$. As a result, L/Θ is a Boolean algebra.

Example 4. Consider relative boosters of elements of lattice L in ideal $I = (\zeta]$. Closed curves in Figure 4 represent the congruence classes of relation Θ which is defined as $a \equiv b(\Theta)$ iff $(a)^{bo} \cap (\delta)^{bo} = (b)^{bo} \cap (\delta)^{bo}$ for all $a, b \in L$. It is equivalent to $a \equiv b(\Theta)$ iff $(a)^{bo} = (b)^{bo}$, since $\delta \in D(L)$. Figure 5 shows the Boolean algebra of L/Θ.

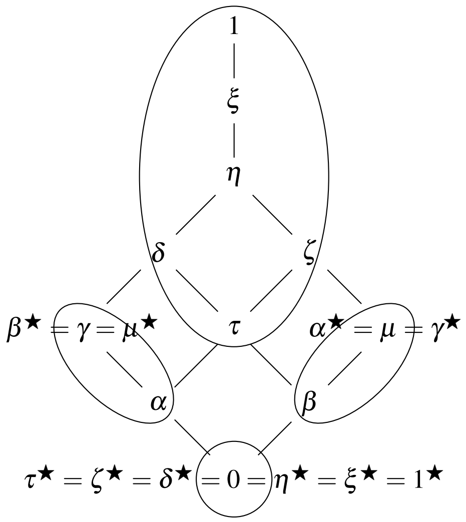

Figure 4: Congruence classes of relation Θ

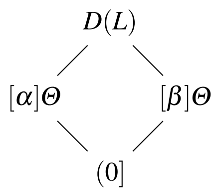

Figure 5: The Boolean algebra $(L/\Theta; \wedge, \vee, ^c, (0], D(L))$

Corollary 6. *Let $B^{bo}(I)$ be a Boolean algebra of relative boosters of fixed ideal I of L and Θ be a congruence defined as*

$a \equiv b(\Theta)$ iff $(a)^{bo} = (b)^{bo}$ for all $a, b \in L$.

Then, $L/\Theta \cong B^{bo}(I)$.

6 Conclusion

(1) The structure of relative booster ideals of a distributive p-algebra L forms a Boolean algebra, and hence it is a generalization of the booster ideals of L. Moreover, there is a closure operator generated by $I(B^{bo}(I))$ on the lattice ideal of L.

(2) A distributive p-algebra L is disjunctive booster iff L is a Boolean algebra, and L is normal booster iff the structure of booster ideals is a sublattice of the structure of all ideals of L.

(3) If the congruence relation Θ defined via the relative booster ideal of a distributive p-algebra L. Then the structure $(L/\Theta; \wedge, \vee, ^c, (0], D(L)))$ forms a Boolean algebra isomorphic to the Boolean algebra of relative boosters.

Conflicts of Interests

The authors declare that they have no conflicts of interests

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