Complexity of the Integration on Hölder-Nikolskii Classes with Mixed Smoothness

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Abstract: We study the information complexity of the numerical integration on the Hölder-Nikolskii classes \(M_{\alpha_p}^r\) in the randomized setting. We adopt classical Monte Carlo method to approximate this integration and derive the corresponding convergence rate. Comparing our results with the previous known results in the deterministic setting, we see that the randomized algorithms have faster convergence rates.

Keywords: Information complexity, Randomized algorithm, Hölder-Nikolskii class, Mixed smoothness, Convergence rate

1 Introduction

The calculation of \(d\)-dimensional integrals occurs in numerous applications including physics, chemistry, finance, and the computational sciences, where the \(d\) may be in the hundreds or even in the thousands, see [1,6,7,8,9]. For most integrands we could not compute the integral utilizing the fundamental theorem of the calculus since there is no closed form expression of the antiderivatives. We have to approximate the integral numerically. Algorithms for solving this problem are given by using function values on finite points. The information complexity is the minimal number of the function values, needed to solve the problem to within a threshold \(\varepsilon\). It is a lower bound of computational complexity which is defined as the minimal number of information operations and combinatory operations to obtain a solution to within \(\varepsilon\). However it is proved that for the integration problem the information complexity is proportional to the computational complexity, cf. [15]. Thus as the computational complexity, the information complexity is a fundamental invariance of computer science. In this paper we concentrate on the information complexity of integration on the classes of functions with mixed smoothness.

A central issue of information complexity theory is to investigate how the information complexity of a given problem depends on \(\varepsilon^{-1}\) and \(d\). If it depends exponentially on \(\varepsilon^{-1}\) or \(d\), we say the problem is intractable. This intractability is also called the curse of dimensionality, cf. [1,4,6,9]. It is well-known that the integration problem defined on the usual Sobolev classes of functions suffers from the curse of dimensionality in the deterministic setting. It is also known that the randomization can break the intractability, that is, the complexity in the randomized setting depends polynomially on \(\varepsilon^{-1}\) and \(d\), cf. [5,6,7, 9]. This shows the advantage of the randomized methods. In this paper we study the efficiency of randomized method in the computation of the high dimensional integration. We consider Hölder-Nikolskii class of functions with mixed smoothness. cf. [3,5,14]. This class plays an important role in the study of the complexity of many numerical problems, such as integration and function approximation, since the bounds of deterministic complexity of these problems depend weakly on the dimension, cf. [2,4,10,11,12,13]. In what follows, we will use randomized method to approximate the integration on this class and derive the corresponding convergence rates. Our results show the randomized algorithms have faster convergence rates than the deterministic ones.

The remaining part of this paper is organized as follows. In Section 2 we formulate the problem of integration in the framework of information-based complexity theory and present the main results. In Section 3 we prove our main results. Finally, in Section 4 we
We would like to approximate the integral $C \int_T \ldots$ by some quadrature formula, where $\lambda^d$ denotes the normalized Lebesgue measure on $T^d$. When $F$ is specified, we write down the integration problem as $(F, \text{Int})$.

We now describe the deterministic quadrature formula. For any given $k \in \mathbb{N}$, we choose $k$ points $t_1, \ldots, t_k \in T^d$ and weights $c_1, \ldots, c_k \in \mathbb{R}$ to compose a mapping $q$ via

$$q(f) := \sum_{j=1}^{k} c_j f(t_j), \quad f \in F,$$

where $F$ is some class of continuous functions. The number $k$ is called the cardinality of the quadrature formula. Denote the set of all $k$-point quadrature rules, by $Q_k$. We introduced the classes of admissible quadrature formulas

$$\mathcal{Q}^n(F, \mathbb{R}) := \bigcup_{k \leq n} Q_k,$$

and

$$\mathcal{Q}(F, \mathbb{R}) := \bigcup_{n \in \mathbb{N}} \mathcal{Q}^n(F, \mathbb{R}) = \bigcup_{k \geq 0} Q_k.$$

The error of a quadrature rule $q \in \mathcal{Q}(F, \mathbb{R})$ on the class $F$ is defined as

$$e(F, \text{Int}, q) := \sup \{|\text{Int}(f) - q(f)|, \quad f \in F\}.$$
numbers, the order inequality \( a_n \ll b_n \) means that there is a number \( c > 0 \) such that, for all \( n \), we have \( a_n \leq cb_n \); and the relation \( a_n \approx b_n \) means \( a_n \approx b_n \) and \( b_n \approx a_n \).

Now we recall the results about the deterministic complexity of the integration on the classes \( MH_p^r \).

**Theorem 1**[14] Let \( r > 1 \) be a real number, \( 1 < p \leq \infty \). Then
\[
e_n^{det}(MH_p^r, \text{Int}) \approx n^{-r}(\log(d-1)n).
\]

Our main results are the following two theorems.

**Theorem 2.** Let \( r > 1 \) be a real number, \( 1 < p < \infty \). Then
\[
e_n^{mc}(MH_p^r, \text{Int}) \approx \begin{cases} n^{-r-1/2}(\log(d-1)n)^{p+1}, & 2 \leq p < \infty; \\ n^{-r-1/p}(\log(d-1)n)^{1/p}, & 1 < p < 2. \end{cases}
\]

**Theorem 3.** Let \( r > 1 \) be a real number, \( 1 < p < \infty \). Then
\[
e_n^{mc}(MH_p^r, \text{Int}) \gg \begin{cases} n^{-r-1/2}, & 2 \leq p < \infty; \\ n^{-r-1/p}, & 1 < p < 2. \end{cases}
\]

We see from Theorem 2 and 3 that modulo a power of logarithm, the sharp bound of \( e_n^{mc}(MH_p^r, \text{Int}) \) has been determined. Comparing Theorem 1 with Theorem 2 and 3, one can see that the randomized method provides a better convergence rate than the deterministic one. Quantitatively, the improvement amounts to the factor \( n^{-1/2} \) if \( p \geq 2 \) and \( n^{-1/p} \) if \( 1 < p < 2 \).

### 3 Proofs of main results

In this section, we shall prove our main results. To this end, we need some auxiliary lemmas. First we invoke a result of approximation of functions from \( MH_p^r \) by trigonometric polynomials.

For \( m \in \mathbb{N} \), we define the de la Vallee-Poussin kernel \( \mathcal{V}_m(x) \) by
\[
\mathcal{V}_m(x) = 1 + 2 \sum_{k=1}^{m} \cos(kx) + 2 \sum_{k=m+1}^{2m} \left( \frac{m-k}{m} \right) \cos(kx),
\]
where \( x \in \mathbb{R} \). Denote \( x(l) = \frac{\pi l}{2m} \), \( l = 1, \ldots, 2m \), we define the linear operator
\[
R_m(f, x) = (4m)^{-1} \sum_{l=1}^{4m} f(x(l)) \mathcal{V}_m(x-x(l))
\]
and
\[
\Delta_n(f, x) = R_{2^n}(f, x) - R_{2^{n-1}}(f, x), \quad n \geq 1,
\]
where \( \Delta_0(f, x) = R_1(f, x) \). Then we define the operator \( \Delta_n(f, \mathbf{x}) \) as the composition of the one-dimensional operators
\[
\Delta_n(f, \mathbf{x}) = \Delta_{x_d}(\Delta_{x_{d-1}} \cdots \Delta_{x_1}(f, x_1) \cdots x_d),
\]
where \( \Delta_i \) acts as a one-dimensional operator on a function depending on the variable \( x_i \). Now the approximation operator \( T_{Q_n} \) is defined as follows:
\[
T_{Q_n} = \sum_{|\mathbf{s}| \leq n} \Delta_n(f, \mathbf{x}),
\]
where \( Q_n \) denotes the number of function values used in \( T_{Q_n} \). It is clear that \( Q_n \approx 2^{m}n^{-1} \).

**Lemma 1.**[14] Let \( 1 \leq p \leq \infty, r > 1/p \). For \( f \in MH_p^r \), we have
\[
\|f - T_{Q_n}(f)\|_p \ll 2^{-n}n^{d-1}.
\]

The proof of Theorem 2 is based on variance reduction. To carry out this process, we will use the classical Monte Carlo quadrature formula which is defined as follows. Let \( (\xi_k^i)_{k=1}^m \) be independent, \( \mathcal{T}^d \)-valued, uniformly distributed over \( \mathcal{T}^d \) random variables on some probability measure space \( (\Omega, \Sigma, \mu) \). For \( f \in C(\mathcal{T}^d) \), we put
\[
Q_\omega(f) = \frac{1}{n} \sum_{i=1}^{n} f(\xi^i(\omega)), \quad \omega \in \Omega.
\]

**Lemma 2.**[3,6] Let \( 1 \leq p \leq \infty \). Then for all \( f \in C(\mathcal{T}^d) \),
\[
\int_{\Omega} |\text{Int}(f) - Q_\omega(f)|d\mu(\omega) \leq \begin{cases} n^{-1/2}\|f\|_p, & 2 \leq p \leq \infty; \\ c_p n^{-1/p-1}\|f\|_p, & 1 \leq p < 2, \end{cases}
\]
where \( c_p = 2^{2/p-1} \).

**Proof of Theorem 2.** We first construct a randomized algorithm based on the classical Monte Carlo quadrature rule.

Let \( (\Omega, \Sigma, \mu) \) and \( Q_\omega \) be defined as above, where \( m = Q_n \). For \( f \in MH_p^r \), we set for \( \omega \in \Omega \),
\[
A_\omega(f) = Q_\omega(f - T_{Q_n}(f)) + \text{Int}(T_{Q_n}(f)) = Q_\omega(f - T_{Q_n}(f)) + q_m(f).
\]

It is easy to see that \( A = (\Omega, \Sigma, \mu), (A_\omega)_{\omega \in \Omega} \) is a randomized method, where the required measurability follows from the fact that the mapping
\[
(f, \mathbf{x}) \longrightarrow f(\mathbf{x}) - (T_{Q_n}(f)(\mathbf{x})
\]
from \( C(\mathcal{T}^d) \times \mathcal{T}^d \) into \( \mathbb{R} \) is continuous. Obviously, \( A \in \mathcal{M}^{2m}(MH_p^r, \mathbb{R}) \). Then we derive a upper estimate of the error of \( A_\omega \) which leads to the required bound. Using Lemma 2, we get for \( f \in MH_p^r \)
\[
\int_{\Omega} |\text{Int}(f) - A_\omega(f)|d\mu(\omega)
\]
\[
= \int_{\Omega} |\text{Int}(f) - Q_\omega(f - T_{Q_n}(f)) - \text{Int}(T_{Q_n}(f))|d\mu(\omega)
\]
\[
= \int_{\Omega} |\text{Int}(f - T_{Q_n}(f)) - Q_\omega(f - T_{Q_n}(f))|d\mu(\omega)
\]
\[
\leq m^{-1/p}\|f - T_{Q_n}(f)\|_p,
\]
(3)
where \( \alpha_p = 1/2 \) for \( 2 \leq p < \infty \) and \( \alpha_p = 1 - 1/p \) for \( 1 < p < 2 \). Combining the relationship \( m = Q_n \propto 2^n n^{d-1} \), Lemma 1 and (3), we get
\[
e_n^{mc}(MH_p^n, \mathrm{Int}) \ll n^{-r-1/(1+p/2)}(\log(n))^{r+1}.
\]

The proof of Theorem 2 is complete.

Next we turn to the proof of the lower bounds. We will reduce the lower estimate in the randomized setting to that of the average case setting. For this purpose we recall the definition of the \( n \)-th minimal error in the average setting.

Let \( X \) be a Banach space and \( F \) be a closed bounded subset of \( X \). Assume that the set \( F \) is equipped with a Borel field \( \mathcal{B}(F) \) which is the \( \sigma \)-algebra containing all open subsets of \( F \). Let \( \mu \) be a probability measure defined on \( (F, \mathcal{B}(F)) \). Denote the subset of those \( q \in \mathcal{L}^m(F, \mathbb{R}) \) which are \( (\mathcal{B}(F), \mathcal{B}(\mathbb{R})) \) measurable by \( \mathcal{L}^m(F, \mathbb{R}) \). Then, for \( q \in \mathcal{L}^m(F, \mathbb{R}) \), we define the average error with respect to \( \mu \) as
\[
e(\mathrm{Int}, q, \mu) := \int_F \|\mathrm{Int}(f) - q(f)\|d\mu(f).
\]

Minimizing the errors with respect to the choice of \( q \in \mathcal{L}^m(F, \mathbb{R}) \), we yield the \( n \)-th minimal average error with respect to \( \mu \)
\[
e^{avg}_{n+1}(\mathrm{Int}, \mu) := \inf_{q \in \mathcal{L}^m(F, \mathbb{R})} e(\mathrm{Int}, q, \mu).
\]

**Lemma 3.** For every probability measure \( \mu \) on \( F \) and for all \( n \in \mathbb{N} \),
\[
e_n^{mc}(F, \mathrm{Int}) \geq e^{avg}_{2n-1}(\mathrm{Int}, \mu)/2.
\]

**Proof of Theorem 3.** In the proof of the lower bounds, we separate two cases, \( 2 \leq p \leq \infty \) and \( 1 \leq p < 2 \).

First, consider the case \( 2 \leq p \leq \infty \). It suffices to consider the case \( p = \infty \). We shall define a probability measure on \( F = MH_{\infty}^n \). For this purpose, we need construct \( n := 2n \) functions with mutually disjoint supports. For a given \( n \in \mathbb{N} \), we divide the torus \( T^d \) into \( 2n \) equal sub-intervals \( \{G_i\}_{i=1}^{2n} \) with mutually disjoint interiors,
\[
G_i = \left\{ x \in T^d : x_1 \in \tilde{T}, x_j \in T, \quad j = 2, \ldots, d \right\},
\]
where
\[
\tilde{T} = \left\{ x_1 : \frac{(i-1)\pi}{n} \leq x_1 < \frac{(i-1)\pi}{n} + \frac{\pi}{n} \right\},
\]
and \( i = 1, \ldots, n \). Then we choose \( \phi \) to be a fixed bump function in \( C_c^\infty(\mathbb{R}) \) with support contained in \( T_0 \), where \( T_0 \) is the interior of \( T \), such that \( 0 \leq \phi(t) \leq 1, t \in T \), \( \phi(t) = 1 \), when \( t \in [\pi/2, 3\pi/2] \). The function \( f_i \) is defined to have a bump only in the rectangle \( G_i \) as follows:
\[
f_i(x) = a_p C_p(2n)^{-r+1/p} \phi(2n(x_1 - (i - 1)\pi/n)) \phi(x_2) \cdots \phi(x_d),
\]
for \( x \in T^d \), where \( 1 \leq p \leq \infty \). It is easily seen that
\[
\int_{T^d} f_i(x) d\lambda^d(x) = \begin{cases} a_p C_p(2n)^{1-r+1/p}, & 1 < p < \infty, \\ a_p C_p(2n)^{-r+1}, & p = \infty, \end{cases}
\]
where \( \int_T \phi(t) dt = C \). Furthermore, let \( \{\varepsilon_i\}_{i=1}^n \) be a sequence of independent, \( \{-1, 1\} \)-valued random variables on some probability space \( (\Omega_1, \Sigma_1, \mu_1) \) with
\[
\mu_1(\{\varepsilon_i = 1\}) = \mu_1(\{\varepsilon_i = -1\}) = 1/2, \quad i = 1, \ldots, n.
\]
For \( p = \infty \), we choose a constant \( a_p > 0 \) such that \( f_i \in MH_{\infty}^n \) for \( i = 1, \ldots, n \) and put
\[
\mu = \text{dist} \left( \sum_{i=1}^n \varepsilon_i f_i \right),
\]
where \( \text{dist} \) means the distribution of the \( MH_{\infty}^n \)-valued random variable. For any system of points \( x_1, \ldots, x_n \), let us define \( I \) by
\[
I = \{ i : 1 \leq i \leq n, \quad \{x_1, \ldots, x_n\} \cap G_i = \emptyset \}.
\]
It is clear that the cardinality of the set \( I \) satisfies
\[
|I| \geq n - n = n.
\]
Further, \( q(f_i) = 0 \) for \( i \in I \), and we can estimate
\[
\int_{MH_{\infty}^n} |\mathrm{Int}(f) - q(f)|d\mu(f)
\]
\[
= \int_{\Omega_1} \left| \int T_{f_i} \sum_{i=1}^n \varepsilon_i(\omega) f_i - q \left( \sum_{i=1}^n \varepsilon_i(\omega) f_i \right) \right| d\mu_1(\omega)
\]
\[
= \int_{\Omega_1} \left| I_1 + I_2 - q \left( \sum_{i \in I} \varepsilon_i(\omega) f_i \right) \right| d\mu_1(\omega),
\]
where
\[
I_1 := \sum_{i \in I} \varepsilon_i(\omega) \mathrm{Int}(f_i), \quad I_2 := \sum_{i \notin I} \varepsilon_i(\omega) \mathrm{Int}(f_i).
\]
The distribution of \( \{\varepsilon_i\}_{i=1}^n \) does not change if we replace \( \varepsilon_i \) by \( -\varepsilon_i \) for \( i \in I \) and leave it unchanged for \( i \notin I \). Consequently, we can continue (6) as follows
\[
= \int_{\Omega_1} \left| -I_1 + I_2 - q \left( \sum_{i \notin I} \varepsilon_i(\omega) f_i \right) \right| d\mu_1(\omega)
\]
\[
\geq \int_{\Omega_1} \sum_{i \notin I} \varepsilon_i(\omega) \mathrm{Int}(f_i) \left| d\mu_1(\omega) \right| \left( \sum_{i \notin I} |\mathrm{Int}(f_i)|^2 \right)^{1/2}
\]
\[
= |I|^{1/2} |\mathrm{Int}(f_i)| \approx n^{r-1/2},
\]
which together with Lemma 3 completes the lower estimates for \( 2 \leq p \leq \infty \). In the above proof, we used the Khintchine’s inequality, and the relations (4), (5). Now let \( 1 \leq p < 2 \) and let \( \mu \) be the equi-distribution on the set
\[
\{ \pm f_i : i = 1, \ldots, n \},
\]
where this time we choose a constant $a_p > 0$ such that $f_i \in MH_p$, for $i = 1, \ldots, n$. With $I$ as above, using the relations (4) and (5) again, we get

$$
\int_{MH_p} |I(f) - q(f)| d\mu(f)
\leq \frac{1}{2n} \sum_{i=1}^{n} \sum_{\sigma \in \{\pm 1\}^n} |\sigma I(f_i) - q(\sigma f_i)|
\geq \frac{1}{2n} \sum_{i \in I} \sum_{\sigma \in \{\pm 1\}^n} |\sigma f_i|
\geq \frac{1}{2n} \sum_{i \in I} |f_i| \gg n^{-r+1/p-1},
$$

which together with Lemma 3 again completes the lower estimates for $1 \leq p < 2$. The proof of Theorem 3 is complete.

4 Conclusion:

We determine the information complexity of the integration over the class $MH_p$. Our results show that this problem is tractable in the randomized setting. Moreover if we neglect the logarithmic factor, then we find that the convergence rate does not depend on $d$. This property again shows the great advantage of the randomized methods. It allows us to use randomized methods to approximate high dimensional integration when the integrand is taken from the class $MH_p$. In particular we can approximate the path integration by randomized algorithms. In this case the dimension $d$ can be arbitrarily large. Since path integration lies at the foundation of quantum mechanics, statistical mechanics and mathematical finance. Our results may have potential applications in these fields.

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References


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