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Stability and Spatial Chaos in 2D Hénon System

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Abstract: This paper is concerned with two-dimensional(2-D) discrete system of the following form

$$x_{m+1,n} + ax_{m,n+1} = f(\mu, (1+a)x_{m,n}, bx_{m-1,n}),$$

where a, μ, b is a real parameters. We investigate the fixed planes, stability of the fixed planes and spatial chaos behavior for this system. A stability condition for the fixed plane is given, and it is proven analytically that for some parameter values the system has a transversal homoclinic orbit, which is a verification of this system to be chaotic in the sense of Li-Yorke. These results extend the corresponding results in the one-dimensional (1-D) Hénon system:

$$x_{m+1,n_0} = f(\mu, x_{m,n_0}, bx_{m-1,n_0}),$$

where n_0 is a fixed integer. These results also extend the corresponding results in the 2-D Logistic system:

$$x_{m+1,n} + ax_{m,n+1} = f(\mu, (1+a)x_{m,n,}),$$

Keywords: Spatial Logistic system, spatial Hénon system, two-dimensional discrete system

1 Introduction

In the engineering applications, particularly in the fields of digital filtering, imaging, and spatiotemporal dynamical systems, two-dimensional (2-D) discrete systems have been a focused subject for investigation (see, for example, [1,2,3,4,5] and [9,10,11,12] and the references cited therein). In this paper, we consider the following 2-D discrete system:

$$x_{m+1,n} + ax_{m,n+1} = f(\mu, (1+a)x_{m,n}, bx_{m-1,n}), \quad (1)$$

where $m, n \in N_r = \{r, r+1, r+2, ...,\}$ r is an integer and $r \leq 0$, a, μ, b is a real parameters, and $f : R^2 \to R$ is a nonlinear function with $R = (-\infty, +\infty)$.

First, observe that in the particular case where $a = 0, b \neq 0, \mu \neq 0$ and

$$f(x,y) \equiv \mu x(1-x),$$

or

$$f(x,y) \equiv 1-\mu x^2,$$

or

$$f(x,y) \equiv 1-\mu x^2 + by,$$

then system (1) becomes,respectively,

$$x_{m+1,n} = \mu x_{m,n}(1-x_{m,n}), \quad (2)$$

or

$$x_{m+1,n} = 1-\mu x^2_{m,n}, \quad (3)$$

or

$$x_{m+1,n} = 1-\mu x^2_{m,n} + bx_{m,n-1}, \quad (4)$$

system (2)-(3)are regular 2-D discrete logistic systems in different forms.

Let n_0 be a fixed integer. if $n \equiv n_0$, then systems (2) and (3) become

$$x_{m+1,n_0} = \mu x_{m,n_0}(1-x_{m,n_0}), \quad (5)$$

or

$$x_{m+1,n_0} = 1-\mu x^2_{m,n_0}, \quad (6)$$

$$x_{m+1,n_0} = 1-\mu x^2_{m,n_0} + bx_{m-1,n_0}. \quad (7)$$

Systems (5) and (6) are the standard 1-D logistic systems. System (4) reduces to (7), and systems (7) is the

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standard 1-D hénon system. Hence, system (1) is quite general. The focus of this paper is on system (1), the stability and spatial chaos behavior in the sense of Li-Yorke is studied.

Moreover, system (1) can be regarded as a discrete analog of the following functional partial differential equation:

$$\frac{\partial u}{\partial x}+a\frac{\partial u}{\partial y}=F[\mu,(1+a)u(x,y),bu(x-1,y)], \quad (8)$$

In fact, this system is a convection equation with a forced term in physics. Therefore, some useful information for analyzing this companion partial differential system can be derived.

The purpose of this work is to study the fixed planes, stability of the fixed planes and dynamic behavior of system (1), give a stability condition for the fixed plane, and provide just such a mathematical proof of the complex behavior. In particular, we shall show analytically that (1) satisfies sufficient conditions for the system to be chaotic.

2 Li-Yorke chaos and Marotto Theorems

Li and Yorke [6] introduced the first precise definition of discrete chaos and established a simple criterion for chaos in 1-D difference equation in 1975. Then Marotto [7,8] generalized the result to n-dimensional difference equations, and showing that the existence of a snap-back repeller implies the existence of chaos in the sense of Li-Yorke. With the recent corrections by several researchers on its original condition [9,10], Marotto Theorems are still the best one in predicting and analyzing discrete chaos in higher-dimensional difference equations today, which are described as follows.

Lemma 2.1. Let $I\subset R$ be an interval and $f:I\to I$ be a continuous map. Assume that there is a point $a\in I$, satisfying

$$f^3(a)\le a<f(a)<f^2(a)$$

or

$$f^3(a)\ge a>f(a)>f^2(a).$$

Then:

(1)For every $i=1,2,\cdots$, there is a periodic point of f^i with period k in I.

(2)There are an uncountable set $S\subset I$ (containing no periodic points) and an uncountable subset $S_0\subset S$, such that

(A)for every $p,q\in S_0$ with $p\neq q$,

$$\limsup_{k\to\infty}\left|f^k(p)-f^k(q)\right|>0$$

and

$$\liminf_{k\to\infty}\left|f^k(p)-f^k(q)\right|=0,$$

(B)for every $p\in S$ and periodic point $q\in I$ with $p\neq q$

$$\limsup_{k\to\infty}\left|f^k(p)-f^k(q)\right|>0.$$

The 1D dynamical system $x_{i+1}=f(x_i)$ that satisfies the above conditions is said to be chaotic in the sense of Li and Yorke.

2.1 Marotto Theorem

Consider the n-dimensional difference equation

$$x_{k+1}=F(x_k), x_k\in R^n \quad (9)$$

Suppose that Eq.(12) has a fixed point x^*. This fixed point x^* is called a *snap-back repeller* if

(i) $F(\cdot)$ is differentiable in a neighborhood $B(x^*,r)$ of x^*, with radius $r>0$, such that all eigenvalues of the Jacobian are strictly larger than one in absolute values;

(ii) there exists a point $x_0\in B(x^*,r)$, with $x_0\neq x^*$, such that for some integer $m>0$, $F^m(x_0)=x^*$ and $F^m(\cdot)$ is differentiable at x_0 with $\det DF^m(x_0)\neq 0$.

If Eq.(9) has a snap-back repeller, then Eq.(9) is chaotic in the sense of Li-Yorke.

2.2 Marotto Theorem

Consider the following difference equation

$$x_{k+1}=f(x_k,by_k), \quad (10)$$
$$y_{k+1}=x_k.$$

where $f:R^2\to R$ is differentiable, when b is close to 0, can be reduced to the one-dimensional equation:

$$x_{k+1}=f(x_k,0), \quad (11)$$

Suppose Eq.(11) has a snap-back repeller. Then Eq.(10)has a transversal homoclinic orbit for all $|b|<\varepsilon$ for some $\varepsilon>0$

Notes that when $f(\mu,(1+a)x_{m,n},bx_{m-1,n})=1-\mu[(1+a)x_{m,n}]^2+bx_{m-1,n}$, system (1) becomes,

$$x_{m+1,n}+ax_{m,n+1}=1-\mu[(1+a)x_{m,n}]^2+bx_{m-1,n}, \quad (12)$$

and it is called a **spatially generalized Hénon system**, that is

$$x_{m+1,n}+ax_{m,n+1}=1-\mu[(1+a)x_{m,n}]^2+b(1+a)y_{m,n}$$
$$y_{m+1,n}=\frac{1}{1+a}x_{m,n}. \quad (13)$$

Note that when $b=0$ in system then system becomes

$$x_{m+1,n}+ax_{m,n+1}=1-\mu[(1+a)x_{m,n}]^2 \quad (14)$$

which was the case studied in [1,2,3,4] Since Eq.(12) is a continuously differentiable perturbation of Eq.(14), We therefore employ Marotto's Theorems to state the stability and spatial chaos conditions of the system (12) from an analysis of the system (14) in the next section.

3 A Fixed Plane of the Spatial System

The following result is well known, a zero point x^* of $x - f(\mu, x) = 0$ is said to be a fixed point of f. The fixed point of the 1-D logistic system (5) can be written

$$x^* = \frac{-1 \pm \sqrt{1+4\mu}}{2\mu}. \quad (15)$$

Definition 3.1 Let $x_{mn} = x^*$ for all $m, n \in N_r$. Then, x^* is called a fixed plane of the 2-D system if x^* is a zero solution of $(1+a)x - f(\mu, (1+a)x, bx) = 0$. A fixed plane is denoted by $L(2D)$.

Next, using Definition 3.1, it is easy to obtain the following results about the fixed plane of system (12)

Theorem 3.1 A fixed plane of spatially generalized Hénon system (12) can be written

$$x_{mn} = x^* = \frac{-(1+a-b) \pm \sqrt{(1+a-b)^2 + 4\mu(1+a)^2}}{2(1+a)^2\mu}. \quad (16)$$

Note that when $b = 0$, system (12) reduces to the spatially generlized logistic system, and when $a = 0$, the spatially generalized hénon system reduces to the 1-D hénon system. Then the following result can be obtained:

Corollary 3.1. (i) When $a = 0$, take $n = n_0$ (constant), the fixed point of system (7) can be written

$$x^* = \frac{b-1 \pm \sqrt{(1-b)^2 + 4\mu}}{2\mu}. \quad (17)$$

(ii) When $b = 0$, the fixed plane of system (14) is given by

$$x_{mn} = x^* = \frac{-1 \pm \sqrt{1+4\mu}}{2(1+a)\mu}, m, n \in N_r. \quad (18)$$

Remark 1. It is clear that Eq.(17) and Eq.(18) are the special case of Eq.(16), respectively, corresponding to that the 1-D Hénon system (7) and the spatially logistic system (14), are the special case of system (12).

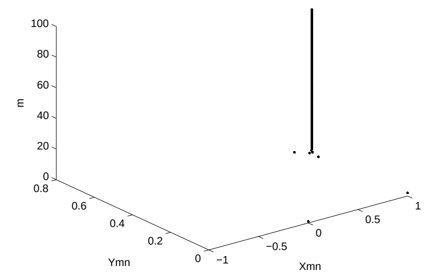
Example 1. Consider system (12), and take $\mu(1+a)^2 = 0.18$, $\mu = 0.045, a = 1, b = 0.01$. Then system (12) becomes to

$$x_{m+1,n} + x_{m,n+1} = 1 - 0.18x_{mn}^2 + 0.01x_{m-1,n} \quad (19)$$

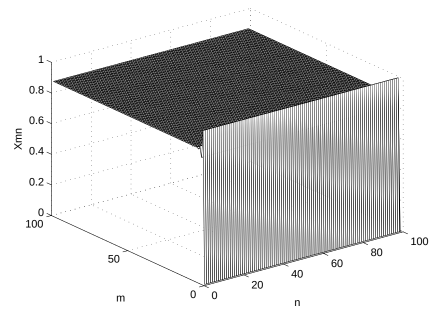
and the fixed planes of system are given by $x_{mn} = x^* = 0.87$ as shown in Fig.1.

4 Periodic Orbits and Stabilities of the Spatially Fixed Plane

In this section, more period points and stability condition for the fixed plane will be given.



(a)



(b)

Fig. 1: (a) Stability case of the solutions for system (19). (b) The fixed plane of system (19).

Let

$$\begin{aligned} x_{m+1,n} + ax_{m,n+1} &= f(\mu, x_{mn} + ax_{mn}, bx_{m-1,n}), \\ x_{m+2,n} + ax_{m,n+2} &= f(\mu, x_{m+1,n} + ax_{m,n+1}, bx_{m,n}), \\ \ldots \quad & \quad \ldots \quad \ldots \\ x_{mn} + ax_{mn} &= f(\mu, x_{m+(k-1),n} + ax_{m,n+(k-1)}, bx_{m+k-2,n}). \end{aligned} \quad (20)$$

For any $m, n \in N_r$, the sequence

$\{x_{mn} + ax_{mn}, x_{m+1,n} + ax_{m,n+1}, \ldots, x_{m+k-1,n} + ax_{m,n+k-1}, x_{mn} + ax_{mn}\}$

is called a spatially periodic k orbit, denoted by

$$r_i(a, m, n) = x_{m+(i-1),n} + ax_{m,n+(i-1)}, i = 1, 2, \ldots, k.$$

Then, it follows from (20), that the $k-$periodic points $r_1(a, m, n)$, $r_2(a, m, n)$, ... , $r_k(a, m, n)$ are all fixed points of the composite function

$$f^k(\mu, a, x, y) = \underbrace{f(\mu, a, f(\mu, a, \ldots, f(\mu, a, x, y)\ldots))}_{ktimes}.$$

Theorem 4.1 Assume that $a \in (-\infty, +\infty) \setminus \{-1\}$, $|b| < \varepsilon$, a stability condition for the fixed plane x^* of the 2-D spatial system Eq.(16) is

$$\mu < \frac{1}{4(a+1)^2}\left[\frac{2(1+a-b)(b^2+1+|a|)}{|1+a|} + \frac{(b^2+1+|a|)^2}{(1+a)^2}\right] \quad (21)$$

Proof. For any $m,n \in N_r$, take $x^* \in L(2D)$, and start the orbit with a small distance away from x^*. Define a nearby point by

$$x_{mn} = x^* + \varepsilon_{mn} \tag{22}$$

where ε_{mn} is the small quantities, and similarly, we also have

$$\begin{cases} x_{m+1,n} = x^* + \varepsilon_{m+1,n}, \\ x_{m,n+1} = x^* + \varepsilon_{m,n+1}. \\ x_{m-1,n} = x^* + \varepsilon_{m-1,n}, \end{cases} \tag{23}$$

Then

$$x_{m+1,n} + a x_{m,n+1} = (1+a)x^* + (\varepsilon_{m+1,n} + a\varepsilon_{m,n+1}). \tag{24}$$

Substituting Eq.(23) and Eq.(24) into Eq.(1), we obtain

$$(1+a)x^* + (\varepsilon_{m+1,n} + a\varepsilon_{m,n+1}) = f[\mu,(1+a)(x^*+\varepsilon_{mn}), b(x^*+\varepsilon_{m-1,n})].$$

and 2-D discrete system stability simply means that when approaching the fixed point x^*, the difference between the kth iterate x_{mn} and x^* keeps decreasing. In other words, it follows that

$$\left|\frac{\varepsilon_{m+1,n}}{\varepsilon_{mn}}\right| < 1, \left|\frac{\varepsilon_{m,n+1}}{\varepsilon_{mn}}\right| < 1, \left|\frac{\varepsilon_{m,n}}{\varepsilon_{m-1,n}}\right| < 1. \tag{25}$$

Therefore, when m and n are sufficiently large, applying (25) yields

$$\left|\frac{\varepsilon_{m+1,n}}{(1+a)\varepsilon_{mn}}\right| = \frac{1}{|1+a|}\left|\frac{\varepsilon_{m+1,n}}{\varepsilon_{mn}}\right| < \frac{1}{|1+a|} \tag{26}$$

and

$$\left|\frac{\varepsilon_{m,n+1}}{(1+a)\varepsilon_{mn}}\right| = \frac{1}{|1+a|}\left|\frac{\varepsilon_{m,n+1}}{\varepsilon_{mn}}\right| < \frac{1}{|1+a|}. \tag{27}$$

$$\left|\frac{b^2\varepsilon_{m-1,n}}{(1+a)\varepsilon_{mn}}\right| = \frac{b^2}{|1+a|}\left|\frac{\varepsilon_{m-1,n}}{\varepsilon_{mn}}\right| > \frac{b^2}{|1+a|}. \tag{28}$$

One obtains

$$\left|\frac{\varepsilon_{m+1,n} + a\varepsilon_{m,n+1}}{(1+a)\varepsilon_{mn}}\right| \le \frac{1+|a|}{|1+a|}$$

$$= \begin{cases} 1 \text{ when } a \in [0,\infty), \\ \frac{1-a}{1+a} \text{ when } a \in (-1,0), \\ \frac{a-1}{1+a} \text{ when } a \in (-\infty,-1). \end{cases} \tag{29}$$

Next expanding f to its Taylor series for the first order in $\varepsilon_{mn}, \varepsilon_{m-1,n}$, one gets

$$(1+a)x^* + (\varepsilon_{m+1,n} + a\varepsilon_{m,n+1}) = f[\mu,(1+a)x^*, bx^*] + \left(\frac{\partial}{\partial x}(1+a)\varepsilon_{mn} + \frac{\partial}{\partial y} b\varepsilon_{m-1,n}\right) f[\mu,(1+a)x^*, bx^*] + . \tag{30}$$

Note that x^* is a fixed plane of system (1) so

$$x^* + ax^* = (1+a)x^* = f[\mu,(1+a)x^*, bx^*]. \tag{31}$$

Substituting Eq.(31) into Eq.(30) and suppressing from both sides the same items, one can obtain

$$\varepsilon_{m+1,n} + a\varepsilon_{m,n+1} = \left(\frac{\partial}{\partial x}(1+a)\varepsilon_{mn} + \frac{\partial}{\partial y} b\varepsilon_{m-1,n}\right) f[\mu,(1+a)x^*, bx^*] + \ldots = -2\mu(1+a)^2 x^*(1+a)\varepsilon_{mn} + b^2\varepsilon_{m-1,n}$$

applying Eq.(29)

$$\left|-2\mu(1+a)x^* + \frac{b^2\varepsilon_{m-1,n}}{(1+a)\varepsilon_{m,n}}\right| < \frac{1+|a|}{|1+a|} \tag{32}$$

substituting Eq.(16) into Eq.(32) gives,

$$\left|1+a-b-\sqrt{(1+a-b)^2+4\mu(1+a)^2} + \frac{b^2}{|1+a|}\right| < \frac{1+|a|}{|1+a|} \tag{33}$$

solving for μ to yield the stability condition for the fixed point x^* as

$$\mu < \frac{1}{4(a+1)^2}\left[\frac{2(1+a-b)(b^2+1+|a|)}{|1+a|} + \frac{(b^2+1+|a|)^2}{(1+a)^2}\right].$$

This completes the proof of Theorem 4.1.

Corollary 4.1 (i) When $a = b = 0$ and $n = n_0$ (constant), the stability condition for the 1-D system is

$$\mu < \frac{3}{4}. \tag{34}$$

(ii) When $b = 0, a = 1$, the stability condition for the spatially general logistic system is

$$\mu < \frac{1}{4}\left[\left(\frac{1+|a|+(a+1)^2}{(a+1)^2}\right)^2 - 1\right] < \frac{5}{16}.$$

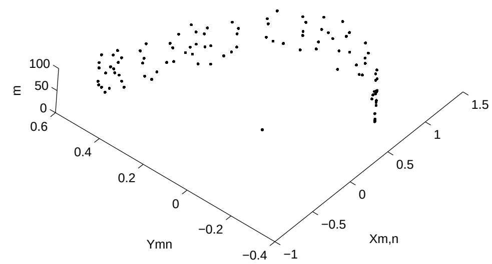
Remark 1. Note that Eq.(34) is just the familiar stability condition for 1-D Logistic system, which shows again that the 2-D system is a natural extension of the 1-D system.

Example 2. Take $\mu = 0.345 > \frac{5}{16}, \mu(1+a)^2 = 1.38$, $a = 1, b = 0.1$. Then we obtain the fixed plane and the stability condition for the case of spatially Hénon system, and Fig.2 shows the behavior of the solution and the unstable case of the solution within the fixed plane.

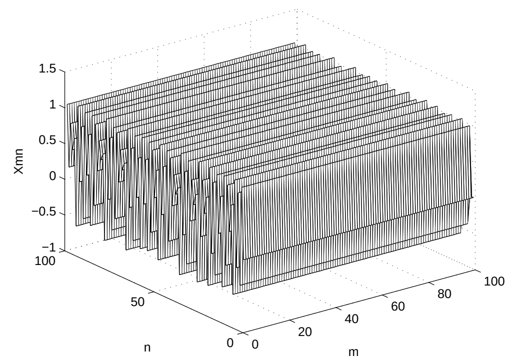
5 A Proof of Spatial Chaotic Behavior for the General 2-D Hénon System

In this section, the chaotic behavior in the general 2-D Hénon system Eq.(12) is investigated, a verification of the system to be chaotic in the sense of Li and Yorke is derived.

Let $V = R^3$, and take $V_0 = R = (-\infty, \infty)$. So $V_0 \subset V$. According to Theorem 2.2, Eq.(12) will have a



(a)



(b)

Fig. 2: (a) Unstable case of the solutions for system (19). (b) The unstable case of the fixed plane.

transversal homoclinic orbit for all $|b| < \varepsilon$ for some $\varepsilon > 0$, if the equation:

$$\begin{aligned} x_{m+1,n} + ax_{m,n+1} &= 1 - \mu\left[(1+a)x_{mn}\right]^2 \\ &= h(\mu, a, x_{in} + ax_{mj}), i, j, m, n \in N_r. \end{aligned} \quad (35)$$

has a snap-back repeller. We shall show this for appropriate values of μ, in particular for $\mu > 1.55$

First take $x^* = \frac{-1\pm\sqrt{1+4\mu}}{2(1+a)\mu} \in L(2D)$, and then observe that $(1+a)x^* = \frac{-1+\sqrt{1+4\mu}}{2\mu}$ is an unstable fixed plane of system (35), i.e., $(1+a)x^* = h(\mu, a, (1+a)x^*)$, due to $h'(\mu, a, (1+a)x^*) = -2\mu(1+a)x^*$. Hence

$$h'(\mu, a, x^*) = 1 - \sqrt{1+4\mu} < -1, \mu > \xi,$$

where

$$\xi = \frac{1}{4}\left[\left(\frac{1+|a|+(a+1)^2}{(a+1)^2}\right)^2 - 1\right].$$

Next, for all $m, n \in N_r$, if one can find a solution $\{x_{in} + ax_{mj}\}_{i,j=-\infty,-\infty}^{+\infty,+\infty}$, with not all $x_{in} + ax_{mj} = (1+a)x^*$ satisfying: (i)$x_{in} + ax_{mj} = (1+a)x^*$ for all $i \geq M, j \geq N$ for some M, N, (ii) $x_{in} + ax_{mj} \to (1+a)x^*$ as $i \to -\infty, j \to -\infty$ (iii) $h'(x_{in} + ax_{mj}) \neq 0$ for all i, j, then $(1+a)x^*$ is a snap-back repeller. Such a sequence can be generated in the following manner. If let $x_{0n} + ax_{m0} = (1+a)x^*$, then, since $(1+a)x^*$ is a fixed plane of system (35), $x_{i,n} + ax_{m,j} = (1+a)x^*$ for all $i \geq 0$, $j \geq 0$. Note that $x_{i,n} + ax_{m,j}$ (for all $i < 0,\ j < 0$) can be constructed by iterating the multi-valued inverse of system (35), that is, if $x_{i+1,n} + ax_{m,j+1} = 1 - \mu(x_{in} + ax_{mj})^2$, then

$$\mu(x_{in} + ax_{mj})^2 = 1 - (x_{i+1,n} + ax_{m,j+1}).$$

Hence

$$\begin{aligned} x_{i-1,n} + ax_{m,j-1} &= \pm\left(\frac{1-(x_{i,n}+ax_{m,j})}{\mu}\right)^{\frac{1}{2}} \\ &= h_{\pm}^{-1}(\mu, a, x_{in} + ax_{mj}) \end{aligned} \quad (36)$$

provided that $x_{in} + ax_{mj} \leq 1$. With $x_{0n} + ax_{m0} = (1+a)x^*$ we have two choices for $x_{-1,n} + ax_{m,-1}$ according to Eq.(36). Choosing the plus sign for this initial point with indices $i = 0, j = 0$ will not yield an appropriate sequence, because one would get

$$\begin{aligned} &x_{-1,n} + ax_{m,-1} = h_+^{-1}(\mu, a, x_{0n} + ax_{m0}) \\ &\quad = h_+^{-1}(\mu, a, (1+a)x^*) = (1+a)x^*. \end{aligned}$$

Hence, define

$$x_{-1,n} + ax_{m,-1} = h_-^{-1}(\mu, a, x_{0n} + ax_{m0}).$$

Note that

$$\begin{aligned} &x_{-1,n} + ax_{m,-1} = h_-^{-1}(\mu, a, x_{0n} + ax_{m0}) \\ &\quad = h_-^{-1}(\mu, a, (1+a)x^*) = -(1+a)x^*. \end{aligned}$$

So, let $x_{i-1,n} + ax_{m,j-1} = h_+^{-1}(\mu, a, x_{in} + ax_{mj})$ in for all $i \leq -1,\ j \leq -1$.

Now we shall show that this solution $\{x_{in} + ax_{mj}\}_{i,j=-\infty,-\infty}^{0,0}$ satisfies $x_{in} + ax_{mj} \to (1+a)x^*$ as $i,\ j \to -\infty$ (for appropriate values of μ). Note that since $x_{-1,n} + ax_{m,-1} = -(1+a)x^* < (1+a)x^*$, so have

$$\begin{aligned} &x_{-2,n} + ax_{m,-2} = h_+^{-1}(\mu, a, x_{-1,n} + ax_{m,-1}) \\ &\quad \in h_+^{-1}\left[-\infty, (1+a)x^*\right] \subset ((1+a)x^*, \infty). \end{aligned}$$

We shall find those values of μ for which $x_{-2,n} + ax_{m,-2} < 1$, note that

$$x_{-1,n} + ax_{m,-1} = -(1+a)x^* = \frac{1-(1+4\mu)^{\frac{1}{2}}}{2\mu},$$

then by

$$\begin{aligned} &x_{-2,n} + ax_{m,-2} = h_+^{-1}(\mu, a, x_{-1,n} + ax_{m,-1}) \\ &\qquad = \left[\frac{1+(1+a)x^*}{\mu}\right]^{\frac{1}{2}}, \end{aligned}$$

so, $x_{-2,n}+ax_{m,-2}<1$ implies

$$[1+(1+a)x^*/\mu]^{\frac{1}{2}}<1,$$

that is, $(1+a)x^*<\mu-1$, or

$$(-1+(1+4\mu)^{\frac{1}{2}})/2\mu<\mu-1,$$

this problem can be written

$$\mu^3-2\mu^2+2\mu-2>0.$$

It is easy to observed that all values of $\mu>1.55$ satisfy this equation, and so

$$x_{-2,n}+ax_{m,-2}\in((1+a)x^*,1)$$

for these values of μ. Let us restrict the remaining of the discussion problem when $\mu>1.55$.

Since $x_{-2,n}+ax_{m,-2}\in((1+a)x^*,1)$ for these μ values, one gets

$$\begin{aligned}x_{-3,n}+ax_{m,-3}&=h_+^{-1}(\mu,a,x_{-2,n}+ax_{m,-2})\\&\in h_+^{-1}((1+a)x^*,1)\subset(0,(1+a)x^*)\end{aligned}$$

and consequently $x_{-3,n}+ax_{m,-3}\in(0,(1+a)x^*)$. Also gets

$$\begin{aligned}x_{-4,n}+ax_{m,-4}&=h_+^{-1}(\mu,a,x_{-3,n}+ax_{m,-3})\\&\in h_+^{-1}(0,(1+a)x^*)\\&\subset h_+^{-1}[(x_{-1,n}+ax_{m,-1},(1+a)x^*)]\\&\subset((1+a)x^*,x_{-2,n}+ax_{m,-2})\end{aligned}$$

and thus $x_{-4,n}+ax_{m,-4}\in((1+a)x^*,x_{-2,n}+ax_{m,-2})$. This implies

$$\begin{aligned}x_{-5,n}+ax_{m,-5}&=h_+^{-1}(\mu,a,x_{-4,n}+ax_{m,-4})\\&\in h_+^{-1}[((1+a)x^*,x_{-2,n}+ax_{m,-2})]\\&\subset(x_{-3,n}+ax_{m,-3},(1+a)x^*).\end{aligned}$$

Hence, $x_{-5,n}+ax_{m,-5}\in(x_{-3,n}+ax_{m,-3},(1+a)x^*)$. Using this manner, the sequence $\{x_{in}+ax_{mj}\}_{i,j=-\infty,-\infty}^{0,0}$ thus constructed satisfies the following properties: $x_{-2i,n}+ax_{m,-2j}$ is a decreasing sequence bounded below by $(1+a)x^*$, and $x_{-2i-1,n}+ax_{m,-2j-1}$ is an increasing sequence bounded above by $(1+a)x^*$ (Fig.3). Hence, there must exist a plane $\rho\in(0,(1+a)x^*)$ that is the limit of $x_{-2i-1,n}+ax_{m,-2j-1}$, and a plane $\sigma\in[(1+a)x^*,1]$ that is the limit of $x_{-2i,n}+ax_{m,-2j,}$ as $i,j\to\infty$. It can also see that such a sequence generated in manner from the following Fig.3, where $r_{-i}=x_{-i,n}+ax_{m,-i}$, $r_*=(1+a)x^*$.

Next let us show that $\rho=\sigma=(1+a)x^*$. Since

$$h\left(\mu,a,x_{-2i-1,n}+ax_{m,-2j-1}\right)=x_{-2i,n}+ax_{m,-2j}$$

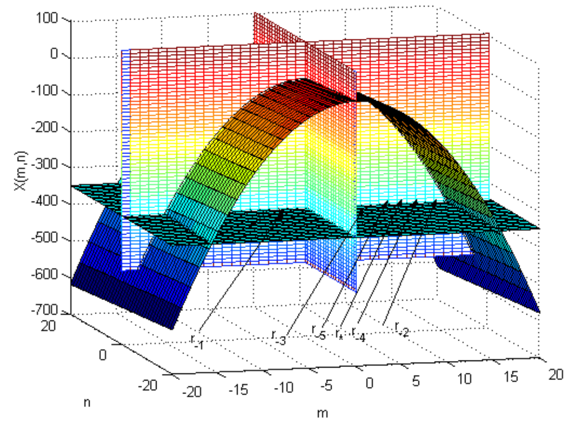
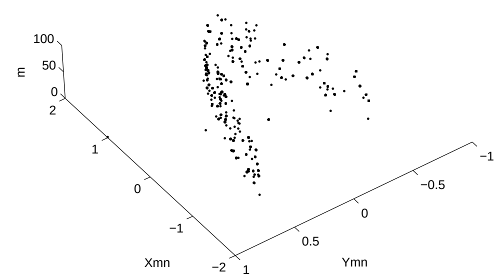
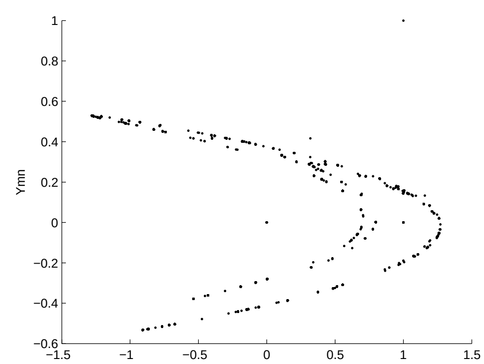


Fig. 3: Inverse iterates of $x_{m+1;n}+x_{m,n+1}=1-\mu x_{mn}^2$ for $a=1$ and $\mu>1.55$.

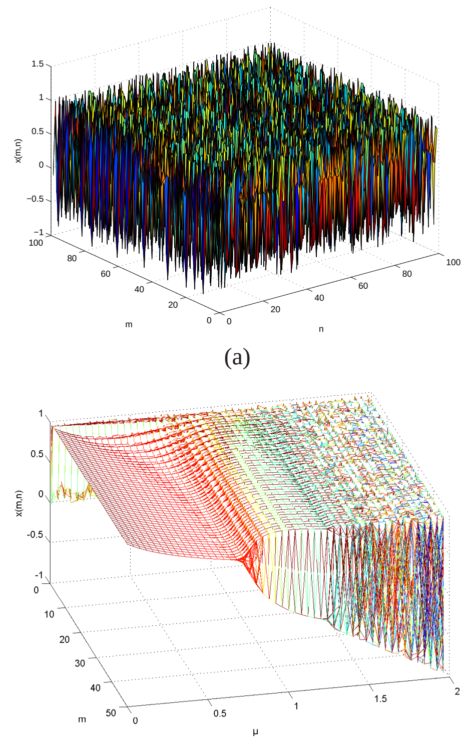


(a)



(b)

Fig. 4: (a) chaotic behavior of system (37). (b) Section View of Chaotic behavior of system (37).



(a)

(b)

Fig. 5: (a) Chaotic behavior of system(38). (b) Bifurcation behavior of system(38).

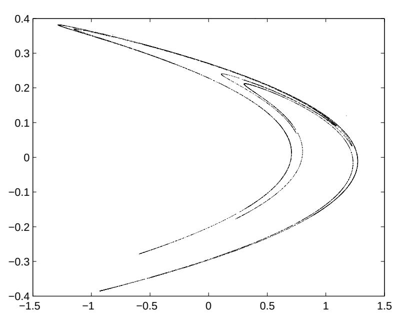


Fig. 6: Chaotic behavior of system(39).

and

$$h\left(\mu, a, x_{-2i,n} + ax_{m,-2j}\right) = x_{-2i+1,n} + ax_{m,-2j+1},$$

it must be that $h(\mu, a, \rho) = \sigma$ and $h(\mu, a, \sigma) = \rho$. Consequently, $h(h(\rho)) = \rho$, and ρ is thus a fixed plane of the function $h \circ h$. But, for $\mu > 1.55$, there are precisely four fixed planes of $h \circ h$ (since this function is a quartic polynomial), each of which can be computed exactly:

$$\frac{-1 \pm (1+4\mu)^{\frac{1}{2}}}{2\mu}, \frac{-1 \pm (4\mu-3)^{\frac{1}{2}}}{2\mu}.$$

It is easy to see that, for

$$\mu > 1.55, -1-(1+4\mu)^{\frac{1}{2}}/2\mu$$

and $-1-(4\mu-3)^{\frac{1}{2}}/2\mu$ are both negative, and thus neither of these can equal $\rho \in (0, (1+a)x^*)$. Consider that

$$\frac{-1+(4\mu-3)^{\frac{1}{2}}}{2\mu} \leq (1+a)x^* = \frac{-1+(1+4\mu)^{\frac{1}{2}}}{2\mu},$$

then, $(1+4\mu)^{\frac{1}{2}} \geq 2+(4\mu-3)^{\frac{1}{2}}$. Squaring each side of this inequality, one gives $4(4\mu-3)^{\frac{1}{2}} \leq 0$, which is a contradiction for $\mu > 1.55$. This implies $1+(4\mu-3)^{\frac{1}{2}}/2\mu \geq (1+a)x^*$. Therefore, there must exists that $\rho = -1+(1+4\mu)^{\frac{1}{2}}/2\mu = (1+a)x^*$ and $\sigma = h(\rho) = (1+a)x^*$. So, $x_{in} + ax_{mj} \to (1+a)x^*$ as i, $j \to -\infty$.

So far we have verified that the sequence $\{x_{in} + ax_{mj}\}_{i,j=-\infty,-\infty}^{0,0}$ satisfies (i) and (ii) from the theorem of Marotto above. It can be easily verified that it also satisfies (iii). Since $h'(\mu, a, x_{m,n}) = -2\mu(1+a)x_{m,n}$, the only possibility for $h'(\mu, a, x_{in} + ax_{mj}) = 0$ is when $x_{in} + ax_{mj} = 0$ for some i and j. But the sequence was constructed in this manner, one gets $x_{in} + ax_{mj} = (1+a)x^*$ for $i \geq 0$, $j \geq 0$, $x_{-1,n} + ax_{m,-1} = -(1+a)x^* < 0$, $x_{-2,n} + ax_{m,-2} > (1+a)x^* > 0$, $x_{-3,n} + ax_{m,-3} > 0$, and

$$x_{in} + ax_{mj} \in (x_{-3,n} + ax_{m,-3}, x_{-2,n} + ax_{m,-2}) \subset (0,1),$$

for all $i < -3$ and $j < -3$. Hence the sequence also satisfies (iii), and $(1+a)x^*$ is therefore a snap-back repeller of system Eq.(35). According to the *Marotto Theorem 2.2*, we shall have the conclusion that the 2D system Eq.(12) has a transversal homoclinic orbit for all $|b| < \varepsilon$ for some $\varepsilon > 0$, so is chaotic in the sense of *Li* and *Yorke*. Following the above analysis, it can be shown that each of these is a snap-back repeller of $h \circ h$ for all $\mu > 1.55$. Continuing in similar manner, the region of μ values can be extended for even smaller values.

6 Illustrative Examples

Example 3. When $a = 1.4, b = 0.3, \mu(1+a)^2 = 1.4$, Consider the following form of Eq.(10):

$$x_{m+1,n} + 1.4x_{m,n+1} = 1 - 1.4x_{m,n}^2 + 0.72y_{m,n}$$
$$y_{m+1,n} = \frac{1}{2.4}x_{m,n}. \quad (37)$$

the system spatial dynamics behavior are demonstrated by Fig.4, it is shown that system (37) is the extension of 1D Hénon System.

Example 4. Consider again system (37) and take $a = -18/25, b = 0,\ \mu = 375/196$. Then, the system becomes

$$x_{m+1,n} - \frac{18}{25}x_{m,n+1} = 1 - \frac{3}{20}x_{mn}^2, \tag{38}$$

with $\mu = 375/196 > 1.55$. that system is chaotic in the sense of Li-Yorke. The chaotic behavior of system is demonstrated by Fig.5.
On the other hand, when $a = 0$ and $n_0 = 0$, system reduces to

$$x_{m+1} = 1 - 1.4x_m^2 + 0.3x_{m-1}. \tag{39}$$

Clearly, system (39) is just a special case of simple Hénon system. Fig.6 shows its chaotic behavior.

7 Conclusions

In this paper, we study the fixed planes, the stable condition of the fixed planes, and analytically prove the existence of a transversal homoclinic orbit of (12)for small values of b and appropriate values of a, μ by applying Marotto theorem, it is to show that for those values of μ for which Eq.(12) has a snap-back repeller, for those values of μ for which chaos occurs. Numerical studies of Eq.(12) indicate that this does in fact occur. On the other hand, Eq.(39) is just the familiar 1-D hénon system, and may be viewed as a special case of the higher dimensional discrete system (12), which shows again that the 2-D hénon system is a natural extension of the 1-D hénon system.

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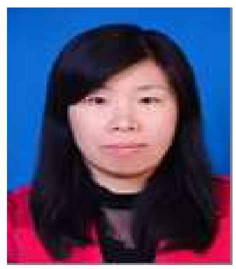

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