Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/080301

The Epsilon-Skew Rayleigh Distribution

John M. Greene[1,*] *and Hassan Elsalloukh*[2]

[1]School of Business, Henderson State University, 1100 Henderson Ave., Arkadelphia, AR, 71999, U.S.
[2]Mathematics and Statistics Department, University of Arkansas at Little Rock, 2801 S University Ave., Little Rock, AR, 72204, U.S.

Received: 23 Sep. 2020, Revised: 15 Feb. 2021, Accepted: 20 Feb. 2021
Published online: 1 Sep. 2021

Abstract: In this paper, we develop the Epsilon-Skew Rayleigh Distribution (ESR). This distribution is a bimodal skew distribution with shape, location, and skewness parameters. This distribution is useful in applications with bimodal and skewed data, such as gender versus age demographics. We develop the properties of a random variable with ESR distribution by determining the mean, median, modes, variance, moments, skewness, excess kurtosis, and the moment generating function.

Keywords: Rayleigh Distribution, Epsilon-Skew, Bimodal, Distribution theory

1 Introduction

In 1888, Lord Rayleigh introduced what would become his eponymous distribution to describe maximum magnitudes during resonance by means of a standardized vector with two normally distributed and independent orthogonal components. The Rayleigh distribution is a special case of the Burr Type X distribution, one of twelve presented by [1] in 1942 as potentially applicable. In 2001, [2] introduced an exponentiated, two-parameter Rayleigh distribution named the Generalized Rayleigh (GR). The GR is a special case of the Exponentiated Weibull distribution previously proposed by [3] in 1993. The Epsilon-Skew Normal distribution (ESN) was introduced in 2000 by [4]. The Epsilon-Skew Exponential Power (ESEP) distribution family was presented by [5] in 2005. The ESEP family includes as special cases, ESN and the Epsilon-Skew Laplace (ESL) distribution. The latter was further developed by [6] in 2008. In 2012, [7] developed the Skewed Double Inverted Weibull distribution. Also in 2012 [8] introduced the Epsilon-Skew Gamma distribution, and [9] presented the Epsilon-Skew Inverted Gamma. In 2014 [10] derived the Skew Rayleigh distribution by applying Azzalini's technique presented in [11]. Most recently, in 2019, [12] introduced the Epsilon-Skew Exponentiated Beta distribution.

2 The Epsilon-Skew Rayleigh Distribution

2.1 The Probability Density Function of the ESR

Definition 1. The random variable X has ESR distribution denoted by $X \sim ESR(\theta,\sigma,\varepsilon)$, if there exist parameters $\theta \in \mathbb{R}$, $\sigma > 0$, and $-1 < \varepsilon < 1$ such that the pdf of X is

$$f(x;\theta,\sigma,\varepsilon) = \frac{1}{2\sigma^2}\begin{cases} \frac{x-\theta}{1-\varepsilon}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{x-\theta}{1-\varepsilon}\right)^2\right) & x \geq \theta \\ \frac{\theta-x}{1+\varepsilon}\exp\left(-\frac{1}{2\sigma^2}\left(\frac{\theta-x}{1+\varepsilon}\right)^2\right) & x < \theta \end{cases}, \tag{1}$$

where θ, σ, and ε are location, scale, and skewness parameters respectively. The density function in (1) satisfies the following two properties.

* Corresponding author e-mail: greene1399@gmail.com

Property 1. $f(x) \geq 0$
Proof. Given that $-1 < \varepsilon < 1$, $\frac{\theta - x}{1+\varepsilon} > 0$ $for\ x < \theta\ and$ $\frac{x-\theta}{1-\varepsilon} > 0$ $for\ x \geq \theta$, the exponential having only positive values completes the proof. Figure 1 verifies Property 1 by varying the shape parameter while the location and skew parameters are constant.

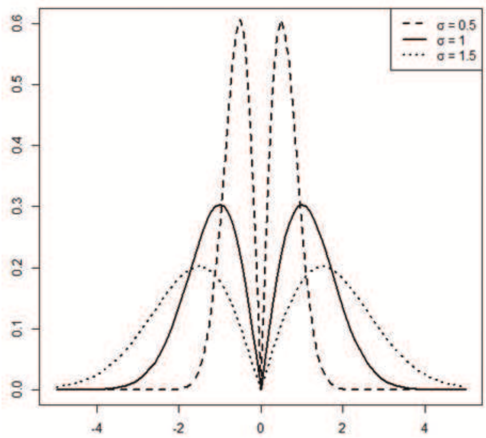

Fig. 1: Epsilon-Skew Rayleigh Distribution with $\theta = 0, \varepsilon = 0$ *and* $\sigma = 0.5,\ 1,\ 1.5$

Property 2. We can conclude that $P(X < \theta) = \frac{1+\varepsilon}{2}$ and $P(X \geq \theta) = \frac{1-\varepsilon}{2}$.
Proof. Substituting $u_1 = \frac{\theta - x}{1+\varepsilon}$ $for\ x < \theta\ and$ $u_2 = \frac{x-\theta}{1-\varepsilon}$ $for\ x \geq \theta$ into (1), results in a weighted Rayleigh distribution with $P_1(X < \theta) = \frac{1+\varepsilon}{2}$ and $P_2(X \geq \theta) = \frac{1-\varepsilon}{2}$. Figure 2 verifies Property 2 by varying the skew parameter while the location and shape parameters are constant.

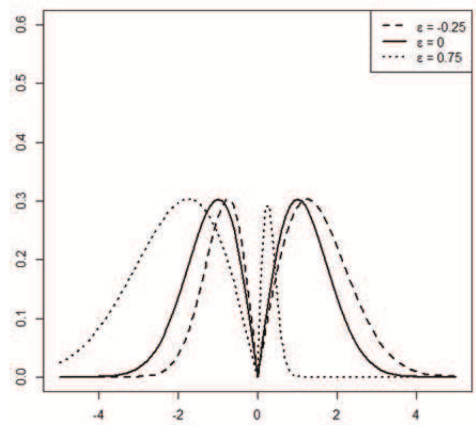

Fig. 2: Epsilon-Skew Rayleigh Distribution with $\theta = 0, \sigma = 1$ *and* $\varepsilon = -0.25,\ 0,\ 0.75$

2.2 Cumulative Distribution Function of the ESR

Definition 2. An Epsilon-Skew Rayleigh random variable $X \sim ESR(\theta, \sigma, \varepsilon)$, has distribution

$$F(x) = \begin{cases} 1 - \frac{1-\varepsilon}{2} e^{-\frac{1}{2\sigma^2}\left(\frac{x-\theta}{1-\varepsilon}\right)^2} & , \ x \geq \theta \\ \frac{1+\varepsilon}{2} e^{-\frac{1}{2\sigma^2}\left(\frac{\theta-x}{1+\varepsilon}\right)^2} & , \ x < \theta \end{cases} \quad , \ -1 < \varepsilon < 1, \ \ \theta \in \mathbb{R}, \ \ \sigma > 0. \tag{2}$$

Proof. Setting θ = 0 and ε = -1 in (2) produces the Rayleigh distribution function. Figure 3 displays the density and distribution functions simultaneously.

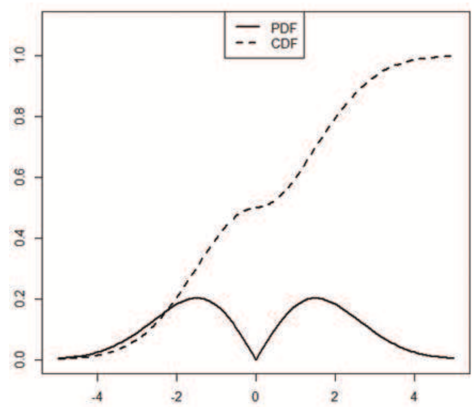

Fig. 3: Epsilon-Skew Rayleigh PDF and CDF with $\theta = 0, \sigma = 1.5$ *and* $\varepsilon = 0$

2.3 Quantile Function of the ESR

Definition 3. The Quantile Function of $X \sim ESR(\theta, \sigma, \varepsilon)$ is given by

$$Q(F;\theta,\sigma,\varepsilon) = \begin{cases} \theta + (1-\varepsilon)\sigma\sqrt{-2\ln\frac{2(1-F)}{1-\varepsilon}} \ , & \frac{1+\varepsilon}{2} \leq F < 1 \\ \theta - (1+\varepsilon)\sigma\sqrt{-2\ln\frac{2F}{1+\varepsilon}} \ , & 0 < F < \frac{1+\varepsilon}{2} \end{cases} . \tag{3}$$

Proof. Substituting F^{-1} for x in (2) and solving for F^{-1} produces (3).

3 Properties of the ESR

3.1 The Mean of the ESR

Proposition 1. If $X \sim ESR(\theta, \sigma, \varepsilon)$, the mean of X is

$$E[X] = \theta - \sigma\varepsilon\sqrt{2\pi} \ . \tag{4}$$

Proof. Integrating (1) around 0 as θ has no effect on the shape of the distribution,

$$E[X] = \frac{1}{2\sigma^2}\int_{-\infty}^{0} x\left(\frac{\theta-x}{1+\varepsilon}\right)e^{-\frac{1}{2\sigma^2}\left(\frac{\theta-x}{1+\varepsilon}\right)^2}dx + \frac{1}{2\sigma^2}\int_{0}^{\infty} x\left(\frac{x-\theta}{1-\varepsilon}\right)e^{-\frac{1}{2\sigma^2}\left(\frac{x-\theta}{1-\varepsilon}\right)^2}dx \ .$$

Making the substitutions

$$u_1 = -\frac{1}{2\sigma^2}\left(\frac{\theta-x}{1+\varepsilon}\right)^2, \ du_1 = \frac{1}{\sigma^2}\frac{\theta-x}{(1+\varepsilon)^2}dx \ , \ u_2 = \frac{1}{2\sigma^2}\left(\frac{x-\theta}{1-\varepsilon}\right)^2, \ du_2 = \frac{1}{\sigma^2}\frac{x-\theta}{(1-\varepsilon)^2}\,dx \ ,$$

and employing $\int_0^\infty \sqrt{u}\,e^{-u}du = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, we arrive at

$$E[X] = \frac{1+\varepsilon}{2}\left[\theta - \sqrt{2}\sigma(1+\varepsilon)\left(\frac{\sqrt{\pi}}{2}\right)\right] + \frac{1-\varepsilon}{2}\left[\theta + \sqrt{2}\sigma(1-\varepsilon)\left(\frac{\sqrt{\pi}}{2}\right)\right] .$$

Simplification produces (4) which completes the proof.

3.2 Median of the ESR

Corollary 1. If $X \sim ESR(\theta,\sigma,\varepsilon)$, the median of X is found by letting F=0.5 in (3).

$$Q(0.5;\theta,\sigma,\varepsilon) = \begin{cases} \theta + (1-\varepsilon)\sigma\sqrt{-2\ln\frac{1}{1-\varepsilon}} \ , & \frac{1+\varepsilon}{2} \leq F < 1 \\ \theta - (1+\varepsilon)\sigma\sqrt{-2\ln\frac{1}{1+\varepsilon}} \ , & 0 < F < \frac{1+\varepsilon}{2} \end{cases} .$$

3.3 Modes of the ESR

Corollary 2. If $X \sim ESR(\theta,\sigma,\varepsilon)$, the modes of X are found by solving $f' = 0$ for x.

$$x_{X<\theta} = \theta - (1+\varepsilon)\sigma \ \ and \ x_{X\geq\theta} = \theta + (1-\varepsilon)\sigma$$

3.4 N^{th} Moments of the ESR

Theorem 1. If $X \sim ESR(\theta,\sigma,\varepsilon)$, then the raw moments of X are given by

$$E[X^n] = \frac{1}{2}\sum_{k=0}^{n}\left(\binom{n}{k}\theta^{n-k}\left(\sigma\sqrt{2}\right)^k \Gamma(1+k/2)\left[(1-\varepsilon)^{k+1} + (-1)^k(1+\varepsilon)^{k+1}\right]\right) . \tag{5}$$

Proof. Letting $u_1 = \frac{1}{2\sigma^2}\left(\frac{\theta-x}{1+\varepsilon}\right)^2$ with $du_1 = \frac{-1}{\sigma^2}\frac{\theta-x}{(1+\varepsilon)^2}dx$ and $u_2 = \frac{1}{2\sigma^2}\left(\frac{x-\theta}{1-\varepsilon}\right)^2$ with $du_2 = \frac{1}{\sigma^2}\frac{x-\theta}{(1-\varepsilon)^2}\,dx$, we integrate around zero to obtain

$$E[X^n] = \frac{1+\varepsilon}{2}\int_{-\infty}^{0}\left(\theta - (1+\varepsilon)\sqrt{2\sigma^2 u_1}\right)^n e^{-u_1}du_1 + \frac{1-\varepsilon}{2}\int_{0}^{\infty}\left(\theta + (1-\varepsilon)\sqrt{2\sigma^2 u_2}\right)^n e^{-u_2}du_2,$$

$$= \frac{1+\varepsilon}{2}\int_{-\infty}^{0}\sum_{k=0}^{n}\left(\binom{n}{k}\theta^{n-k}(-1)^k(1+\varepsilon)^k\sigma^k\sqrt{2u_1}^k\right)e^{-u_1}du_1 + \frac{1-\varepsilon}{2}\int_{0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\theta^{n-k}(1-\varepsilon)^k\sigma^k\sqrt{2u_2}^k\right)e^{-u_2}du_2 \ .$$

Employing $\int_0^\infty \sqrt{u}^c\, e^{-u}du = \Gamma\left(1+\frac{c}{2}\right)$ produces (5) which completes the proof.

3.5 Variance of the ESR

Proposition 2. If $X \sim ESR(\theta,\sigma,\varepsilon)$, the variance of X is given by

$$E\left[(X - E(X))^2\right] = 2\sigma^2\left((3-\pi)\varepsilon^2 + 1\right) . \tag{6}$$

Proof. Using (5) from Theorem 1,

$$E\left[X^2\right] - E^2[X] = \theta^2 - 4\theta\varepsilon\sigma\sqrt{\frac{\pi}{2}} + 6\varepsilon^2\sigma^2 + 2\sigma^2 - \theta^2 + 4\theta\varepsilon\sigma\sqrt{\frac{\pi}{2}} - 2\pi\varepsilon^2\sigma^2 .$$

Simplification produces (6) which completes the proof.

3.6 Central Moments of the ESR

Theorem 2. If $X \sim ESR(\theta,\sigma,\varepsilon)$, then the central moments of X are given by

$$E[(X - E[X])^n] = \sigma^n\sqrt{2}^{n-2}\sum_{k=0}^{n}\left(\binom{n}{k}\left(\varepsilon\sqrt{\pi}\right)^{n-k}\Gamma(1+k/2)\left[(1-\varepsilon)^{k+1} + (-1)^k(1+\varepsilon)^{k+1}\right]\right) . \tag{7}$$

Proof. Letting $u_1 = \frac{1}{2\sigma^2}\left(\frac{\theta-x}{1+\varepsilon}\right)^2$ with $du_1 = -\frac{1}{\sigma^2}\frac{\theta-x}{(1+\varepsilon)^2}dx$ and $u_2 = \frac{1}{2\sigma^2}\left(\frac{x-\theta}{1-\varepsilon}\right)^2$ with $du_2 = \frac{1}{\sigma^2}\frac{x-\theta}{(1-\varepsilon)^2}\ dx$, we integrate around zero to obtain

$$E\left[(X-\mu)^n\right] = \frac{1+\varepsilon}{2}\int_{-\infty}^{0}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^k(\theta-\mu)^{n-k}(1+\varepsilon)^k 2^{k/2}\sigma^k\sqrt{u_1}^k\right)e^{-u_1}du_1$$

$$+\frac{1-\varepsilon}{2}\int_{0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(\theta-\mu)^{n-k}(1-\varepsilon)^k 2^{k/2}\sigma^k\sqrt{u_2}^k\right)e^{-u_2}du_2\ .$$

Employing $\int_0^\infty \sqrt{u}^c\ e^{-u}du = \Gamma\left(1+\frac{c}{2}\right)$ we arrive at

$$E\left[(X-\mu)^n\right] = \frac{1}{2}\sum_{k=0}^{n}\left(\binom{n}{k}(\theta-\mu)^{n-k}2^{k/2}\sigma^k\Gamma(1+k/2)\left[(1-\varepsilon)^{k+1}+(-1)^k(1+\varepsilon)^{k+1}\right]\right)\ .$$

Substituting (4) for μ and simplifying produces (7), which completes the proof.

3.7 The Skewness of the ESR

Corollary 3. If $X \sim ESR(\theta,\sigma,\varepsilon)$, the Skewness of X is derived from (7) in Theorem 2.

$$\gamma_1 = E\left[\left(\frac{X-\mu}{\sqrt{Var(X)}}\right)^3\right] = \frac{E\left[(X-\mu)^3\right]}{\left(E\left[(X-\mu)^2\right]\right)^{3/2}} = \frac{4\sigma^3\varepsilon^3\sqrt{2\pi}(3-\pi)}{\left(2\sigma^2\left((3-\pi)\varepsilon^2+1\right)\right)^{3/2}}\ .$$

3.8 The Excess Kurtosis of the ESR

Corollary 4. If $X \sim ESR(\theta,\sigma,\varepsilon)$, the Excess Kurtosis of X is also derived from (7).

$$\gamma_2 = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]-3 = \frac{E\left[(X-\mu)^4\right]}{\left(E\left[(X-\mu)^2\right]\right)^2}-3 = \frac{-6\pi^2\varepsilon^4+24\pi\varepsilon^4-17\varepsilon^4+2\varepsilon^2-1}{\left(\varepsilon^2(3-\pi)+1\right)^2}$$

3.9 The Moment Generating Function of the ESR

Theorem 2. The Moment Generating Function of the ESR is given by

$$M_X(t) = \frac{e^{t\theta}}{2}\left[1+(1+\varepsilon)^2\sigma t e^{\frac{(1+\varepsilon)^2\sigma^2 t^2}{2}}\sqrt{\frac{\pi}{2}}\left(\operatorname{erf}\left(\frac{(1+\varepsilon)\sigma t}{\sqrt{2}}\right)-1\right)\right] \tag{8}$$

$$+\frac{e^{t\theta}}{2}\left[1+(1-\varepsilon)^2\sigma t e^{\frac{(1-\varepsilon)^2\sigma^2 t^2}{2}}\sqrt{\frac{\pi}{2}}\left(\operatorname{erf}\left(\frac{(1-\varepsilon)\sigma t}{\sqrt{2}}\right)+1\right)\right]\ .$$

Proof. By definition, $M_X(t) = \frac{1}{2\sigma^2}\int_{-\infty}^{\theta} e^{tx}\left(\frac{\theta-x}{1+\varepsilon}\right)e^{-\frac{1}{2\sigma^2}\left(\frac{\theta-x}{1+\varepsilon}\right)^2}dx + \frac{1}{2\sigma^2}\int_{\theta}^{\infty} e^{tx}\left(\frac{x-\theta}{1-\varepsilon}\right)e^{-\frac{1}{2\sigma^2}\left(\frac{x-\theta}{1-\varepsilon}\right)^2}dx,$

which leads to

$$M_X(t) = \frac{1}{2(1+\varepsilon)\sigma^2}\left[(1+\varepsilon)^2\sigma^2 e^{t\theta}\right] - \frac{(1+\varepsilon)t}{2}\left[(1+\varepsilon)\sigma e^{t\theta+\frac{(1+\varepsilon)^2\sigma^2 t^2}{2}}\sqrt{\frac{\pi}{2}}\left(1+\operatorname{erf}\left(-\frac{(1+\varepsilon)\sigma t}{\sqrt{2}}\right)\right)\right]$$

$$+\frac{1}{2(1-\varepsilon)\sigma^2}\left[(1-\varepsilon)^2\sigma^2 e^{t\theta}\right] + \frac{(1-\varepsilon)t}{2}\left[(1-\varepsilon)\sigma e^{t\theta+\frac{(1-\varepsilon)^2\sigma^2 t^2}{2}}\sqrt{\frac{\pi}{2}}\left(\operatorname{erf}\left(\frac{(1-\varepsilon)\sigma t}{\sqrt{2}}\right)+1\right)\right].$$

Simplification produces (8) which completes the proof.

4 Discussion

In this paper, we introduced the *ESR* distribution and developed its properties. A limiting aspect of the *ESR* density function is that it possesses equal value at both modes. This is a result of the ratio of the probabilities on each side of θ being a function of the skew parameter, ε , while the shape parameter is a constant. Reversing this scenario and making the shape parameter σ a function of ε instead, the probability on each side of θ becomes constant, allowing the value at the mode of each side to vary instead. This deserves further development as does the application of the ESR as a model for real-world data by an iterative estimation of parameters. Further research will also include developing the properties of the Epsilon-Skew Exponentiated Rayleigh distribution.

References

[1] I.W. Burr, Annals of Mathematical Statistics, **13**, No.2, 215?232 (1942).
[2] J.G. Surles, W.J. Padgett, Lifetime Data Analysis, **7**, 187-200 (2001).
[3] G.S. Mudholkar, D.K. Srivastava, IEEE Transactions on Reliability, **42**, No. 2, 299-302 (1993).
[4] G.S. Mudholkar, A.D. Hutson, Journal of Statistical Planning and Inference **83**, 291-309 (2000).
[5] H. Elsalloukh, J.H. Guardiola, D.M. Young, Far East Journal of Theoretical Statistics, **17**, No. 1, 97-112 (2005).
[6] H. Elsalloukh. The Epsilon-Skew Laplace Distribution. JSM Proceedings, Biometrics Section: The American Statistical Association Conference, 1174?1180 (2008).
[7] A. Flaih, H. Elsalloukh, Appl. Math. Inf. Sci. **6**, No. 2, 167-171 (2012).
[8] H. Elsalloukh, E. Abdulah, The Epsilon Skew Gamma Distribution, JSM Proceedings, Biometrics Section: The American Statistical Association Conference, 50?59 (2012).
[9] H. Elsalloukh, E. Abdulah, The Epsilon Skew Inverted Gamma Distribution, JSM Proceedings, Biometrics Section: The American Statistical Association Conference, 42-49 (2012).
[10] L. Barman, An Analytical Study of Characteristics of Skewed Distributions, Gauhati University (2014).
[11] A. Azzalini, Scandinavian Journal of Statistics, **12**, No. 2, 171-178 (1985).
[12] C. Akmyradov, H. Elsalloukh, J. Stat. Appl. Pro. Lett. **6**, No. 3, 101-108 (2019).