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π-Power Exponential Odd G-family: A New Family of Probability Distributions

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Abstract: A new continuous family of distributions is introduced, and this study focuses on one specific member within this family, which showcases a hazard function exhibiting distinct J, reverse-J, bathtub, or monotonically increasing shapes. The article explores the essential characteristics of this distribution and employs the maximum likelihood estimation (MLE) method to estimate its associated parameters. To evaluate the accuracy of the estimation procedure, a simulation experiment is conducted, revealing a decrease in biases and mean square errors as sample sizes increase, even when working with small samples. Furthermore, the practical application of the proposed distribution is demonstrated by analyzing COVID-19 and engineering datasets. In this study, we employ MLEs to predict the death rate during the initial phase of the COVID-19 outbreak in China in 2020. By employing model selection criteria and conducting goodness-of-fit test statistics, the article establishes that the proposed model surpasses existing models in performance. The application of this research work can be significant in various fields where modeling and analyzing hazard functions or survival data are essential, while also making contributions to probability theory and statistical inferences.

Keywords: Cramer Rao inequality, Estimation, family of distribution, π-Power transformation

1 Introduction

Statistical models play a crucial role in representing and analyzing datasets in practical applications. While traditional distributions such as Weibull, gamma, Lomax, log-normal, exponential, beta, etc. have been widely used, they may not always provide a satisfactory fit for complex datasets. To address this limitation, researchers have been actively working on developing new models that offer greater adaptability and generality. These advancements often involve techniques such as exponentiation and the T-X approach to generate more flexible distributions. In this research article, we concentrate on an alternative approach called the pi-power transformation (PPT) family, which was introduced by [1]. The PPT family provides a distinct blend of high skewness and flexibility to the base distribution. The authors specifically examined the Pi-exponentiated Weibull as a member of this family.

Before its introduction, the alpha power transformation (APT) approach had gained significant popularity among researchers in the fields of probability theory and survival analysis. Using the APT technique, numerous authors have put forward new generalized models and distribution families. For instance, Nassar et al. [2] employed the APT technique to define the new family of distributions using log transformation. Mead et al. [3] have further studied the APT family by providing some mathematical properties that were not provided in [4]. Further, Lomax distribution was transformed using APT by [5]. Ihtisham et al. [6] and Ihtisham et al. [7] studied the Pareto and inverse Pareto distributions using the APT approach, with the inverse Pareto distribution being applied to model real data related to extreme values. Similarly, Hozaien et al. [8] and Klakattawi and Aljuhani [9] introduced new models using the APT family of distributions. Alotaibi et al. [10] have introduced a new distribution as a weighted form of the APT method, while Gomma et al. [11] introduced the Alpha power of the power Ailamujia distribution, which offers a flexible hazard function. They utilized this distribution to model COVID-19 datasets from Italy and the UK. Singh and Dhar Das [12] introduced a new model using the weighted transformation technique. Furthermore, Nassar et al. [13] defined a new family utilizing the quantile function of the APT

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family, whose cumulative distribution function (CDF) is

$$F(t) = \frac{\log\left[1+(\alpha-1)\,G(t;\phi)\right]}{\log(\alpha)}; \quad t>0, \alpha>0, \alpha\neq 1,$$

where $G(t;\phi)$ is the CDF, and ϕ is the parameter space of the base distribution. Similarly, Elbatal et al. [14] introduced another new APT family whose CDF is

$$F(x) = \frac{\alpha^{G(x)}G(x)}{\alpha}; \alpha>0, x\in\Re.$$

Similarly, Kyurkchiev [15] has introduced a family of distribution based on the Verhulst logistic function and its CDF is

$$F(x) = \frac{2G(x)}{1-G(x)}; x\in\Re.$$

Another new method for transformation can be found in [16] and the CDF of this transformation is

$$F(x) = \frac{e}{e-1}\left\{1-e^{-G(x)}\right\}; x\in\Re.$$

Also using the APT method, Mandouch et al. [17] have reported a new two-parameter family of distributions whose CDF is

$$F(x) = \frac{\alpha^{kW\{G(x)\}}-1}{\alpha-1}; \alpha>0, \alpha\neq 1, x\in\Re.$$

Lone and Jan [1] have introduced another new family using the concept of the APT family and named it the Pi-Exponentiated transformed (PET) family, whose CDF is

$$F(x) = \frac{\pi^{\{G(x)\}^{\alpha}}-1}{\pi-1}; \alpha>0, x\in\Re.$$

Hence, researchers are continuously developing and exploring new models and families of distributions to better capture the characteristics of complex datasets. The PET family has emerged as a popular approach, offering increased skewness and flexibility to the base distribution. These advancements have led to the proposal of various generalized models and distributions, which have been successfully applied to a range of datasets, including those related to COVID-19 and extreme values, such as the odd Lomax-G family [18] have defined odd Lomax generalized exponential distribution and used this model for COVID-19 data analysis. Building upon the concept of the PET, we have introduced a novel method to enhance existing distributions by incorporating an additional parameter, which we refer to as the π-power exponential odd (PiPEO-G) family of distributions. This new family offers increased robustness compared to other compound probability distributions and demonstrates great potential for modeling real-life datasets.

The suggested family possesses two parameters that enable it to capture a broader range of characteristics exhibited by a dataset, including skewness, kurtosis, failure rate, and mathematical tractability. This enhanced flexibility allows for a more accurate representation of complex data patterns and distributional properties. By considering the PiPEO-G family, researchers and practitioners can better account for the intricate nature of real-world datasets, leading to improved modeling outcomes. Among the members of the PiPEO-G family, one distribution stands out as particularly noteworthy- the Weibull distribution. The Weibull distribution has long been employed in reliability theory and life-testing due to its ability to capture failure rates and survival probabilities effectively [19]. With the integration of the PiPEO-G framework, the Weibull distribution can be further adapted and refined to better align with the unique characteristics observed in various applications. The focus of our research lies in introducing a novel, adaptable model and exploring its practical applications. This innovative model is poised to make significant contributions to the realm of statistical theory and modeling.

We have organized the rest of the sections of this paper, which are managed as follows; PiPEO-G family is introduced in Section 2 while its particular member, the PiPEO-Weibull distribution, is presented in Section 3. Some statistical properties are discussed in Section 4 and in Section 5, we discussed statistical inferences of the PiPEOW distribution. The simulation experiment, application, and conclusion of the suggested model are presented in Sections 6, 7, and 8 respectively.

2 PiPEO-G Family and Some Important Functions

Let $Y \sim PiPEO-G$ family, then the CDF and PDF of PiPEO family $U(y;\Psi)$ and $u(y;\Psi)$ for $y \in \Re$, and $\Psi > 0$ is the vector of parameters defined as

$$U(y;\Psi) = \begin{cases} \frac{\pi - \pi^{\exp\left\{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)\right\}}}{\pi-1} & \text{for } y \in \mathbb{R} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$$u(y;\Psi) = \begin{cases} \frac{(\log\pi)}{\pi-1}\pi^{\exp\left\{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)\right\}} exp\left\{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)\right\}\frac{t(y;\Psi)}{[1-T(y;\Psi)]^2} & \text{for } y \in \mathbb{R} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

where $T(y;\Psi)$ and $t(y;\Psi)$ are the CDF and PDF of any continuous distribution. Further reliability, hazard, and quantile functions of the PiPEO-G family can be expressed as

$$R(y;\Psi) = 1 - \left[\frac{1}{\pi-1}\left\{\pi - \pi^{\exp\left\{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)\right\}}\right\}\right]; y \in \mathbb{R},$$

$$h(y;\Psi) = \frac{\frac{(\log\pi)}{\pi-1}\pi^{\exp\left\{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)\right\}} exp\left\{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)\right\}\frac{t(y;\Psi)}{[1-T(y;\Psi)]^2}}{\left[1-\left\{\frac{1}{\pi-1}\left\{\pi - \pi^{\exp\left\{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)\right\}}\right\}\right\}\right]}.$$

$$Q_Y(p) = T^{-1}\left[\log\left\{\frac{\log\{\pi - p(\pi-1)\}}{\log\pi}\right\}^{-1}\left\{1+\log\left\{\frac{\log\{\pi - p(\pi-1)\}}{\log\pi}\right\}^{-1}\right\}^{-1}\right]; p \in (0,1). \tag{3}$$

2.1 Linear form of PiPEO-G family

After some mathematics, the CDF (1) of PiPEO-G can be expressed in the linear form as

$$U(y;\Psi) = \frac{\pi}{\pi-1} - \frac{1}{\pi-1}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k} i^j (\log\pi)^i}{i!j!}\binom{j}{k}T^{j+k}(y;\Psi); y \in \mathbb{R}. \tag{4}$$

Differentiating equation (4)

$$u(y;\Psi) = \frac{1}{\pi-1}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k+1}(j+k) i^j (\log\pi)^i}{i!j!}\binom{j}{k}t(y;\Psi)T^{j+k-1}(y;\Psi); y \in \mathbb{R}.$$

$$u(y;\Psi) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ijk}T^{j+k-1}(y;\Psi)t(y;\Psi); y \in \mathbb{R},$$

where $\Delta_{ijk} = \frac{1}{\pi-1}\frac{(-1)^{j+k+1}(j+k)i^j(\log\pi)^i}{i!j!}\binom{j}{k}$.

3 π-Power Exponential Odd Weibull (PiPEOW) distribution

Let Y be a continuous random variable following the Weibull distribution, then the CDF and PDF are

$$T(y;\Psi) = 1 - e^{-\alpha y^\delta}; \alpha, \delta > 0, y > 0. \tag{5}$$

$$t(y;\Psi) = \alpha\delta y^{\delta-1}e^{-\alpha y^\delta}; y > 0. \tag{6}$$

Now using Equation (5) as a base distribution, we introduce the PiPEOW distribution having CDF

$$U(y;\alpha,\delta)=\frac{\pi-\pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}}{\pi-1};\alpha,\delta>0,y>0. \tag{7}$$

The PDF of the PiPEOW distribution can be expressed as

$$u(y;\alpha,\delta)=\frac{\alpha\delta(\log\pi)}{\pi-1}\pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}y^{\delta-1}e^{\alpha y^{\delta}};y>0. \tag{8}$$

Now some key functions like reliability, hazard, and quantile of PiPEOW distribution can be presented as

$$R(y;\alpha,\delta)=1-\left[\frac{1}{\pi-1}\left\{\pi-\pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}\right\}\right];y\in\mathbb{R}.$$

$$h(y;\alpha,\delta)=\frac{\alpha\delta(\log\pi)y^{\delta-1}e^{\alpha y^{\delta}}exp\left\{-\left(e^{-\alpha y^{\delta}}-1\right)^{\alpha}\right\}\pi^{\exp\left\{-\left(e^{-\alpha y^{\delta}}-1\right)\right\}}}{(\pi-1)\left[1-\left\{\frac{1}{\pi-1}\left\{\pi-\pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}\right\}\right\}\right]}$$

$$Q_{Y}(p)=\left\{\frac{1}{\alpha}\log\left\{1-\log\left\{\frac{\log\left(\pi-p\left(\pi-1\right)\right)}{\log\pi}\right\}\right\}\right\}^{1/\delta}. \tag{9}$$

By employing the formulae of skewness and kurtosis derived from quartiles and octiles, as outlined in the study conducted

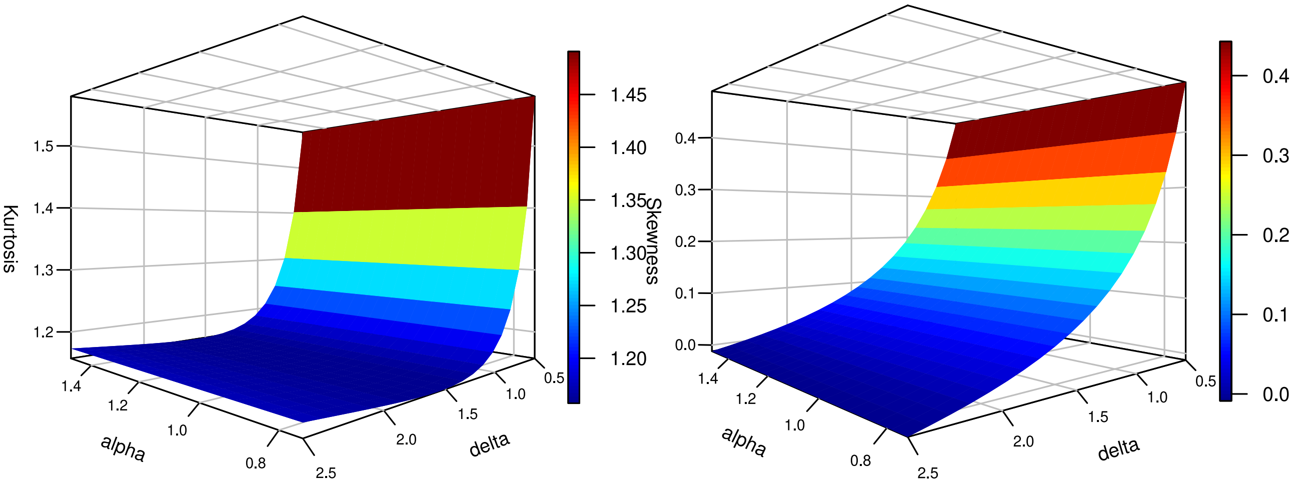


Fig. 1: Kurtosis and Skewness of PiPEOW distribution for various values of parameters α and δ.

by [20], we have displayed graphical representations of the skewness and kurtosis for PiPEOW distribution in Figure 1. Also, random number deviate can be obtained through

$$y=\left\{\frac{1}{\alpha}\log\left\{1-\log\left\{\frac{\log\left(\pi-u\left(\pi-1\right)\right)}{\log\pi}\right\}\right\}\right\}^{1/\delta};u\in(0,1).$$

The PiPEOW distribution has a density plot that can take on a diversity of shapes, including symmetrical, left-skewed, right-skewed, or decreasing, and Figure 2 (left) shows some examples of these shapes. The HRF, on the other hand, can take on the shapes of an increasing, a j, or a reverse-j, bathtub, and Figure 2 (right) shows some examples of these shapes.

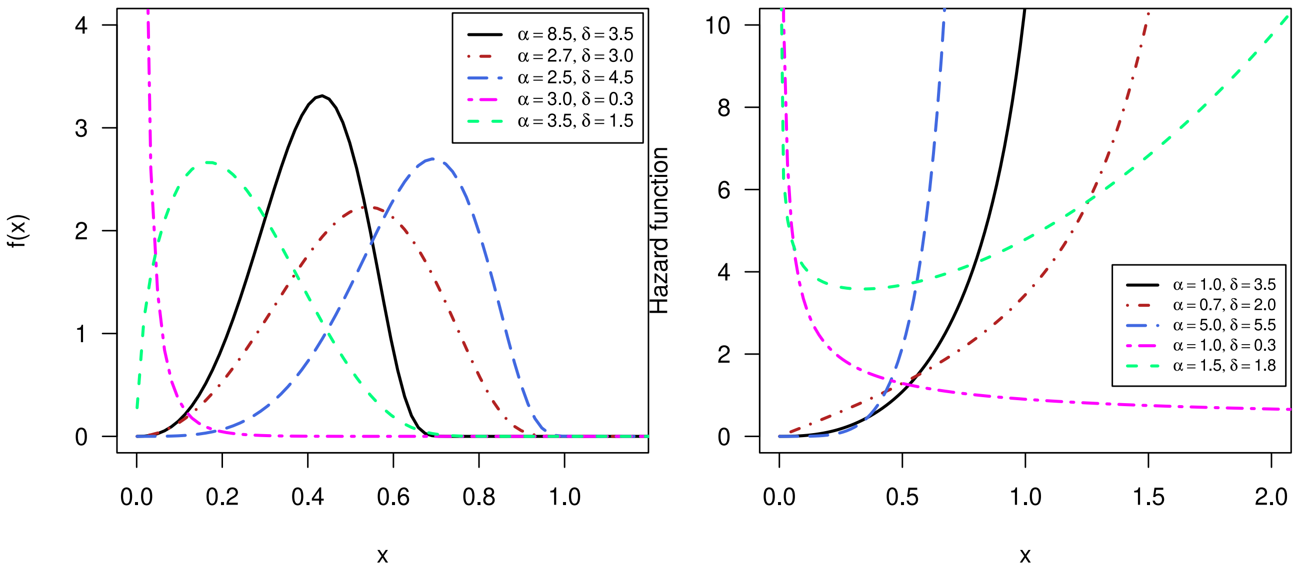


Fig. 2: Shapes of PDF and HRF of PiPEOW distribution.

4 Statistical properties of PiPEOW distribution

4.1 Linear form of PDF of PiPEOW distribution

After some mathematics, the PDF (8) of PiPEOW distribution can be obtained in linear form as

$$u(y;\alpha,\delta)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l-0}^{\infty}\Delta^{*}_{ijkl}y^{\delta-1}e^{-(l+1)\alpha y^{\delta}};y>0$$

where $\Delta^{*}_{ijkl}=(-1)^{l}\alpha\delta\binom{j+k-1}{l}\Delta_{ijk}$ and $\Delta_{ijk}=\frac{1}{\pi-1}\frac{(-1)^{j+k+1}(j+k)i^{j}(\log\pi)^{i}}{i!j!}\binom{j}{k}$

4.2 Moments

The r^{th} moment of PiPEOW distribution is

$$\begin{aligned}E\left[Y^{r}\right]&=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Delta^{*}_{ijkl}\int_{0}^{\infty}y^{\delta+r-1}e^{-(l+1)\alpha y^{\delta}}dy\\&=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Delta^{*}_{ijkl}\int_{0}^{\infty}t^{\frac{r}{\delta}+1-1}e^{-(k+1)\alpha t}dt\\&=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Delta^{*}_{ijkl}\frac{\delta^{-1}\Gamma\left(\frac{r}{\delta}+1\right)}{\{(l+1)\alpha\}^{\frac{r}{\delta}+1}}.\end{aligned}\tag{10}$$

Mean and variance of PiPEOW distribution

$$E\left[Y\right]=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Delta^{*}_{ijkl}\frac{\delta^{-1}\Gamma\left(\frac{1}{\delta}+1\right)}{\{(l+1)\alpha\}^{\frac{1}{\delta}+1}}.$$

and

$$E\left[Y^{2}\right]=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Delta^{*}_{ijkl}\frac{\delta^{-1}\Gamma\left(\frac{2}{\delta}+1\right)}{\{(l+1)\alpha\}^{\frac{2}{\delta}+1}}.$$

$$V[Y]=E\left[Y^2\right]-[E(Y)]^2$$
$$=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ijkl}^{*}\frac{\delta^{-1}\Gamma\left(\frac{2}{\delta}+1\right)}{\{(l+1)\alpha\}^{\frac{2}{\delta}+1}}-\left[\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ijkl}^{*}\frac{\delta^{-1}\Gamma\left(\frac{1}{\delta}+1\right)}{\{(l+1)\alpha\}^{\frac{1}{\delta}+1}}\right]^2.$$

4.3 Moment Generating Function of PiPEOW distribution

$$M_Y(t)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m}{m!}\Delta_{ijkl}^{*}\int_0^{\infty}y^{\delta+m-1}e^{-(l+1)\alpha y^{\delta}}dy$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m}{m!}\Delta_{ijkl}^{*}\int_0^{\infty}t^{\frac{m}{\delta}+1-1}e^{-(l+1)\alpha t}dt$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m}{m!}\Delta_{ijkl}^{*}\frac{\delta^{-1}\Gamma\left(\frac{m}{\delta}+1\right)}{\{(l+1)\alpha\}^{\frac{m}{\delta}+1}}.$$

4.4 Characteristic Function of PiPEOW Distribution

$$\Phi_Y(t)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(vt)^m}{m!}\Delta_{ijkl}^{*}\int_0^{\infty}y^{\delta+m-1}e^{-(l+1)\alpha y^{\delta}}dy$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(vt)^m}{m!}\Delta_{ijkl}^{*}\int_0^{\infty}t^{\frac{m}{\delta}+1-1}e^{-(l+1)\alpha t}dt$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(vt)^m}{m!}\Delta_{ijkl}^{*}\frac{\delta^{-1}\Gamma\left(\frac{m}{\delta}+1\right)}{\{(l+1)\alpha\}^{\frac{m}{\delta}+1}}.$$

where $v=\sqrt{-1}$.

4.5 Incomplete moment of PiPEOW distribution

The incomplete moment for PiPEOW distribution is given by

$$M_r(z)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l-0}^{\infty}\Delta_{ijkl}^{*}\int_0^{Z}y^{r+\delta-1}e^{-(l+1)\alpha y^{\delta}}dy$$
$$=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l-0}^{\infty}\Delta_{ijkl}^{*}\frac{\delta^{-1}\left[\gamma\left(\frac{\delta+r}{\delta},(l+1)\alpha z^{\delta}\right)\right]}{\{(l+1)\alpha\}^{\frac{\delta+r}{\delta}}}.$$

where $\gamma(.)$incomplete gamma function.

4.6 Mean Residual Life Function of PiPEOW Distribution

The mean residual life function for the PiPEOW distribution is given by

$$\overline{M(z)} = \frac{1}{F(z)}\left[\mu - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l-0}^{\infty}\Delta^{*}_{ijkl}\int_{0}^{z} y^{\delta} e^{-(l+1)\alpha y^{\delta}} dy\right] - z$$

$$= \frac{1}{F(z)}\left[\mu - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l-0}^{\infty}\Delta^{*}_{ijkl}\frac{\delta^{-1}\left[\gamma\left(\frac{\delta+1}{\delta},(l+1)\alpha z^{\delta}\right)\right]}{\{(l+1)\alpha\}^{\frac{\delta+1}{\delta}}}\right] - z.$$

where $\gamma(.)$ incomplete gamma function.

4.7 Order Statistics

Let $y_i(i=1,\ldots,n) \sim PiPEOW(y;\alpha,\delta)$ with CDF $U(y_i;\alpha,\delta)$ and PDF $u(y_i;\alpha,\delta)$. If $u_r(y)$ denote the PDF of r^{th} order statistic $Y_{(r)}$, then their CDF and PDF are given by

$$U_r(y) = I_{U(y)}(r,n-r+1).$$

$$u_r(y) = \frac{d}{dy}[U_r(y)] = \frac{d}{dy}\left[I_{U(y)}(r,n-r+1)\right] = \frac{1}{B(r,n-r+1)}U^{r-1}(y)u(y)\left[1-U(y)\right]^{n-r}.$$

$$u_r(y) = \frac{1}{B(r,n-r+1)}\frac{\alpha\delta(\log\pi)}{\pi-1}\pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}y^{\delta-1}e^{\alpha y^{\delta}}\left[C_y\right]^{r-1}$$
$$\left[1-\{C_y\}\right]^{n-r},$$

here $I_U(a,b) = \int_0^t t^{a-1}(1-t)^{b-1}dt$ and $B(a,b)$ are incomplete and standard beta functions respectively and $C_y = \frac{\pi - \pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}}{\pi-1}$. The CDF and PDF of first-order statistic $Y_{(1)}$ are given by

$$U_1(y) = 1 - \left[1-\{C_y\}\right]^{n}; y > 0.$$

$$u_1(y) = \frac{n\alpha\delta(\log\pi)}{\pi-1}\left[1-C_y\right]^{n-1}\pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}y^{\delta-1}e^{\alpha y^{\delta}}; y > 0.$$

The CDF and PDF of n^{th} order statistic $Y_{(n)}$ are given by

$$U_n(y) = \left[C_y\right]^{n}; y > 0.$$

$$u_n(y) = \frac{n\alpha\delta(\log\pi)}{\pi-1}\left[C_y\right]^{n-1}\pi^{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}}exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}y^{\delta-1}e^{\alpha y^{\delta}}; y > 0.$$

The joint PDF of r^{th} and s^{th} order statistic is given by

$$u_{rs}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}U^{r-1}(x).u(x)\left[U(y)-U(x)\right]^{s-r-1}u(y).\left[1-U(y)\right]^{n-s}.$$

$$u_{rs}(x,y) = \frac{n!\pi^{\left\{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}+\exp\left\{-\left(e^{\alpha x^{\delta}}-1\right)\right\}\right\}}}{(r-1)!(s-r-1)!(n-s)!}exp\left\{-\left\{\left(e^{\alpha y^{\delta}}-1\right)+\left(e^{\alpha x^{\delta}}-1\right)\right\}\right\}$$
$$(xy)^{\delta-1}e^{\alpha\left(y^{\delta}+x^{\delta}\right)}\left[\frac{\alpha\delta(\log\pi)}{\pi-1}\right]^{2}\left[C_y\right]^{r-1}\left[C_y-C_x\right]^{s-r-1}\left[1-C_y\right]^{n-s}; (x,y)\in\Re_{+}\times\Re_{+}.$$

The Joint PDF of 1st and n^{th} order statistics is given by

$$u_{1n}(x,y) = n(n-1)\left[U(y)-U(x)\right]^{n-2} u(x).u(y).$$

$$u_{1n}(x,y) = n(n-1)\left(\frac{\alpha\delta(\log\pi)}{\pi-1}\right)^2 [C_y - C_x]^{n-2}\, \pi^{\left\{\exp\left\{-\left(e^{\alpha y^{\delta}}-1\right)\right\}+\exp\left\{-\left(e^{\alpha x^{\delta}}-1\right)\right\}\right\}}$$
$$\exp\left\{-\left\{\left(e^{\alpha y^{\delta}}-1\right)+\left(e^{\alpha x^{\delta}}-1\right)\right\}\right\}(xy)^{\delta-1} e^{\alpha\left(y^{\delta}+x^{\delta}\right)};(x,y)\in\Re_+\times\Re_+.$$

5 Statistical Inference

5.1 Estimation

Let $y_i(i=1,...,n) \sim PiPEOW(y_i;\alpha,\delta)$ with PDF $u(y_i;\alpha,\delta)$ then the log-likelihood function can be calculated as

$$\begin{aligned}\ell(\underline{y};\alpha,\delta) =& n\log(\alpha\delta)+n\log(\log\pi)-n\log(\pi-1)+log\pi\sum_{i=1}^{n}\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}\\ &-\sum_{i=1}^{n}\left(e^{\alpha y_i^{\delta}}-1\right)+(\delta-1)\sum_{i=1}^{n}\log y_i+\alpha\sum_{i=1}^{n}y_i^{\delta}.\end{aligned} \tag{11}$$

Differentiating Equation (11) with respect to associated parameters, we get

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-log\pi\sum_{i=1}^{n}\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}\left\{e^{\alpha y_i^{\delta}}y_i^{\delta}\right\}-\sum_{i=1}^{n}e^{\alpha y_i^{\delta}}y_i^{\delta}+\sum_{i=1}^{n}y_i^{\delta}.$$

$$\frac{\partial^2\ell}{\partial\alpha^2}=-\frac{n}{\alpha^2}-log\pi\sum_{i=1}^{n}\left\{\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}e^{\alpha y_i^{\delta}}y_i^{\delta}-e^{2\alpha y_i^{\delta}}y_i^{\delta}\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}\right\}y_i^{\delta}-\sum_{i=1}^{n}e^{\alpha y_i^{\delta}}\left(y_i^{\delta}\right)^2.$$

$$\frac{\partial\ell}{\partial\delta}=\frac{n}{\delta}-log\pi\sum_{i=1}^{n}\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}\left\{e^{\alpha y_i^{\delta}}\alpha y_i^{\delta}\log y_i\right\}-\alpha\sum_{i=1}^{n}e^{\alpha y_i^{\delta}}y_i^{\delta}\log y_i+\alpha\sum_{i=1}^{n}y_i^{\delta}\log y_i.$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\delta^2}=&-\frac{n}{\delta^2}-log\pi\alpha\sum_{i=1}^{n}\left\{\frac{\partial}{\partial\delta}\left\{\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}\left\{e^{\alpha y_i^{\delta}}y_i^{\delta}\right\}\right\}\right\}\log y_i\\ &-\alpha\sum_{i=1}^{n}\left\{\alpha e^{\alpha y_i^{\delta}}\left(y_i^{\delta}\right)^2\log y_i+e^{\alpha y_i^{\delta}}y_i^{\delta}\log y_i\right\}\log y_i+\alpha\sum_{i=1}^{n}y_i^{\delta}\left(\log y_i\right)^2.\end{aligned}$$

By solving the above three non-linear equations using suitable software one can obtain the estimates under the maximum likelihood estimation (MLE) method.

5.2 Cramer-Rao (CR) Inequality

If $T(y_1,...,y_n)$ is an unbiased estimator for $g(\psi)$, a function of parameter ψ,then

$$Var[T(y_1,...,y_n)]\geqslant\frac{\left\{\frac{d}{d\psi}g(\psi)\right\}^2}{E\left(\frac{\partial}{\partial\psi}logL\right)}=\frac{\{g'(\psi)\}^2}{I(\psi)},$$

where $I(\psi)$ is the information on ψ, supplied by the sample. To define CR lower bound (CRLB) for α when δ is supposed to be known, then CRLB for an unbiased estimator $T_1(y_1,...,y_n)$ of a parameter α is given by $\frac{1}{I(\alpha)}$, where

$$\begin{aligned}I(\alpha)&=-E\left[\frac{\partial^2\ell}{\partial\alpha^2}\right]\\ &=\frac{n}{\alpha^2}+log\pi\sum_{i=1}^{n}E\left[\left\{\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}e^{\alpha y_i^{\delta}}y_i^{\delta}-e^{2\alpha y_i^{\delta}}y_i^{\delta}\exp\left\{-\left(e^{\alpha y_i^{\delta}}-1\right)\right\}\right\}y_i^{\delta}\right]\\ &+\sum_{i=1}^{n}E\left\{e^{\alpha y_i^{\delta}}\left(y_i^{\delta}\right)^2\right\}.\end{aligned}$$

and

$$\frac{\partial^2 \ell}{\partial \alpha^2}=-\frac{n}{\alpha^2}-log\pi\sum_{i=1}^{n}\left\{\exp\left\{-\left(e^{\alpha y_i^\delta}-1\right)\right\}e^{\alpha y_i^\delta}y_i^\delta-e^{2\alpha y_i^\delta}y_i^\delta\exp\left\{-\left(e^{\alpha y_i^\delta}-1\right)\right\}\right\}y_i^\delta-\sum_{i=1}^{n}e^{\alpha y_i^\delta}\left(y_i^\delta\right)^2.$$

Again CRLB for δ when α is supposed to be known, then CRLB for an unbiased estimator $T_2(y_1,...,y_n)$ of a parameter δ is given by $\frac{1}{I(\delta)}$, where

$$I(\delta)=-E\left[\frac{\partial^2 \ell}{\partial \delta^2}\right]=\frac{n}{\delta^2}+log\pi\alpha\sum_{i=1}^{n}E\left[\left\{\frac{\partial}{\partial\delta}\left\{\exp\left\{-\left(e^{\alpha y_i^\delta}-1\right)\right\}\left\{e^{\alpha y_i^\delta}y_i^\delta\right\}\right\}\right\}logy_i\right]$$
$$+\alpha\sum_{i=1}^{n}E\left[\left\{\alpha e^{\alpha y_i^\delta}\left(y_i^\delta\right)^2\log y_i+e^{\alpha y_i^\delta}y_i^\delta\log y_i\right\}\log y_i\right]-\alpha\sum_{i=1}^{n}E\left\{y_i^\delta(\log y_i)^2\right\}.$$

and

$$\frac{\partial^2 \ell}{\partial \delta^2}=-\frac{n}{\delta^2}-log\pi\alpha\sum_{i=1}^{n}\left\{\frac{\partial}{\partial\delta}\left\{\exp\left\{-\left(e^{\alpha y_i^\delta}-1\right)\right\}\left\{e^{\alpha y_i^\delta}y_i^\delta\right\}\right\}\right\}\log y_i$$
$$-\alpha\sum_{i=1}^{n}\left\{\alpha e^{\alpha y_i^\delta}\left(y_i^\delta\right)^2\log y_i+e^{\alpha y_i^\delta}y_i^\delta\log y_i\right\}\log y_i+\alpha\sum_{i=1}^{n}y_i^\delta(\log y_i)^2.$$

5.3 Asymptotical Properties

A consistent likelihood equation solution is asymptotically normally distributed around true value θ_0. Thus, $\hat{\theta}$ is asymptotically $N\left(\theta_0,\frac{1}{I(\theta_0)}\right)$ as $n\to\infty$. Particularly $\hat{\alpha}$ and $\hat{\delta}$ are distributed asymptotically $N\left(\alpha,\frac{1}{I(\alpha)}\right)$, $N\left(\delta,\frac{1}{I(\delta)}\right)$ respectively as $n\to\infty$.

5.4 Pivotal Quantity (PQ)

Let $y_i(i=1,2,...,n)\sim PiPEOW(y;\alpha,\delta)$ with CDF $U(y_i;\alpha,\delta)$ then pivotal quantity is defined as

$$-2\sum_{i=1}^{n}\ln\left[U(y_i;\alpha,\delta)\right]\sim\chi^2_{2n}\quad and-2\sum_{i=1}^{n}\ln\left[1-U(y_i;\alpha,\delta)\right]\sim\chi^2_{2n}.$$

$$PQ=-2\sum_{i=1}^{n}\ln\left[\frac{\pi-\pi^{\exp\left\{-\left(e^{\alpha y_i^\delta}-1\right)\right\}}}{\pi-1}\right]\sim\chi^2_{2n}$$

and

$$PQ^*=-2\sum_{i=1}^{n}\ln\left[1-\left\{\frac{\pi-\pi^{\exp\left\{-\left(e^{\alpha y_i^\delta}-1\right)\right\}}}{\pi-1}\right\}\right]\sim\chi^2_{2n}.$$

where χ^2_{2n} represent the chi-square distribution with 2n degree of freedom. Let $x_i(i=1,2,...,m)\sim PiPEOW(x;\alpha,\delta)$ and $y_i(i=1,2,...,n)\sim PiPEOW(y;\alpha,\delta)$ are two independent random variable with CDF $U(x_i;\alpha,\delta)$ and $U(y_i;\alpha,\delta)$ respectively, then $\frac{PQ_1}{PQ_2}\sim Beta_2(m,n)$, $\frac{PQ_1}{PQ_1+PQ_2}\sim Beta_1(m,n)$, and $\frac{n}{m}\frac{PQ_1}{PQ_2}\sim F(m,n)$, where

$$PQ_1=-2\sum_{i=1}^{n}\ln\left[\frac{\pi-\pi^{\exp\left\{-\left(e^{\alpha x_i^\delta}-1\right)\right\}}}{\pi-1}\right]\quad\text{and}$$

$$PQ_2=-2\sum_{i=1}^{n}\ln\left[\frac{\pi-\pi^{\exp\left\{-\left(e^{\alpha y_i^\delta}-1\right)\right\}}}{\pi-1}\right].$$

Generally, PQ is used to construct CI for a single parametric model. For this model, we used PQ to construct CI for a small sample for one parameter while keeping others constant.

5.5 Confidence interval for Large Sample

The first derivative of the logarithm of the likelihood function with respect to parameter θ viz., $\frac{\partial \log L}{\partial \theta}$, which is asymptotically normal with mean zero and variance, can be computed under regularity conditions as

$$Var\left(\frac{\partial \log L}{\partial \theta}\right) = E\left(\frac{\partial \log L}{\partial \theta}\right)^2 = -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right).$$

Hence for large n, $Z = \frac{\frac{\partial \log L}{\partial \theta}}{\sqrt{Var\left(\frac{\partial \log L}{\partial \theta}\right)}} \sim N(0,1)$. By utilizing the obtained results, we can derive a confidence interval for the parameter θ when dealing with a substantial sample. Consequently, for a large sample, we can establish the confidence interval for θ, incorporating a confidence coefficient of $(1-b)\%$, by transforming the inequalities involved. $P(|Z| \leqslant \gamma_b) = 1-b$, where γ_b is given by

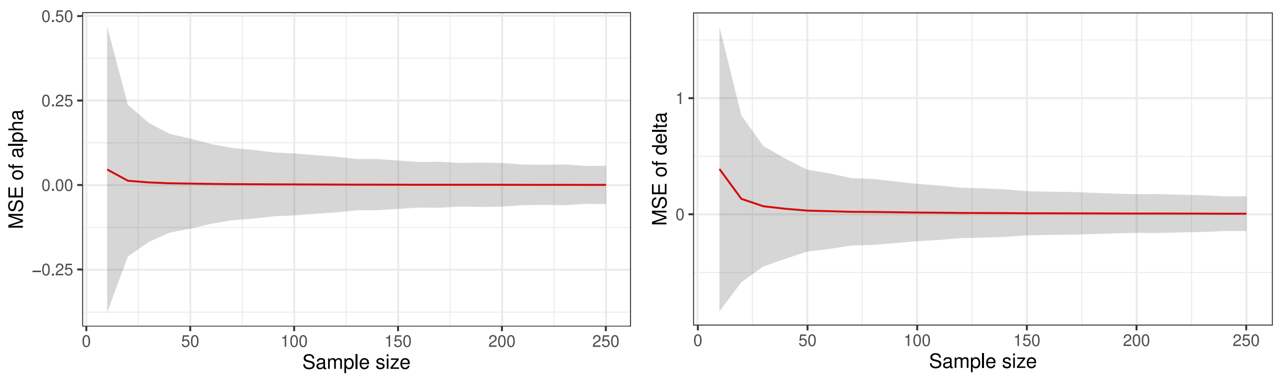
$$\frac{1}{(2\pi)^{1/2}} \int_{-\gamma_b}^{\gamma_b} \exp\left(-t^2/2\right) dt = 1-b.$$

Thus confidence interval for α and δ are given by $\widehat{\alpha} \pm \gamma_{b/2} SE(\widehat{\alpha})$ and. $\widehat{\delta} \pm \gamma_{b/2} SE\left(\widehat{\delta}\right)$ at the confidence coefficient $(1-b)\%$.

6 Simulation

In our research study, we employed the maxLik R package, developed by [21], to generate samples from the quantile function described in Equation (9) for various combinations of parameters of the PiPEOW distribution. The MLEs were then computed for each sample using the `maxLik()` function and the BFGS algorithm. This analysis allowed us to investigate issues related to parameter estimation and determine the direction and magnitude of bias in the MLEs, whether it be overestimation or underestimation.

In our simulation, we utilized sample sizes ranging from 10 to 250 with increments of 10. The entire process was repeated 1000 times to obtain estimates of the mean square error (MSE). The MSEs for the four different parameter combinations, namely set-I $(\alpha = 0.5, \delta = 1.5)$, set-II $(\alpha = 0.75, \delta = 1.25)$, set-III $(\alpha = 0.25, \delta = 0.75)$ and set-IV $(\alpha = 1.25, \delta = 0.5)$, are presented in Figure 3. The results demonstrate that as the sample size increases, the corresponding MSEs decrease toward zero. This finding suggests that the MLE method exhibits asymptotic efficiency, and consistency, and follows the invariance property.



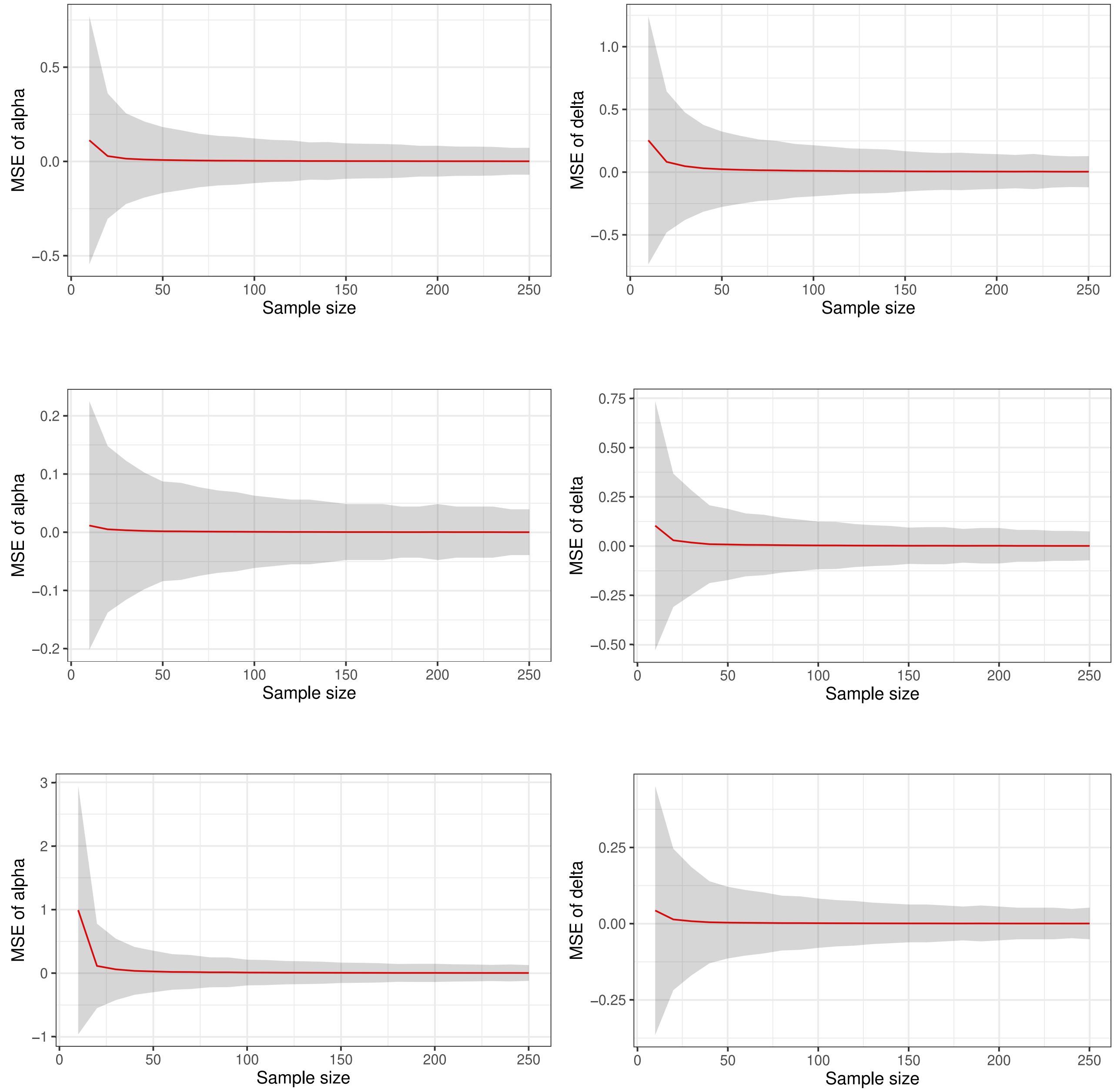


Fig. 3: Graphs of MSE for α and δ with 95% confidence bound.

7 Application

In this section, we demonstrate the application of the PiPEOW distribution using two real datasets. The datasets utilized for applying the suggested distribution are presented below.

Dataset-I

We have used the data presented in [22] which illustrates the daily number of new COVID-19-related deaths in China spanning from January 23, 2020, to March 28, 2020. The data are as follows

"8, 16, 15, 24, 26, 26, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97, 108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 29, 44, 47, 35, 42, 31, 38, 31, 30, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6,9, 7, 4, 6, 5, 3, 5".

Dataset-II

The second data set is on the Aircraft windshield failures (thousands of hours) reported by [23].
"0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663".
In Figure 4, we present the distributions of the two datasets and note that both exhibit a right-skewed pattern. To examine the shape of the hazard function, we utilize the total-time-on test (TTT) plot (Figure 5), as introduced by [24]. This plot illustrates the scaled TTT transform of the empirical data. Our observations reveal that the first dataset displays a bathtub hazard rate, while the second dataset exhibits a concave curve in the TTT plot. Consequently, we conclude that the hazard function is increasing.

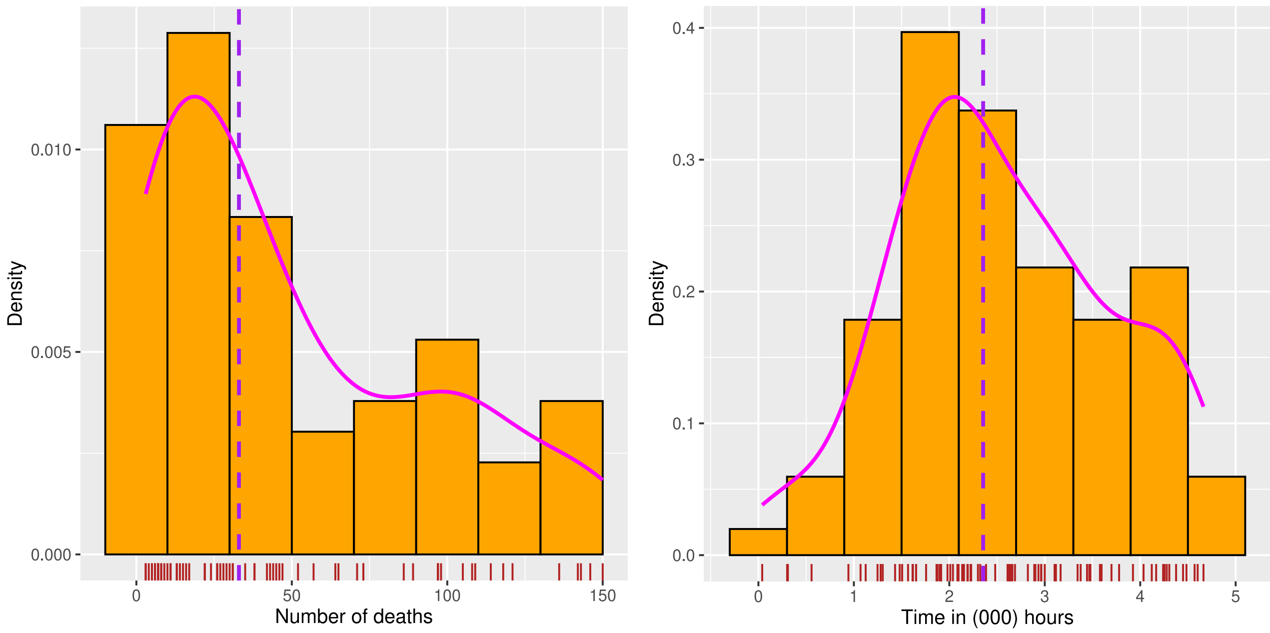


Fig. 4: Histograms of Dataset I (left) and II (right) with median lines (purple).

7.1 Model Analysis

We computed several well-known goodness-of-fit statistics to analyze the data sets I and II. The fitted models were evaluated using various metrics, which included the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), and Kolmogorov-Smirnov (KS) test with corresponding p-values. All the essential computations and graphical plots were performed using the R software, for more detail see [25] and [26]. To compare the fitting capability of the PIPEOW model, we have selected several models such as the alpha power exponential (APE) by [4], inverse Weibull (IW), Weibull, alpha power Weibull quantile (APWQ) [13], APT-Weibull (APTW) [2], and new APT-Weibull (NAPTW) [14]. We have presented contour and QQ (Quantile-Quantile) plots for both datasets in Figures 6 and 7 respectively. By examining the contour plots and the associated log-likelihood values, we gain insights into the behavior of the likelihood function and the relationship between the parameters. This information can guide parameter estimation, model selection, and inference in statistical analysis. Likewise, QQ plots indicate that the suggested model can effectively fit the real datasets.

The estimated values of the parameters and their associated standard errors (SE) for both datasets were presented in Tables 1 and 3, which were obtained using the MLE method. Additionally, Tables 2 and 4 showcase model selection criteria such as log-likelihood, HQIC, and AIC, and goodness of fit statistics such as KS along with p-value for both datasets. Our observations show that the PIPEOW model has the least statistics compared to the APE, IW, Weibull,

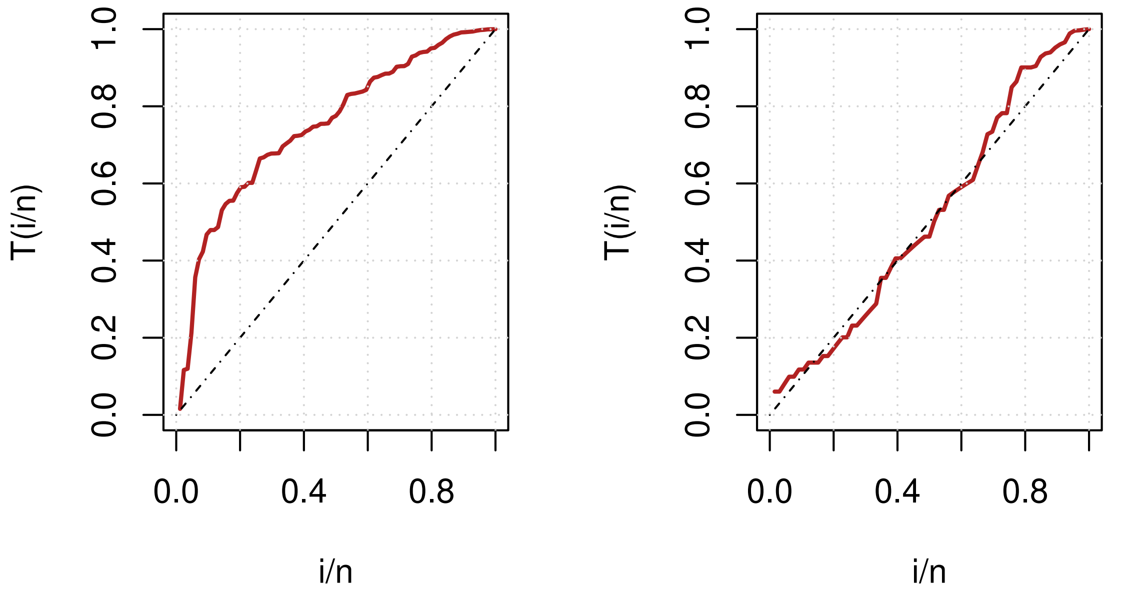


Fig. 5: TTT plots of Dataset I (right) and II (left).

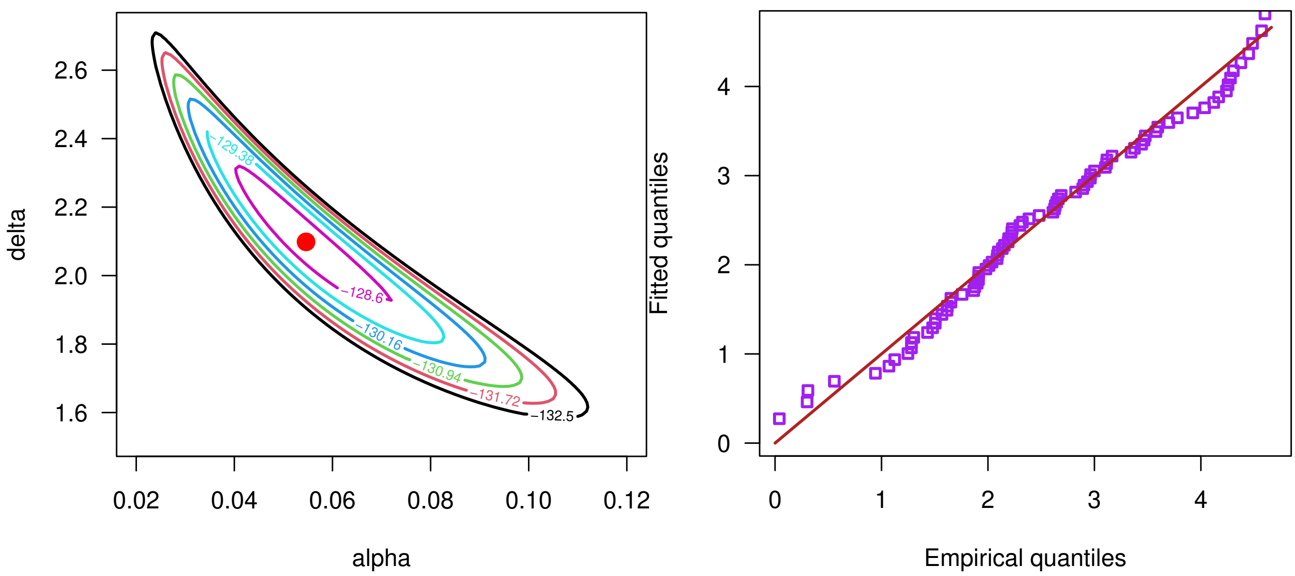


Fig. 6: Contour and QQ plots of PIPEOW distribution (dataset-I).

APIP, APWQ, APTW, and NAPTW distributions, along with the corresponding highest p-values. This indicates that the PiPEOW model is more flexible and provides a good fit. Furthermore, we have provided graphical illustrations of the fitted models in Figures 8 and 9, which support our findings that the PiPEOW model outperforms the other candidate models.

Based on the maximum likelihood estimates (MLEs) obtained from the first dataset of COVID-19, we utilized them to make projections on the daily number of deaths attributed to COVID-19 in China spanning from January 23, 2020, to March 28, 2020. The analysis outcomes are presented in Table 5, which exhibits the probabilities associated with various ranges of daily deaths. The findings indicate that if the prevailing conditions observed during the initial phase of the COVID-19 pandemic in China persisted, there was an estimated 35.56% probability of experiencing 0 to 25 deaths among infected individuals on a specific day. Similarly, the probability of encountering deaths within the range of 25 to 50 infected people was approximately 24.41%, and so forth.

Table 1: Estimated parameters using the MLE method for the data set-I.

Model	parameter	SE	parameter	SE	parameter	SE
PIPEOW(δ,λ)	0.0109	0.0049	0.9671	0.0967	–	–
APE(δ,λ)	1.3901	1.0578	0.0217	0.0045	–	–
IW(β,λ)	0.9158	0.0747	13.5272	2.6413	–	–
Weibull(β,λ)	0.0137	0.0066	1.0891	0.1071	–	–
APWQ(α,β,λ)	52.5186	4.1427	1.7266	0.1444	3.00E-04	2.00E-04
APIE(δ,λ)	6.8693	4.4839	9.8413	2.2419	–	–
APTW(β,λ,θ)	1.0000	1.9053	1.0753	0.2404	0.0145	0.0213
NAPTW(α,δ,θ)	11.0505	7.7978	0.6547	0.0986	0.1583	0.0775

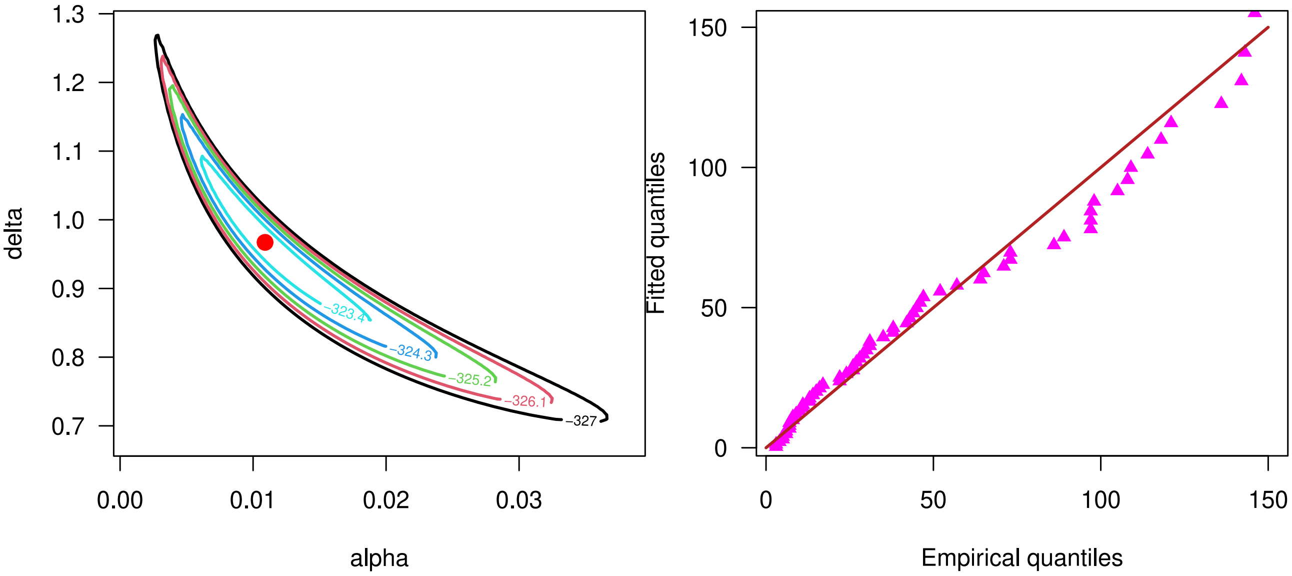


Fig. 7: Contour and QQ plots of PIPEOW distribution (dataset-II).

Table 2: Fitted statistics for the data set-I.

Model	-2logL	AIC	HQIC	KS	p-value
PiPEOW	645.189	649.189	650.919	0.0884	0.6810
APE	647.521	651.521	653.251	0.0895	0.6737
IW	662.203	666.203	667.934	0.1285	0.2262
Weibull	646.977	650.977	652.707	0.0916	0.6375
APWQ	643.394	649.394	651.989	0.0989	0.5391
APIE	659.027	663.027	664.758	0.1348	0.1815
APTW	646.993	652.993	655.589	0.0906	0.6506
NAPTW	650.278	656.278	658.874	0.0924	0.6256

Table 3: Estimated parameters using the MLE method for the data set-II

Model	parameter	SE	parameter	SE	parameter	SE
PiPEOW(δ,λ)	0.0544	0.0142	2.1043	0.1868	–	–
APE(δ,λ)	139.6919	2.9644	0.8469	0.0517	–	–
IW(β,λ)	0.8387	0.0521	1.3646	0.1516	–	–
Weibull(β,λ)	0.0823	0.0227	2.3744	0.2096	–	–
APWQ(α,β,λ)	1.0000	2.7645	2.1081	0.3348	0.1090	0.1251
APIE(δ,λ)	106.6261	2.9635	0.3873	0.0468	–	–
APTW(β,λ,θ)	23.5220	4.5066	1.5909	0.1512	0.3718	0.0735
NAPTW(α,δ,θ)	6.3683	5.8343	1.6552	0.2756	0.3253	0.1481

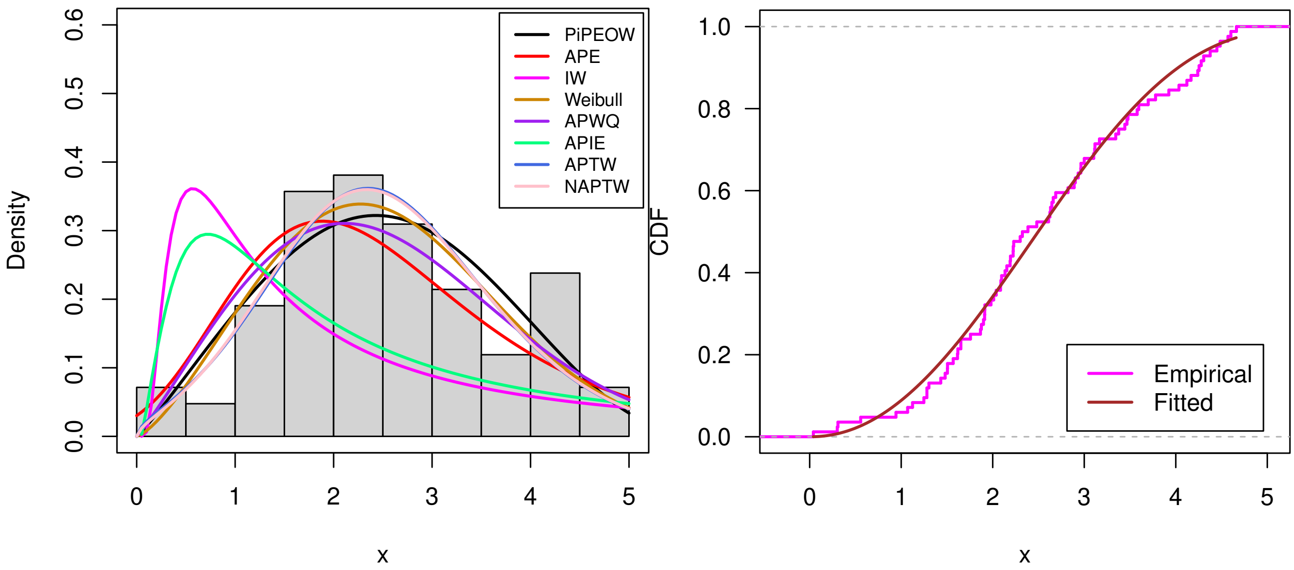


Fig. 8: Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-I)

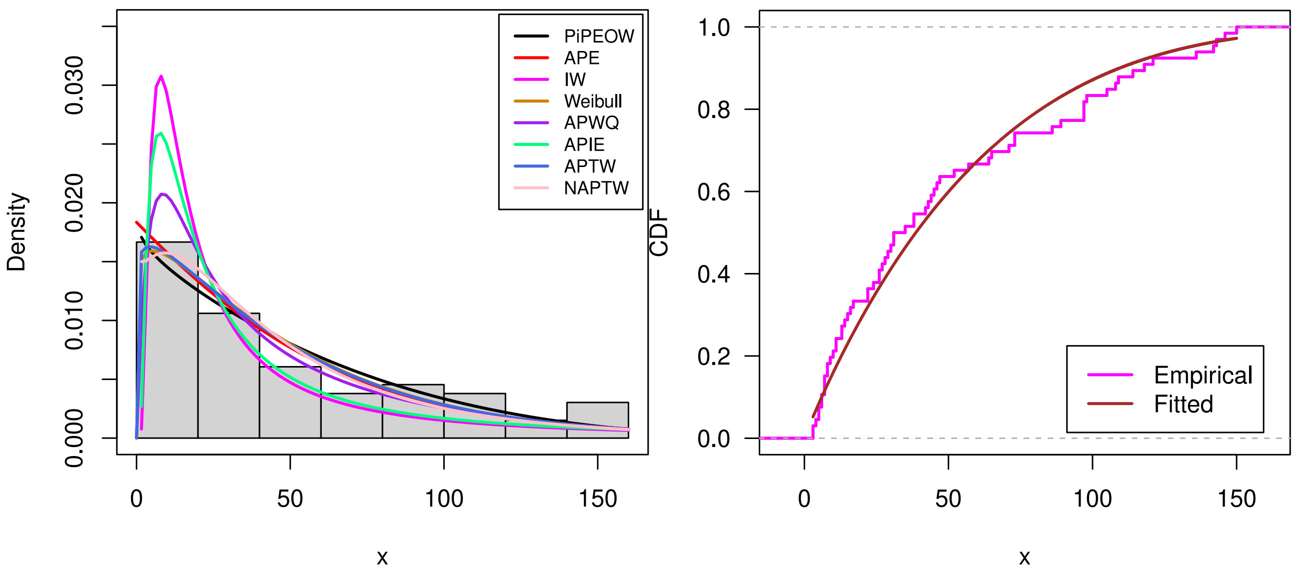


Fig. 9: Fitted PDF (left) and fitted CDF vs empirical CDF (right) (dataset-II)

Table 4: Fitted statistics for the data set-II

Model	-2logL	AIC	HQIC	KS	p(KS)
PiPEOW	255.9265	259.9265	261.8808	0.0624	0.8996
APE	266.4053	270.4053	272.3596	0.1071	0.2906
IW	389.0733	393.0733	395.0277	0.3128	0.0000
Weibull	260.1067	264.1067	266.061	0.0537	0.9689
APWQ	261.9421	267.9421	270.8736	0.0837	0.5983
APIE	359.6204	363.6204	365.5747	0.3137	0.0000
APTW	257.0685	263.0685	266.0000	0.0673	0.8417
NAPTW	257.3307	263.3307	266.2622	0.0658	0.8598

Table 5: Prediction of the number of deaths on a particular day in China.

Death rate	0-25	25-50	50-75	75-100	100 and above
Probability	0.3556	0.2441	0.1652	0.1068	0.1282

8 Conclusion

In this study, we have introduced an innovative distribution family called the π-power exponential odd transformation family. Drawing inspiration from the PPT methodology, we selected the Weibull distribution as the foundation for this new family. The PiPEOW distribution offers a wide range of hazard function shapes, including increasing, bathtub, J-shaped, and reverse-J-shaped. By employing the maximum likelihood estimation technique, we explored the statistical properties of this distribution and estimated its parameters. To assess the accuracy of our estimation method, we conducted a Monte Carlo simulation. The results revealed that the mean square errors decrease as the sample size increases, even when dealing with small samples.

To demonstrate the practical utility of the PiPEOW distribution, we applied it to two real-world datasets. Through model selection criteria and goodness-of-fit tests, we compared its performance against seven existing models. Our findings strongly support the superiority of the PiPEOW distribution over the alternative models, suggesting its potential application in various fields, such as medical and life sciences, reliability engineering, actuarial science, and survival analysis. Additionally, the π-power transformation family of distributions holds promise as a foundation for developing novel models in the future.

References

[1] M. A. Lone and T. R. Jan, *A New Pi-Exponentiated Method for Constructing Distributions with an Application to Weibull Distribution*, Reliability: Theory & Applications, **18**(1 (72)), 94-109, (2023).

[2] M. Nassar, A. Alzaatreh, O. Abo-Kasem, M. Mead, and M. Mansoor, *A new family of generalized distributions based on alpha power transformation with application to cancer data*, Annals of Data Science, **5**, 421-436, (2018a).

[3] M. E. Mead, G. M. Cordeiro, A. Z. Afify and H. Al Mofleh, *The alpha power transformation family: properties and applications*, Pakistan Journal of Statistics and Operation Research, 525-545, (2019).

[4] A. Mahdavi and D. Kundu, *A new method for generating distributions with an application to exponential distribution*, Communications in Statistics-Theory and Methods, **46**(13), 6543-6557, (2017).

[5] S. K. Maruthan and N. Venkatachalam, *Alpha Power Transformation of Lomax Distribution: Properties and Applications*, Thailand Statistician, **20**(3), 669-685, (2022).

[6] S. Ihtisham, A. Khalil, S. Manzoor, S. A. Khan, and A. Ali, *Alpha-Power Pareto distribution: Its properties and applications*, PloS one, **14**(6), e0218027, (2019).

[7] S. Ihtisham, S. Manzoor, A. Khalil, S. Badshah, M. Ijaz and H. Atta, *Modeling Extreme Values with Alpha Power Inverse Pareto Distribution*, Measurement Science Review, **23**(2), 55-62, (2023).

[8] H. E. Hozaien, G. A. Dayian and A. A. EL-Helbawy, *Kumaraswamy Distribution Based on Alpha Power Transformation Methods,* Asian Journal of Probability and Statistics, **11**(1), 14-29, (2021).

[9] H. S. Klakattawi and W. H. Aljuhani, *A new technique for generating distributions based on a combination of two techniques: alpha power transformation and exponentiated TX distributions family,* Symmetry, **13**(3), 412, (2021).

[10] R. Alotaibi, H. Okasha, H. Rezk, and M. Nassar, *A new weighted version of alpha power transformation method: Properties and applications to COVID-19 and software reliability data*, Physica Scripta, **96**(12), 125221, (2021).

[11] R. S. Gomaa, E. A. Hebeshy, M. M. El Genidy and B. S. El-Desouky, *alpha-Power of the Power Ailamujia Distribution: Properties and Applications.*, (2023).

[12] B. P. Singh and U. Dhar Das, *On an Induced Distribution and its Statistical Properties*, REVSTAT-Statistical Journal, **21**(3), 303–319, (2023).

[13] M. Nassar, S. Dey and D. Kumar, *Logarithm transformed Lomax distribution with applications*. Calcutta Statistical Association Bulletin, **70**(2), 122-135, (2018b).

[14] I. Elbatal, Z. Ahmad, M. Elgarhy, and A. M. Almarashi, *A new alpha power transformed family of distributions: properties and applications to the Weibull model,* Journal of Nonlinear Science and Applications,**12**(1), 1-20, (2018).

[15] N. Kyurkchiev, *A new transmuted cumulative distribution function based on the Verhulst logistic function with application in population dynamics*, Biomath Communications, **4**(1) (2017).

[16] P. Kavya and M. Manoharan, *Some parsimonious models for lifetimes and applications,* Journal of Statistical Computation and Simulation, **91**(18), 3693–3708, (2021).

[17] R. M. Mandouh, E. M. Sewilam, M. M. Abdel-Zaher and M. R. Mahmoud, *A New Family of Distributions in the Class of the Alpha Power Transformation with Applications to Income*, Asian Journal of Probability and Statistics, **19**(1), 41-55, (2022).

[18] L. P. Sapkota, and V. Kumar, *Odd Lomax Generalized Exponential Distribution: Application to Engineering and COVID-19 data*, Pakistan Journal of Statistics and Operation Research, **18**(4), 883-900, (2022).
[19] J. I. McCool, *Using the Weibull distribution: reliability, modeling, and inference* (Vol. 950), John Wiley & Sons, (2012).
[20] L. P. Sapkota, and V. Kumar, *Half Cauchy Generalized Rayleigh Distribution: Theory and Application*, South East Asian J. of Mathematics and Mathematical Sciences, **19**(1), 335-360, (2023).
[21] A. Henningsen and O. Toomet, *maxLik: A package for maximum likelihood estimation in R,* Computational Statistics, **26**, 443-458, (2011).
[22] F. Chipepa, T. Moakofi and B. Oluyede, *The Marshall-Olkin-Odd Power Generalized Weibull-G Family of Distributions with Applications of COVID-19 Data,* Journal of Probability and Statistical Science, **20**(1), 1-20, (2022).
[23] S. Konishi and G. Kitagawa, *Information criteria and statistical modeling*. Springer, New York, 2008.
[24] M. V. Aarset, *How to identify bathtub hazard rate*. IEEE Transactions Reliability, **36**, 106-108, (1987).
[25] R. McElreath, *Statistical rethinking: A Bayesian course with examples in R and Stan* (2nd ed.). CRC Press, (2020).
[26] R Core Team *R: A language and environment for statistical computing*. R Foundation for Statistical Computing, Vienna, Austria, (2023). URL https://www.R-project.org/.