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A Study of the Sequential Test for the Parameter of the Odd Generalized Exponential Gompertz Distribution

Surinder Kumar[1] *and Mradula Sharma*[*]

Department of Statistics, School of Physical & Decision Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow - 226 025, India

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Abstract: In this paper, we propose Sequential Probability Ratio Test $(SPRT)$ which is developed for testing the simple hypothesis regarding the shape parameter of Odd Generalized Exponential Gompertz distribution. The expressions for the Operating Characteristic (OC) and Average Sample Number (ASN) functions are derived. In order to obtain the numerical values of OC and ASN functions different methods are suggested and results are presented through graphs and tables.

Keywords: Odd Generalized Exponential Gompertz distribution, SPRT, OC and ASN functions, Approximation method and Newton-Raphson method.

1 Introduction

A.Wald (1947)[7], has given a significant contribution in the area of a sequential test of the statistical hypothesis. He developed sequential probability ratio test (SPRT) for testing the simple hypothesis against simple alternative hypothesis and obtained the expression of Operating Characteristic (OC) and Average Sample Number (ASN) functions of the SPRT. Various authors have studied the robustness of the SPRT when the distribution under consideration has undergone a change. Barlow and Proschan (1967)[1], Harter and Moore (1976) [4], Montagne and Singpurwalla (1985)[5], and others have studied this problem with different probability models. Chaturvedi, Kumar and Surinder (2000)[3] developed sequential test of simple and composite hypothesis regarding the parameters of a class of distributions representing various life testing models. Surinder and Vaish (2015)[6] study the robustness of the sequential test for the scale parameter of nakagami distribution. Also, Surinder and Mukesh kumar worked on sequential tesing (2013)[7] and (2016)[8]
In this paper, we have developed the sequential test for testing the simple hypothesis against simple alternatives hypothesis for the shape parameter of the Odd Generalized Exponential Gompertz distribution. The Odd Generalized Exponential-Gompertz (OGE-G) distribution is exponentiated exponential distribution and Gompertz distribution using new family of univariate distributions proposed by [9].

2 Set-up of the problem

Let the random variable X follows the Odd Generalized Exponential Gompertz distribution with probability density function (pdf) given by

$$f(x:\Theta) = \alpha\beta\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]} \left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]}\right\}^{\beta-1} \tag{1}$$

* Corresponding author e-mail: mradulakrish@gmail.com

where $x > 0,\ \alpha > 0,\ \lambda > 0,\ c > 0,\ \beta > 0.$

Given a sequence of observations $X_1, X_2, X_3, \ldots$ from (1), suppose one wishes to test the simple null hypothesis $H_0 : \beta = \beta_0$ against the simple alternative hypothesis $H_1 : \beta = \beta_1 (> \beta_0)$. The expression for the OC and ASN functions are derived and their behavior is studied through obtaining the numerical values of the expressions.

3 SPRT for testing the hypothesis regarding "β"

The SPRT for testing H_0 is define as follows:

$$Z = ln \frac{f(x;\lambda,\alpha,\beta_1)}{f(x;\lambda,\alpha,\beta_0)}$$

$$Z = ln \left\{ \frac{\alpha\beta_1\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]} \left\{1 - e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]}\right\}^{\beta_1-1}}{\alpha\beta_0\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]} \left\{1 - e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]}\right\}^{\beta_0-1}} \right\}$$

On solving, we get

$$Z = ln \left\{ \left(\frac{\beta_1}{\beta_0}\right) \left(1 - e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]}\right)^{\beta_1-\beta_0} \right\}$$

$$Z = ln\left(\frac{\beta_1}{\beta_0}\right) + (\beta_1 - \beta_0)\, ln \left\{1 - e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}\right]} - 1\right\} \tag{2}$$

Taking exponential on both side, we get

$$e^z = \left\{ \left(\frac{\beta_1}{\beta_0}\right) \left(1 - e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right]}\right)^{\beta_1-\beta_0} \right\} \tag{3}$$

Now, we choose two numbers A and B such that $0 < B < 1 < A$. At the n^{th} stage of sampling, accept null hypothesis H_0 if $\sum_{i=1}^{n} z_i \leq lnB$, reject null hypothesis H_0 if $\sum_{i=1}^{n} z_i \geq lnA$, otherwise continue sampling by taking the $(n+1)^{th}$ observation. If $\alpha \in (0,1)$ and $\beta \in (0,1)$ are Type I and Type II errors respectively, then according to Wald (1947)[10], A and B are approximately given by

$$A \approx \frac{1-\beta}{\alpha}\ and\ B \approx \frac{\beta}{1-\alpha} \tag{4}$$

The OC function $L(\beta)$ is given by

$$L(\beta) = \frac{A^h - 1}{A^h - B^h} \tag{5}$$

where 'h' is the non-zero solution of

$$E[e^{hz}] = 1 \tag{6}$$

$$\int_0^\infty \left[\frac{f(x;\lambda,\alpha,\beta_1)}{f(x;\lambda,\alpha,\beta_0)}\right]^h f(x;\lambda,\alpha,\beta)\, dx = 1 \tag{7}$$

On simplifying (7), we get

$$\beta = \frac{h(\beta_1 - \beta_0)}{\frac{\beta_1}{\beta_0} - 1} \tag{8}$$

The ASN function is approximately given by

$$E\left(N\mid\beta\right)=\frac{L\left(\beta\right)lnB+\left\{1-L\left(\beta\right)\right\}lnA}{E\left(Z\right)} \tag{9}$$

provided $E(Z)\neq 0$

$$E\left(Z\right)=E\left[ln\left\{\frac{f\left(x,\lambda,\alpha,\beta_1\right)}{f\left(x,\lambda,\alpha,\beta_0\right)}\right\}\right]$$

$$=E\left[\left(\frac{\beta_1}{\beta_0}\right)+\left(\beta_1-\beta_0\right)ln\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}}\left(e^{cx}-1\right)\right]}-1\right\}\right]$$

Thus,

$$E\left(Z\right)=\left[ln\left(\frac{\beta_1}{\beta_0}\right)-\left(\frac{\beta_1-\beta_0}{\beta}\right)\right) \tag{10}$$

From equation (10) the ASN function under H_0 and H_1 is respectively, given by

$$E_0\left(N\right)=\frac{\left(1-\alpha\right)lnB+\alpha lnA}{\left[ln\left(\frac{\beta_1}{\beta_0}\right)-\left(\frac{\beta_1-\beta_0}{\beta}\right)\right]}$$

and

$$E_1\left(N\right)=\frac{\beta lnB+\left(1-\beta\right)lnA}{\left[ln\left(\frac{\beta_1}{\beta_0}\right)-\left(\frac{\beta_1-\beta_0}{\beta}\right)\right]}$$

Now, in order to obtain the numerical values of OC and ASN function, some methods are provide below.

4 Methods

4.1 (a) By Approximation Method

Taking the logarithms of equation (8) and using the expression $log\left(1+x\right)$, we get

$$log\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$$

On retaining terms upto third degree in 'h' and ignoring higher order in equation (11), we get

$$hlog\left(\frac{\beta_1}{\beta_0}\right)=\left(\frac{\beta_1-\beta_0}{\beta}\right)h-\frac{1}{2}\left(\frac{\beta_1-\beta_0}{\beta}\right)^2h^2+\frac{1}{3}\left(\frac{\beta_1-\beta_0}{\beta}\right)^3h^3 \tag{11}$$

$$\frac{1}{3}\left(\frac{\beta_1-\beta_0}{\beta}\right)^3h^2-\frac{1}{2}\left(\frac{\beta_1-\beta_0}{\beta}\right)^2h+\left\{\left(\frac{\beta_1-\beta_0}{\beta}\right)-ln\left(\frac{\beta_1}{\beta_0}\right)\right\}$$

Above equation is a quadratic in 'h' and can be written as

$$Ah^2+Bh+C=0$$

where

$$A=\frac{1}{3}\left(\frac{\beta_1-\beta_0}{\beta}\right)^3,\ B=\frac{1}{2}\left(\frac{\beta_1-\beta_0}{\beta}\right)^2$$

and

$$C=\left\{\left(\frac{\beta_1-\beta_0}{\beta}\right)-ln\left(\frac{\beta_1}{\beta_0}\right)\right\}$$

The solution of above quadratic equation in 'h' is

$$h=\frac{-B\pm\sqrt{B^2-4AC}}{2A} \tag{12}$$

Remarks

For testing the hypothesis $H_0 : \beta_0 = 20$ versus $H_1 : \beta_1 = 25$, for fixed $\alpha = 0.5$ and verifying values of β and using the real roots of 'h'' obtained from equation (12). The OC and ASN functions evaluated from equation (5) and equation (9). It is noted that the values of 'h'' obtained through approximations are given satisfactorily results (Table. 1) and the OC and ASN functions curves are plotted in Fig. 1 and Fig. 2.

Table. 1: Values of OC and ASN functions obtained by approximation method.

$(H_0 : \beta_0 = 20 \text{ against } H_1 : \beta_1 = 25, \alpha = 0.05)$

β	$L(\beta)$	ASN	β	$L(\beta)$	ASN
19.0	0.991361	72.3135	22.6	0.443597	175.3927
19.2	0.987318	76.9926	22.8	0.386905	173.1983
19.4	0.982016	82.0659	23.0	0.333337	170.6221
19.6	0.975152	87.5546	23.2	0.283912	166.8586
19.8	0.966368	93.4725	23.4	0.239301	162.1437
20.0*	0.955244	99.8224	23.6	0.199819	156.7254
20.2	0.941309	106.5909	23.8	0.165483	150.8422
20.4	0.924045	113.7421	24.0	0.136972	144.7059
20.6	0.902911	121.2110	24.2	0.111204	138.4933
20.8	0.877375	128.8965	24.4	0.090408	132.3438
21.0	0.846965	136.6559	24.6	0.073178	126.3606
21.2	0.811339	144.3026	24.8	0.059011	120.6156
21.4	0.770356	151.6087	25.0*	0.047437	115.1547
21.6	0.724155	158.3152	25.2	0.038032	110.0033
21.8	0.673222	164.1502	25.4	0.030422	105.1713
22.0	0.618398	168.8548	25.6	0.024288	100.6577
22.2	0.560876	173.2126	25.8	0.019357	96.4537
22.4	0.502091	177.0767	26.0	0.018012	90.0101

Fig. 1: OC function plot for the Odd Generalized Exponential Gompertz Distribution by approximation method

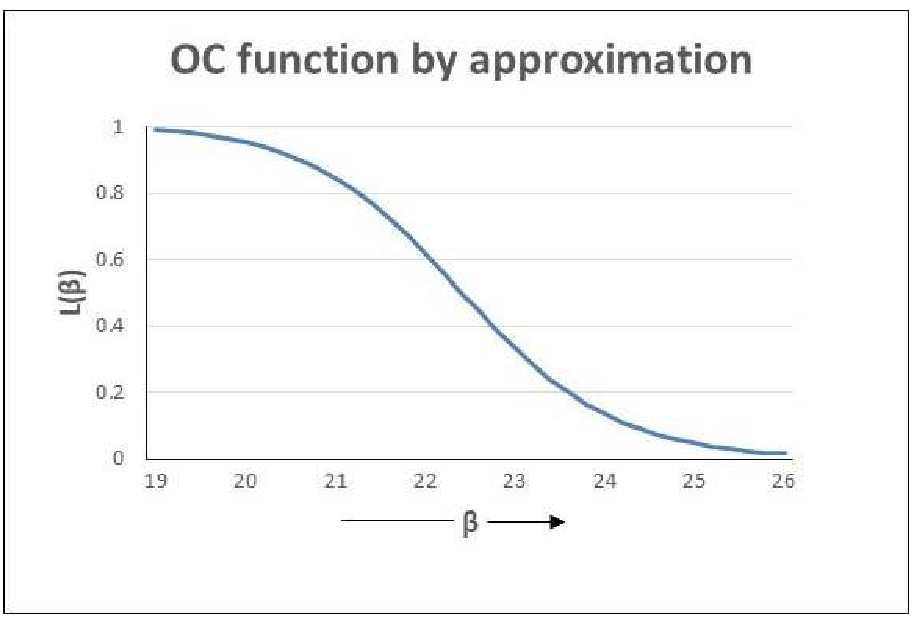
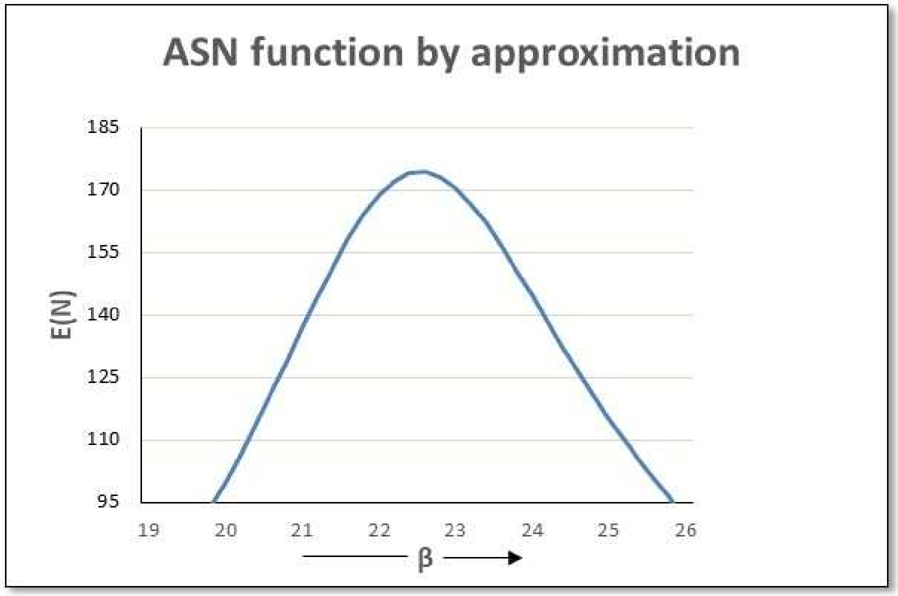


Fig. 2: ASN function plot for the Odd Generalized Exponential Gompertz Distribution by approximation method



4.2 (b) By Newton-Raphson Method

Using the values of 'h' given by (12) as initial value for solving equation (8) through Newton-Raphson method, we written as

$$F = ln\left[1-\frac{h(\beta_0-\beta_1)}{\beta}\right]-hln\left(\frac{\beta_1}{\beta_0}\right) \tag{13}$$

$$FD = \frac{\frac{1}{\beta}(\beta_1-\beta_0)}{\left\{1-\frac{h(\beta_0-\beta_1)}{\beta}\right\}}-ln\left(\frac{\beta_1}{\beta_0}\right) \tag{14}$$

where FD is 1st derivative of $F(\cdot)$.

We use the Newton-Raphson formula

$$h_{n+1} = h_n - \frac{F(h)}{FD(h)}$$

The ASN function is approximately given by

$$E(N\mid\beta) = \frac{L(\beta)\,lnB+\{1-L(\beta)\}lnA}{E(Z)}$$

where

$$E(Z) = ln\left(\frac{\beta_1}{\beta_0}\right)-\left(\frac{\beta_1-\beta_0}{\beta}\right)$$

Remarks

For testing the hypothesis $H_0:\beta_0 = 20\,versus\,H_1:\beta_1 = 25$, fixed values for $\alpha = .05$ and for different values of β. It is noted that the values of 'h' obtained through Newton-Raphson method are given satisfactorily results (Table. 2) and the OC and ASN functions curves are plotted in Fig. 3 and Fig. 4.

Table. 2: Values of OC and ASN functions obtained by Newton-Raphson method.

$(H_0 : \beta_0 = 20 \text{ against } H_1 : \beta_1 = 25, \alpha = 0.05)$

β	$L(\beta)$	ASN	β	$L(\beta)$	ASN
19.0	0.9912	72.2940	22.6	0.4436	175.3928
19.2	0.9872	76.9719	22.8	0.3869	172.1985
19.4	0.9819	82.0447	23.0	0.3333	170.6228
19.6	0.9750	87.0447	23.2	0.2839	166.8601
19.8	0.9663	93.4517	23.4	0.2339	162.1461
$20.0^{\star}$	0.9552	99.8027	23.6	0.1998	156.7289
20.2	0.9412	106.5728	23.8	0.1655	150.8468
20.4	0.9240	113.7261	24.0	0.1361	144.7115
20.6	0.9029	121.1974	24.2	0.1112	138.4998
20.8	0.8773	128.8855	24.4	0.0904	132.3509
21.0	0.8469	136.6476	24.6	0.0732	126.3681
21.2	0.8113	144.2968	24.8	0.0590	120.6234
21.4	0.7703	151.6051	$25.0^{\star}$	0.0474	115.1625
21.6	0.7242	158.3132	25.2	0.0380	110.0109
21.8	0.6732	164.1493	25.4	0.0304	105.1787
22.0	0.6184	168.8546	25.6	0.0243	100.6647
22.2	0.5609	172.2126	25.8	0.0193	96.4602
22.4	0.5021	175.0768	26.0	0.0187	92.9810

Fig. 3: OC function plot for the Odd Generalized Exponential Gompertz Distribution by Newton-Raphson method

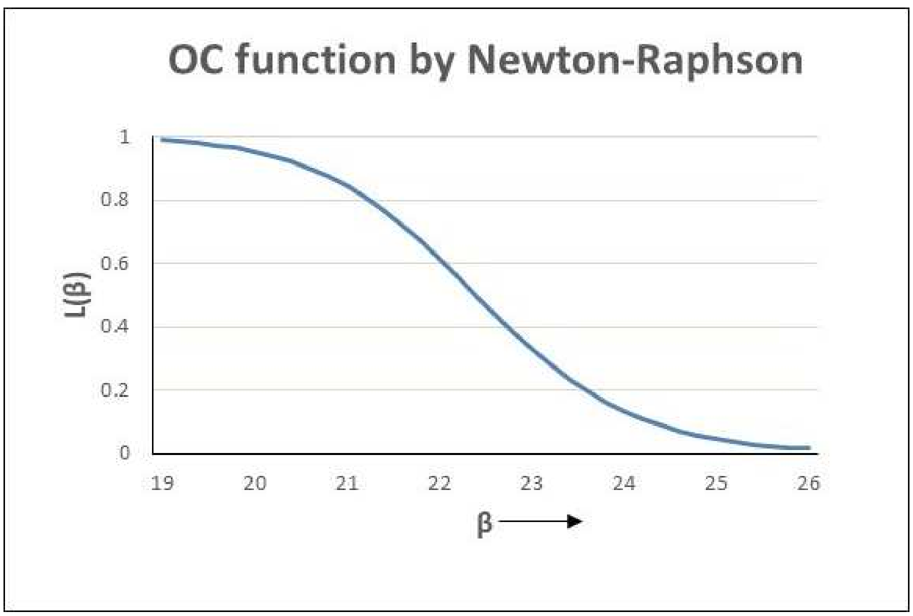
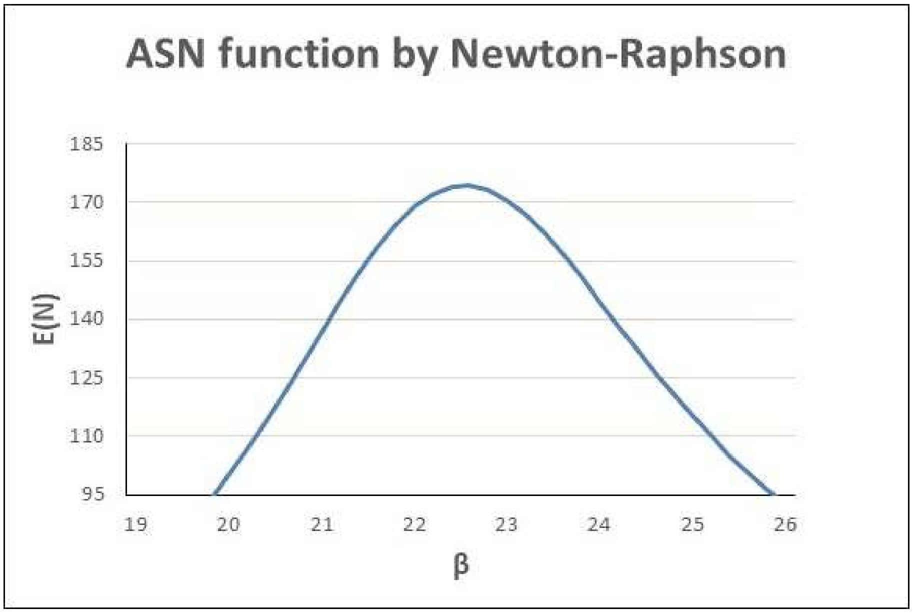


Fig. 4: ASN function plot for the Odd Generalized Exponential Gompertz Distribution by Newton-Raphson method



5 Conclusion

The values of OC and ASN functions obtained through approximation method and Newton-Raphson method are plotted in Fig. 1, Fig. 2, Fig. 3 and Fig. 4, respectively. The values of 'h' obtained through approximation method and through Newton-Raphson method are extremely close to each others. The values of $L(\beta_0)$ and $L(\beta_1)$ are quite close to their theoretical values i.e. 0.95 and 0.05, respectively.

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Surinder Kumar, Professor, Department of Statistics, BBAU (A Central University) Lucknow, India. He is having 22 years research experience in various research fields of Statistics such as Sequential Analysis, Reliability Theory, Business Statistics and Bayesian Inference. Prof. Kumar has published more than 40 research papers in various journals of national and international repute.

Mradula Sharma is research scholar of Department of Statistics, BBAU (A Central University) Lucknow, India. She has research experience of 2 years, her research areas are Sequential Analysis, Optimization Theory.