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On the Geometry of Equiform Normal Curves in the Galilean Space G^4

M. Fakharany[1,2], *A. El-Abed*[2,3], *M. Elzawy*[2,3] *and S. Mosa*[4,5,*]

[1]Mathematics and Statistics Department, College of Science, Taibah University, Yanbu, Saudi Arabia
[2]Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt
[3]Mathematics Department, College of Science, Taibah University, Medina, Saudi Arabia
[4]Department of Mathematics, College of Science, University of Bisha, Bisha, Saudi Arabia
[5]Mathematics Department, Faculty of Science, Damanhour University, Damanhour, Egypt

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Abstract: In our article, we establish the definition of the Equiform Normal curves in Galilean space G^4. To obtain the position vector of an Equiform Normal curve in G^4, we have to solve an integro-differential equation in μ_2, where μ_2 is the position function of a space curve $\gamma(\sigma)$ in the direction of third vector V_3 of the Galilean space. Special cases of Equiform Normal curvatures are discussed. Finally, we prove that there is no equiform normal curve that is congruent to an Equiform Normal curve in G^4.

Keywords: Normal Curves, Curvature, Galilean Space, Equiform Geometry, integro-differential equation.

1 Introduction

We consider a Galilean space as the limiting case of a Pseudo-Euclidean space in which the isotropic cone degenerates into a plane [1]. There are many concepts and principles such as velocity, momentum, kinetic energy, laws of motion, and conversation laws in classical physics defined in Galilean space [1]. Besides, Galilean space has an influential role in non-relativistic physics. More details about Galilean space can be found in [2,3,4,5,6].

From the differential geometric point of view, the study of normal curves in Galilean space has its interest. Many researchers have derived many valuable and interesting results on normal curves in Galilean space, see [7,8].

Cayley-Klein space has an equiform geometry derived when the angles between planes and lines are preserved by the space's similarity group [9].

In [10], equiform Darboux helices in Galilean space G_3, were defined and their explicit parameter equations were obtained. In [11] 4-dimensional Galilean space G^4, an equiform differential geometry of curves was created. They calculate the angle between the equiform Frenet vectors and their derivatives. Also, in [12], the authors introduced the Pseudo-Galilean space G_3^1 and study the equiform differential geometry of curves in this space.

In section 3, we present a summary of Frenet equations in equiform geometry in G^4. In section 4, firstly, in Galilean 4-space G^4, the equiform normal curve, and differentiate the equation of the position vector of this curve are defined. Next, the nonhomogeneous linear system of differential equations with variable coefficients is obtained, and then we solve this system. Secondly, with constant curvatures K_1, K_2 and K_3. We got the position vector of an equiform normal curve in G^4. Finally, the characterizations of equiform normal curves in Galilean space G^4 is introduced.

2 Basic Concepts

In this section, we introduce some basic definitions that are considered in this paper. For more basic concepts, see [4,5, 11,13]. Given $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4)$ be two vectors in G^4. Then the dot(scalar) product g between the two vectors $\mathbf{a}$ and $\mathbf{b}$ in G^4 is defined by the following relation

$$g(\mathbf{a},\mathbf{b}) = \begin{cases} a_1 b_1, \text{ if } a_1 \neq 0 \text{ or } b_1 \neq 0, \\ a_2 b_2 + a_3 b_3 + a_4 b_4, \text{ if } a_1 = 0 \text{ and } b_1 = 0. \end{cases}$$

* Corresponding author e-mail: smosa@ub.edu.sa

Also, the norm of the vector $\mathbf{a}$ with coordinates (a_1,a_2,a_3,a_4) is given by $|\mathbf{a}| = \sqrt{g(\mathbf{a},\mathbf{a})}$.

For a given vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ with the coordinates (x_1,x_2,x_3,x_4), (y_1,y_2,y_3,y_4), (z_1,z_2,z_3,z_4), the cross product is defined by the relation

$$\mathbf{x}\times\mathbf{y}\times\mathbf{z} = \begin{cases} \begin{vmatrix} 0 & \mathbf{e_2} & \mathbf{e_3} & \mathbf{e_4} \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \text{ if } x_1 \neq 0 \text{ or } y_1 \neq 0 \text{ or } z_1 \neq 0, \\ \\ \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} & \mathbf{e_4} \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \text{ if } x_1 = y_1 = z_1 = 0, \end{cases}$$

where $\mathbf{e_i}$ are the standard basis vectors. Suppose that $\beta : I \subset \mathbb{R} \to G^4$ is a curve of C^∞ defined in the Galilean space G^4 such that

$$\beta(t) = (u(t), v(t), w(t), r(t)).$$

when the curve β is parametrized by the Galilean arc-length s, it will be written in the form

$$\beta(s) = (s, v(s), w(s), r(s)).$$

Now, Frenet Serret vector fields of $\beta(s)$ is obtained by the next formulas

$$\begin{aligned} T(s) &= \beta'(s) = (1, v'(s), w'(s), r'(s)), \\ N(s) &= \tfrac{1}{k_1(s)} T'(s) = \tfrac{1}{k_1(s)}(0, v''(s), w''(s), r''(s)), \\ B_1(s) &= \tfrac{1}{k_2(s)}\left(0, \tfrac{d}{ds}\left(\tfrac{v''}{k_1(s)}\right), \tfrac{d}{ds}\left(\tfrac{w''}{k_1(s)}\right), \tfrac{d}{ds}\left(\tfrac{r''}{k_1(s)}\right)\right), \\ B_2(s) &= \varsigma T(s) \times N(s) \times B_1(s), \end{aligned}$$

where the coefficient $\varsigma = \pm 1$, is chosen by the criterion $\det(T(s), N(s), B_1(s), B_2(s)) = 1$. Here $T(s)$, $N(s)$, $B_1(s)$ and $B_2(s)$ are the Tangent, the Normal, the First binormal, and the Second binormal vectors of $\beta(s)$.
Moreover, $k_1(s)$ and $k_2(s)$ are the first, and the second curvatures, that defined by the following formulas

$$k_1(s) = \left|\beta''(s)\right|_{G^4} = \sqrt{(v''(s))^2 + (w''(s))^2 + (r''(s))^2},$$

$$k_2(s) = \left|N'(s)\right|_{G^4} = \sqrt{g(N'(s), N'(s))}.$$

Also, the third curvature function of the curve $\beta(s)$ is given by $k_3(s) = g(B_1'(s), B_2(s))$ and the Frenet equations in G^4 are given by

$$\begin{aligned} T'(s) &= k_1(s) N(s), \\ N'(s) &= k_2(s) B_1(s), \\ B_1'(s) &= -k_2(s) N(s) + k_3(s) B_2(s), \\ B_2'(s) &= -k_3(s) B_1(s). \end{aligned}$$

3 Frenet Equations in Equiform Geometry in G^4

Let $\gamma : I \subset \mathbb{R} \to G^4$ be a parametrized curve and s is the arc length of γ, the equiform parameter (σ) of the parametrized curve $\gamma(s)$ will be defined by

$$\sigma = \int \frac{ds}{\rho},$$

$\rho = \frac{1}{\kappa_1}$ is called the radius of curvature for the curve $\gamma(s)$. Suppose h is a homothety with center at the origin and the coefficient λ. If we take $\widetilde{\gamma} = h(\gamma)$, then $\widetilde{s} = \lambda s$ and $\widetilde{\rho} = \lambda \rho$, where $\widetilde{s}$ and $\widetilde{\rho}$ are the arc length parameter and the radius of curvature of $\widetilde{\gamma}$ respectively. Hence σ is an equiform invariant parameter of γ.
Let $\{V_1, V_2, V_3, V_4\}$ be an equiform invariant tetrahedron of the curve γ. The first derivatives of such vectors with respect to the euiform parameter σ are given by:

$$\begin{aligned} \dot{V}_1 &= K_1 V_1 + V_2, \\ \dot{V}_2 &= K_1 V_2 + K_2 V_3, \\ \dot{V}_3 &= -K_2 V_2 + K_1 V_3 + K_3 V_4, \\ \dot{V}_4 &= -K_3 V_3 + K_1 V_4, \end{aligned} \tag{1}$$

where $K_1 = \dot{\rho}$, $K_2 = \frac{\kappa_2}{\kappa_1}$, $K_3 = \frac{\kappa_3}{\kappa_1}$ are the Equiform curvatures of the curve γ.
These formulas can be written in matrix form as follows:

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix} = \begin{bmatrix} K_1 & 1 & 0 & 0 \\ 0 & K_1 & K_2 & 0 \\ 0 & -K_2 & K_1 & K_3 \\ 0 & 0 & -K_3 & K_1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}. \tag{2}$$

The Equiform curvatures are defined as:

$$K_1 = \frac{1}{\rho^2} < \dot{V}_i, V_i >, \quad i = 1,2,3,4,$$

$$K_2 = \frac{1}{\rho^2} < \dot{V}_2, V_3 > = -\frac{1}{\rho^2} < V_2, \dot{V}_3 >,$$

$$K_3 = \frac{1}{\rho^2} < \dot{V}_3, V_4 > = -\frac{1}{\rho^2} < V_3, \dot{V}_4 > .$$

or

$$K_j = \begin{cases} \frac{1}{\rho^2} g(\dot{V}_i, V_i), \; i = 1,2,3,4, \text{ for } j = 1, \\ \frac{1}{\rho^2} g(\dot{V}_j, V_{j+1}) = -\frac{1}{\rho^2} g(V_j, \dot{V}_{j+1}), \text{ for } j = 2,3. \end{cases}$$

4 Equiform Normal Curves in G^4

Definition 1. *Suppose that $\gamma : I \subset \mathbb{R} \to G^4$ is a parameterized curve in G^4. If the orthogonal components of $V_1(\sigma)$ contains a fixed point for all $\sigma \in I$, the curve γ is called an Equiform Normal curve.*

For an equiform normal curve $\gamma(\sigma)$ in G^4, the position vector is defined by

$$\gamma(\sigma) = \mu_1(\sigma)V_2(\sigma) + \mu_2(\sigma)V_3(\sigma) + \mu_3(\sigma)V_4(\sigma), \quad (3)$$

where $\mu_1(\sigma)$, $\mu_2(\sigma)$, and $\mu_3(\sigma)$ are differentiable functions of σ. Differentiating equation (3), we obtain

$$\begin{aligned}\frac{d\gamma(\sigma)}{d\sigma} &= \mu_1(K_1V_2 + K_2V_3) + \frac{d\mu_1}{d\sigma}V_2 + \mu_2(-K_2V_2 + K_1V_3 \\ &\quad + K_3V_4) + \frac{d\mu_2}{d\sigma}V_3 + \mu_3(-K_3V_3 + K_1V_4) + \frac{d\mu_3}{d\sigma}V_4,\end{aligned}$$

$$\begin{aligned}\frac{d\gamma(\sigma)}{d\sigma} &= (K_1\mu_1 - K_2\mu_2 + \frac{d\mu_1}{d\sigma})V_2(\sigma) \\ &\quad + (K_2\mu_1 + K_1\mu_2 - K_3\mu_3 + \frac{d\mu_2}{d\sigma})V_3(\sigma) \\ &\quad + (K_3\mu_2 + K_1\mu_3 + \frac{d\mu_3}{d\sigma})V_4(\sigma) \\ &= \alpha_1(\sigma)V_2(\sigma) + \alpha_2(\sigma)V_3(\sigma) + \alpha_3(\sigma)V_4(\sigma),\end{aligned}$$

where $\alpha_1(\sigma)$, $\alpha_2(\sigma)$ and $\alpha_3(\sigma)$ are differentiable functions of σ.

Theorem 1. *Let $\gamma(\sigma)$ be an Equiform Normal curve in G^4 with Equiform curvatures K_1, K_2 and K_3. The position vector of $\gamma(\sigma)$ is defined if K_1, K_2 and K_3 satisfy the the following integro-differential equation in μ_2*

$$\begin{aligned}&\frac{d}{d\sigma}\left[\mu_2 \exp\left(\int K_1(\sigma)d\sigma\right)\right] \\ &+ \sum_{j=2}^{3}\left(K_j(\sigma)\int_{\sigma_0}^{\sigma}\left[K_j(\theta)\mu_2(\theta)\exp\left(\int K_1(\theta)d\theta\right)\right]d\theta\right) \\ &= C_0K_3(\sigma) - C_1K_2(\sigma).\end{aligned}$$

Proof. The system of differential equations that constitutes the normal curvatures is given by

$$\frac{d\mu_1}{d\sigma} + K_1(\sigma)\mu_1 = K_2(\sigma)\mu_2 + \alpha_1(\sigma), \quad (4)$$

$$\frac{d\mu_2}{d\sigma} + K_1(\sigma)\mu_2 = -K_2(\sigma)\mu_1 + K_3(\sigma)\mu_3 + \alpha_2(\sigma), \quad (5)$$

$$\frac{d\mu_3}{d\sigma} + K_1(\sigma)\mu_3 = -K_3(\sigma)\mu_2 + \alpha_3(\sigma). \quad (6)$$

The system (4)-(6) is a nonhomogeneous linear system with variable coefficients. First, we solve the corresponding homogeneous system, then obtain a particular solution for the nonhomogeneous term. The associated homogeneous system is given by

$$\frac{d\mu_1}{d\sigma} + K_1(\sigma)\mu_1 = K_2(\sigma)\mu_2, \quad (7)$$

$$\frac{d\mu_2}{d\sigma} + K_1(\sigma)\mu_2 = -K_2(\sigma)\mu_1 + K_3(\sigma)\mu_3, \quad (8)$$

$$\frac{d\mu_3}{d\sigma} + K_1(\sigma)\mu_3 = -K_3(\sigma)\mu_2. \quad (9)$$

Our objective is to convert this system into one differential equation in one unknown function. First, we multiply the equations (7), (8) and (9) by the integrating factor $\exp(\int K_1(\sigma)d\sigma)$, then we have

$$\begin{aligned}\frac{d}{d\sigma}\left[\mu_1 \exp\left(\int K_1(\sigma)d\sigma\right)\right] &= K_2(\sigma)\mu_2 \exp\left(\int K_1(\sigma)d\sigma\right), \\ \frac{d}{d\sigma}\left[\mu_2 \exp\left(\int K_1(\sigma)d\sigma\right)\right] &= (-K_2(\sigma)\mu_1 + K_3(\sigma)\mu_3) \\ &\quad \exp\left(\int K_1(\sigma)d\sigma\right), \\ \frac{d}{d\sigma}\left[\mu_3 \exp\left(\int K_1(\sigma)d\sigma\right)\right] &= -K_3(\sigma)\mu_2 \exp\left(\int K_1(\sigma)d\sigma\right).\end{aligned} \quad (10)$$

By adding the first and third equations of equation (10), one gets

$$\begin{aligned}&\frac{d}{d\sigma}\left[(\mu_1+\mu_3)\exp\left(\int K_1(\sigma)d\sigma\right)\right] = \Big(K_2(\sigma) - K_3(\sigma)\Big)\mu_2 \exp\left(\int K_1(\sigma)d\sigma\right) \Rightarrow \\ &\mu_3(\sigma) = \exp\left(-\int K_1(\sigma)d\sigma\right)\left(C_0 + \int_{\sigma_0}^{\sigma}\left[\Big(K_2(\theta) - K_3(\theta)\Big)\mu_2(\theta)\right.\right. \\ &\qquad \left.\left.\exp\left(\int K_1(\theta)d\theta\right)\right]d\theta\right) - \mu_1(\sigma),\end{aligned} \quad (11)$$

where, C_0 is an integration constant. Form equation (11) and the first equation of (10) into the second equation of (10), then we have the following integro-differential equation in μ_2

$$\begin{aligned}&\frac{d}{d\sigma}\left[\mu_2 \exp\left(\int K_1(\sigma)d\sigma\right)\right] \\ &+ \sum_{j=2}^{3}\left(K_j(\sigma)\int_{\sigma_0}^{\sigma}\left[K_j(\theta)\mu_2(\theta)\exp\left(\int K_1(\theta)d\theta\right)\right]d\theta\right) \\ &= C_0K_3(\sigma) - C_1K_2(\sigma).\end{aligned} \quad (12)$$

Next, we discuss some cases of equation (12).

First case

If the curvatures K_2 and K_3 are considered to be functions in σ, while K_1 is constant and $\sigma_0 = 0$. We study the case when $K_2(\sigma) = e^{a\sigma}$ and $K_3(\sigma) = e^{b\sigma}$, where a, $b \in \mathbb{R}$ and $K_1 = k_1 \in \mathbb{R}$. Let $y(\sigma) = \mu_2(\sigma)e^{k_1\sigma}$, then equation (12) becomes

$$\frac{dy}{d\theta} + \int_0^{\sigma}\left(e^{a(\sigma+\theta)} + e^{b(\sigma+\theta)}\right)y(\theta)d\theta = C_0e^{b\sigma} - C_1e^{a\sigma}. \quad (13)$$

By applying the operator $\Theta(D_\sigma) = (D_\sigma - a)(D_\sigma - b)$ (where $D_\sigma = \frac{d}{d\sigma}$), on equation (13), one gets

$$\begin{aligned}&\Big(D_\sigma^3 - (a+b)D_\sigma^2 + (ab + e^{2a\sigma} + e^{2b\sigma})D_\sigma \\ &\quad -((2b-a)e^{2b\sigma} + (2a-b)e^{2a\sigma})\Big)y = 0,\end{aligned} \quad (14)$$

when $a = b$, the solution of equation (14) is given by

$$\begin{aligned}y(\sigma) &= \hat{C}_1 \cos\left(\frac{\sqrt{2}\,\mathrm{e}^{a\sigma}}{a}\right) + \hat{C}_2 \sin\left(\frac{\sqrt{2}\mathrm{e}^{a\sigma}}{a}\right) \\ &\quad + \hat{C}_3\, G_{1,3}^{3,1}\left(\frac{\mathrm{e}^{2a\sigma}}{2a^2}\,\middle|\,\begin{matrix}0\\0,0,-1/2\end{matrix}\right)\mathrm{e}^{a\sigma},\end{aligned}$$

where $G_{1,3}^{3,1}\left(\frac{e^{2a\sigma}}{2a^2}\middle|\begin{smallmatrix}0\\0,0,-1/2\end{smallmatrix}\right)$ is the Meijer G-function and defined by [14]

$$G_{p,q}^{m,n}\left(z\middle|\begin{smallmatrix}a_1,a_2,\ldots,a_n,a_{n+1},\ldots,a_p\\b_1,b_2,\ldots,b_m,b_{m+1},\ldots,a_q\end{smallmatrix}\right)$$
$$=\frac{1}{2\pi i}\oint_L\frac{\left(\prod_{j=1}^{n}\Gamma(1-a_j+s)\right)\left(\prod_{j=1}^{m}\Gamma(b_j-s)\right)}{\left(\prod_{j=m+1}^{q}(\Gamma(1-b_j-s))\right)\left(\prod_{j=n+1}^{p}\Gamma(a_j-s)\right)}z^s ds.$$

Second case

If K_2 and K_3 are constants, while $K_1(\sigma)$ is an arbitrary function in σ. By substituting into equation (12) and differentiating both sides, we have

$$\frac{d^2}{d\sigma^2}\left[\mu_2\exp\left(\int K_1(\sigma)d\sigma\right)\right]+(K_2^2+K_3^2)\left(\mu_2\exp\left(\int K_1(\sigma)d\sigma\right)\right)=0. \quad (15)$$

Hence, the solution of equation (15) is given by

$$\mu_2(\sigma)=\exp\left(-\int K_1(\sigma)d\sigma\right)\left(\tilde{C}_1\cos\sqrt{K_2^2+K_3^2}\sigma+\tilde{C}_2\sin\sqrt{K_2^2+K_3^2}\sigma\right). \quad (16)$$

Consequently, we can obtain the corresponding solutions of $\mu_1(\sigma)$ by integrating the first equation of system (10) and $\mu_3(\sigma)$ from equation (11). Hence we caculate the solution of the homogeneous system (7)-(9). Based on the variation of parameters, a particular solution of the non-homogeneous system (4)-(6) is obtained.

Third case

When all the curvatures K_1, K_2, and K_3 are constants, it is treated as in the second case considering K_1 is a constant. Then the solutions $\mu_1(\sigma)$, $\mu_2(\sigma)$, and $\mu_3(\sigma)$ of the homogeneous system (4)-(6) are given by

$$\begin{aligned}\mu_1(\sigma)&=\frac{K_2e^{-K_1\sigma}}{\sqrt{K_2^2+K_3^2}}\left(\tilde{C}_1\sin\sqrt{K_2^2+K_3^2}\sigma-\tilde{C}_2\cos\sqrt{K_2^2+K_3^2}\sigma+\tilde{C}_3\right)\\ \mu_2(\sigma)&=e^{-K_1\sigma}\left(\tilde{C}_1\cos\sqrt{K_2^2+K_3^2}\sigma+\tilde{C}_2\sin\sqrt{K_2^2+K_3^2}\sigma\right) \qquad (17)\\ \mu_3(\sigma)&=\frac{-K_3e^{-K_1\sigma}}{\sqrt{K_2^2+K_3^2}}\left(\tilde{C}_1\sin\sqrt{K_2^2+K_3^2}\sigma-\tilde{C}_2\cos\sqrt{K_2^2+K_3^2}\sigma-\frac{K_2^2}{K_3^2}\tilde{C}_3\right).\end{aligned}$$

Corollary 1. *Let $\gamma(\sigma)$ be an Equiform Normal curve in G^4 with non vanishing constant Equiform curvatures K_1, K_2, and K_3 for the homogeneous system (4)-(6). Then the following statements are satisfied.*

$1-<\gamma(\sigma),V_2(\sigma)>=$
$$\rho^2\frac{K_2e^{-K_1\sigma}}{\sqrt{K_2^2+K_3^2}}\left(\tilde{C}_1\sin\sqrt{K_2^2+K_3^2}\sigma-\tilde{C}_2\cos\sqrt{K_2^2+K_3^2}\sigma+\tilde{C}_3\right),$$
$2-<\gamma(\sigma),V_3(\sigma)>=$
$$\rho^2e^{-K_1\sigma}\left(\tilde{C}_1\cos\sqrt{K_2^2+K_3^2}\sigma+\tilde{C}_2\sin\sqrt{K_2^2+K_3^2}\sigma\right),$$
$3-<\gamma(\sigma),V_3(\sigma)>=$
$$\rho^2\frac{-K_3e^{-K_1\sigma}}{\sqrt{K_2^2+K_3^2}}\left(\tilde{C}_1\sin\sqrt{K_2^2+K_3^2}\sigma-\tilde{C}_2\cos\sqrt{K_2^2+K_3^2}\sigma-\frac{K_2^2}{K_3^2}\tilde{C}_3\right)$$

where $\tilde{C}_1,\tilde{C}_2$, and $\tilde{C}_3$ are constants.

Proof. Let $\gamma(\sigma)$ be an Equiform Normal curve in G^4 with non vanishing constant Equiform curvatures K_1,K_2, and K_3. Then the Equiform curve $\gamma(\sigma)$ can be written in the following form

$$\begin{aligned}\gamma(\sigma)=&\frac{K_2e^{-K_1\sigma}}{\sqrt{K_2^2+K_3^2}}\left(\tilde{C}_1\sin\sqrt{K_2^2+K_3^2}\sigma-\tilde{C}_2\cos\sqrt{K_2^2+K_3^2}\sigma+\tilde{C}_3\right)V_2(\sigma)\\ &+e^{-K_1\sigma}\left(\tilde{C}_1\cos\sqrt{K_2^2+K_3^2}\sigma+\tilde{C}_2\sin\sqrt{K_2^2+K_3^2}\sigma\right)V_3(\sigma)\\ &+\frac{-K_3e^{-K_1\sigma}}{\sqrt{K_2^2+K_3^2}}\left(\tilde{C}_1\sin\sqrt{K_2^2+K_3^2}\sigma-\tilde{C}_2\cos\sqrt{K_2^2+K_3^2}\sigma-\frac{K_2^2}{K_3^2}\tilde{C}_3\right)V_4(\sigma)\end{aligned}$$

Taking the dot product of the both sides with $V_2(\sigma),V_3(\sigma)$, and $V_4(\sigma)$, we will obtain the result.

Theorem 2. *Let $\gamma(\sigma)$ be an Equiform Normal curve in G^4 with non vanishing constant Equiform curvatures K_1,K_2, and K_3 for the homogeneous system (4)-(6). Then there aren't any curves that are congruent to $\gamma(\sigma)$.*

Proof. Let us define $\zeta(\sigma)$ as

$$\zeta(\sigma)=\gamma(\sigma)-\mu_1(\sigma)V_2(\sigma)-\mu_2(\sigma)V_3(\sigma)-\mu_3(\sigma)V_4(\sigma). \quad (18)$$

By differentiate equation (18) with respect to σ, we will obtain the following result

$\dot{\zeta}(\sigma)=V_1(\sigma)-\dot{\mu}_1(\sigma)V_2(\sigma)-\mu_1(\sigma)\dot{V}_2(\sigma)-\dot{\mu}_2(\sigma)V_3(\sigma)-\mu_2(\sigma)\dot{V}_3(\sigma)-\dot{\mu}_3(\sigma)V_4(\sigma)-\mu_3(\sigma)\dot{V}_4(\sigma).$

By using equation (1), we have

$\dot{\zeta}(\sigma)=V_1(\sigma)+[-\dot{\mu}_1(\sigma)-K_1\mu_1(\sigma)+K_2\mu_2(\sigma)]V_2(\sigma)+[-\dot{\mu}_2(\sigma)-K_2\mu_1(\sigma)-K_1\mu_2(\sigma)+K_3\mu_3(\sigma)]V_3(\sigma)+[-\dot{\mu}_3(\sigma)-K_3\mu_2(\sigma)-K_1\mu_3(\sigma)]V_4(\sigma).$

Using equation (17) gives

$$\dot{\zeta}(\sigma)=V_1(\sigma)+\left(\frac{\sqrt{K_2^2+K_3^2}-1}{K_3\sqrt{K_2^2+K_3^2}}\right)K_1K_2^2e^{-K_1\sigma}\tilde{C}_3V_3(\sigma)$$

i.e., $\dot{\zeta}(\sigma)$ not equal zero, whence $\zeta(\sigma)$ is not a constant vector. Thus, $\gamma(\sigma)$ does not congruent to a Normal curve.

Conflict of Interest

All authors declare that there is no conflict regarding the publication of this paper.

References

[1] I. M. Yaglom, *A simple Non-Euclidean Geometry and its physical Basis*, Springer, New York, NY, USA, (1979).
[2] M. ELzawy and S. Mosa, Quaternionic Bertrand Curves in the Galilean Space, *Filomat*, **34** (2020) 59-66.
[3] M. ELzawy and S. Mosa, Razzaboni Surfaces in the Galilean Space G^3, *Far East Journal of Mathematical Sciences*, **108** (2018) 13-26.

[4] M. ELzawy and S. Mosa, Smarandache curves in the Galilean 4-space G_4, *Journal of the Egyptian Mathematical society*, **25** (2017) 53-56.
[5] S. Mosa, M. Fakharany, and M. Elzawy, Normal curves in 4-dimensional Galilean space G^4, *Frontiers in Physics*, **9** (2021) 1-6.
[6] S. Mosa and M. ELzawy, Helicoidal surfaces in Galilean Space with density, *Frontiers in Physics*, **8** (2020) 1-6.
[7] H. Öztekin, Normal and rectifying curves in Galilean space G_3, *Proceedings of IAM*, **5** (2016) 98-109.
[8] S. Kiziltuğ, S. Yurttançikmaz and A. Çakmak, Normal ve rectifying curves in the equiform differential geometry of G_3, *Poincare Journal of Analysis & Applications*, **2** (2014) 55-61.
[9] Z. Erjavec, B. Divjak and D. Horvat, The general solutions of Frenet's system in the equiform geometry of the Galilean, Pseudo-Galilean, Simple Isotropic and Double Isotropic Space, *International Mathematical Forum*, **6** (2011) 837-856.
[10] U. Öztürk, E. B. Koç Öztürk and E. Nešović, On equiform Darboux helices in Galilean 3-space, *Math. Commun.*, **23** (2018) 145-159.
[11] M. E. Aydin and M. Ergut, The eqiform differential geometry of curves in 4-dimensional Galilean space G_4, *Stud. Univ. Babes-Bolyai Math.*, **58** (2013) 393-400.
[12] Z. Erjavec and B. Divjak, The equiform differential geometry of curves in the Pseudo-Galilean space, *Mathematical communications*, **13** (2008) 321-332.
[13] H. Gün Bozok and H. Öztekin, Inextensible Flows of curves in the equiform geometry of 4-dimensional Galilean space G_4, *Facta Universitatis(NIŠ), Ser. Math. Inform.*, **30** (2015) 209-216.
[14] A. M. Mathai and H. J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, NY, USA, (2008).