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The Optimal Retailer's Economic Production Quantity (EPQ) Policies with Two-Level Trade Credit under Alternate Due Date of Payment and Limited Storage Capacity

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Abstract: In recent years, many researchers investigated and developed the Economic Production Quantity (EPQ) model under permissible delay in payment. There are two payment terms which are usually being used. If a customer buys one item from a retailer at time t belonging to the time-interval $[0,N]$, then the customer will have a trade-credit period $N-t$ and will make the payment at time N. The other payment term (alternate due date) is when a customer buys one item from a retailer at time t belonging to $[0,T]$, the customer will have a trade-credit period N and will make the payment at time $N+t$. This paper develops a two-level trade-credit model with a finite replenishment rate by considering an alternate due date of payment and limited storage capacity together. Four theorems are developed in this investigation with a view to characterizing the optimal solutions according to cost-minimization strategy. Finally, sensitivity analyses are executed to investigate the effects of the various parameters on the ordering policies and the annual total relevant costs. Several interesting results are also considered in order to make several managerial suggestions.

Keywords: Economic Production Quantity (EPQ), permissible delay in paymentayment, trade-credit model, alternate due date of payment, trade-credit period, limited storage capacity, sensitivity analyses, optimal solutions, cost-minimization strategy.

1 Introduction

As long ago as 1913, Harris [1] designed the first Economic Order Quantity (EOQ) model. Subsequently, Taft [2] established the Economic Production Quantity (EPQ) model which integrated the EOQ model with the idea of continuous production in order to make it more practical. Hartley [3] proposed the inventory model of two warehouses for solving the capacity problem, which distinguished between the owned and the rented warehouses. Goyal [4] studied an EOQ model under permissible delay in payments by assuming that the supplier would offer the retailer a fixed-delay period and the retailer could sell the goods and accumulate revenue and earning within the trade-credit period. Huang [5] as well as Teng and Goyal [6] extended Goyal's EOQ model in [4] to provide a fixed trade-credit period M between the supplier and the retailer, and a trade-credit period N between the retailer and the customer. Huang's payment terms (see [5]): If a customer buys one item from a retailer at time t belonging to the time-interval $[0,N]$, then the customer will have a trade-credit period $N-t$ and will make the payment at time N. So, the retailer allows a maximum trade-credit period N for customers to settle the account. Huang's concept of payment terms in [5] was extended by many researchers (see, [7,8,9,10,11,12,13,14,15,16,17,18,19]). The Teng-Goyal payment terms (see [6]): If a customer buys one item from a retailer at

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time t belonging to the time-interval $[0,T]$, then the customer will have a trade-credit period N and will make the payment at time $N+t$. So, the retailer allows each customer to settle the account within the same trade period N. Related researches were present in several subsequent works including [20,21,22,23,24,25].
We remark that Yen *et al.* [18] presented an EOQ model by considering two warehouses system together with the payment terms considered earlier by Teng and Goyal. Motivated essentially by the above-mentioned developmentsm our attempt in this paper is to develop a generalization of the EPQ model considered by Yen *et al.* [18]. Four easy-to-use theorems are developed in order to characterize the optimal solutions according to the cost-minimization strategy. Finally, sensitivity analyses are executed to investigate the effects of the various parameters on ordering policies and annual total relevant costs (TRCs) of the inventory system. The conclusion in the work by Yen *et al.* [18] will thus become a special case of the results presented in this paper. For various other recent developments on the subject, the interested reader is referred also to such related works as those in [26,27,28,29,30].

2 Formulation of the Mathematical Model

$$tw_2 - tw_1 = \begin{cases} \frac{DT\rho - W}{P-D} + \frac{DT\rho - W}{D} & (DT\rho > W) \\ 0 & (DT\rho \leqq W) \end{cases}$$

T^* the optimal solution of $\mathrm{TRC}(T)$

2.1 Assumptions

We considered some assumptions for the proposed, both of the of Demand rate D and eplenishment rate P are known and constant. Shortages are not allowed and time period is infinite. Moreover, $k \geqq h$, $M \geqq N$ and $s \geqq c$. When the order quantity is larger than the retailer's OW storage capacity W, the retailer would rent a warehouse to store these exceeding items, and the RW storage capacity is unlimited. When the demand occurs, the ordered items would be replenished from the RW which stores those exceeding items. During the period of time when the account is not settled, generated sales revenue is deposited in an interest-bearing account. When $T \geqq M$, the account is settled at $T = M$, the retailer pays off all units sold and keeps his/her profits, and starts paying for a higher interest charge on the items in stock with rate I_k. When $T \leqq M$, the account is settled at $T = M$ and the retailer does not need to pay any interest charge. The retailer can accumulate revenue and earn interest after a customer pays for the amount of purchasing cost to the retailer until the end of the trade-credit period offered by the supplier. That is, the retailer can accumulate revenue and earn interest between the period N and the period M with rate I_e under the condition of trade credit. If a customer buys an item from the retailer at time $t \in [0,T]$, then the customer will have a trade-credit period N and make the payment at time $N+t$.

2.2 The Annual Total Relevant Cost

The annual total relevant cost (TRC) consists of two elements. First, the annual ordering cost is given by

$$\text{Annual ordering cost} = \frac{A}{T}. \tag{1}$$

Second, annual stock-holding cost including owned warehouse and rented warehouse). Two cases occur in costs of owned warehouse:

1. $DT\rho \leqq W$, shown in Figure 1. The annual stock-holding cost in owned warehouse is given by

$$\text{Annual stock-holding cost in owned warehouse} = \frac{DTh\rho}{2}. \tag{2}$$

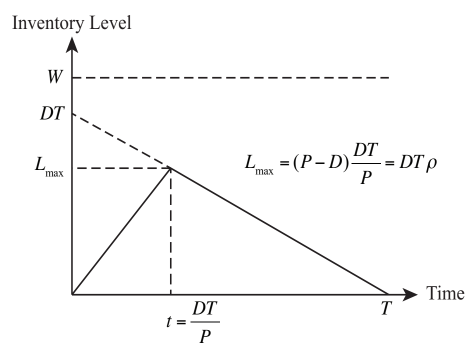

Fig. 1: Annual stock-holding cost when $DT\rho \leqq W$

2. $DT\rho > W$, shown in Figure 2. The annual stock-holding cost in owned warehouse is given by

$$\text{Annual stock-holding cost in owned warehouse} = Wh - \frac{PW^2}{2DT(P-D)}. \tag{3}$$

Two cases occur in costs of rented warehouse:

1. $DT\rho \leqq W$, shown in Figure 1. The annual stock-holding cost in rented warehouse is given by

$$\text{Annual stock-holding cost in rented warehouse} = 0. \tag{4}$$

2. $DT\rho > W$, shown in Figure 2. The annual stock-holding cost in rented warehouse given by

$$\text{Annual stock-holding cost in rented warehouse} = \frac{Pk(DT\rho - W)^2}{2DT(P-D)}. \tag{5}$$

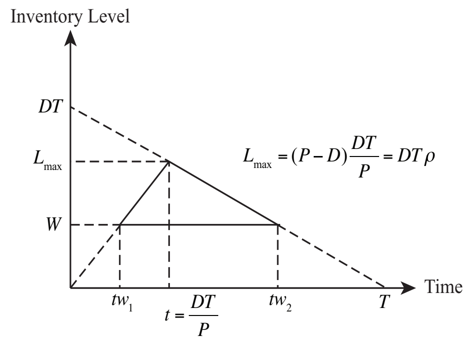

Fig. 2: Annual stock-holding cost when $DT\rho > W$

Four cases are expected to occur each year in the costs of interest charges for the items kept in stock.

1. $T \leqq M - N$. The annual interest charged is given, in this case, by

$$\text{Annual interest charged} = 0. \tag{6}$$

2. $M - N \leqq T < M$. The annual interest charged is given, in this case, by

$$\text{Annual interest charged} = 0. \tag{7}$$

3. $M \leqq T < \frac{PM}{D}$, shown in Figure 3. The annual interest charged is given, in this case, by

$$\text{Annual interest charged} = \frac{cI_k D(T-M)^2}{2T}. \tag{8}$$

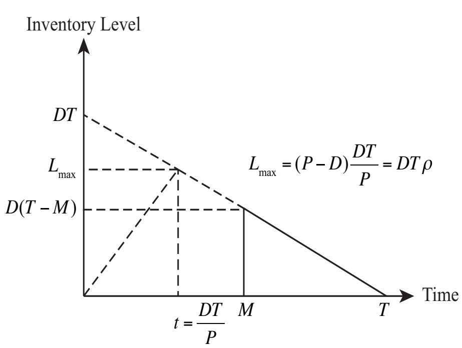

Fig. 3: Annual interest charged when $M < T \leqq \frac{PM}{D}$

4. $\frac{PM}{D} < T$, shown in Figure 4. The annual interest charged is given, in this case, by

$$\text{Annual interest charged} = \frac{cI_k\rho(DT^2 - PM^2)}{2T}. \tag{9}$$

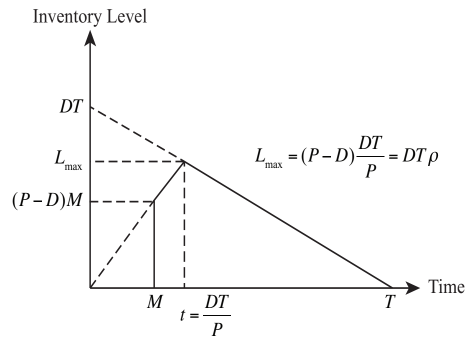

Fig. 4: Annual interest charged when $\frac{PM}{D} < T$

There are four cases to occur in interest earned per year.

1. $T < M - N$, shown in Figure 5. The annual interest earned is given, in this case, by

$$\text{Annual interest earned} = \frac{sI_e D(2M - 2N - T)}{2}. \tag{10}$$

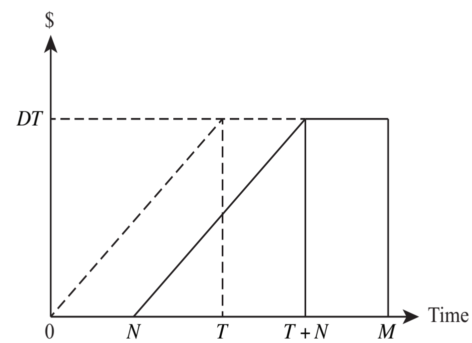

Fig. 5: Annual interest earned when $T \geqq M - N$

2. $M - N \leqq T < M$, shown in Figure 6. The annual interest earned is given, in this case, by

$$\text{Annual interest earned} = \frac{sI_e D(M-N)^2}{2T}. \tag{11}$$

3. $M \leqq T < \dfrac{PM}{D}$, shown in Figure 7. The annual interest earned is given, in this case, by

$$\text{Annual interest earned} = \frac{sI_e D(M-N)^2}{2T}. \tag{12}$$

4. $\dfrac{PM}{D} \leqq T$. The annual interest earned is given, in this case, by

$$\text{Annual interest earned} = \frac{sI_e D(M-N)^2}{2T}. \tag{13}$$

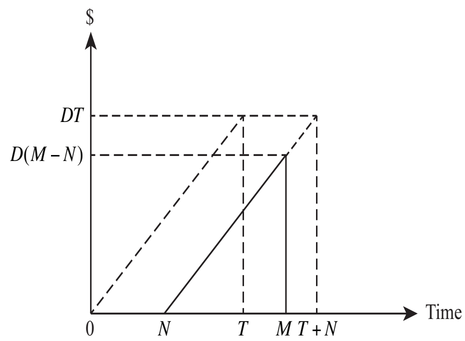

Fig. 6: Annual interest earned when $M - N < T \leqq M$

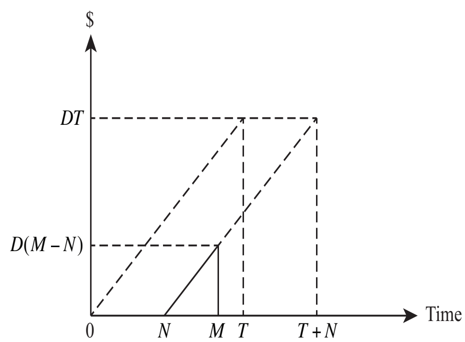

Fig. 7: Annual interest earned when $M < T \leqq \frac{PM}{D}$

From the above arguments, the annual total relevant cost for the retailer can be expressed as follows:

$$\mathrm{TRC}(T) = \text{ordering cost} + \text{stock-holding} + \text{interest charged} - \text{interest earned}.$$

The following four cases arise:

1. $\dfrac{W}{D\rho} < M - N.$
2. $M - N \leqq \dfrac{W}{D\rho} < M.$
3. $M \leqq \dfrac{W}{D\rho} < \dfrac{PM}{D}.$
4. $\dfrac{PM}{D} \leqq \dfrac{W}{D\rho}.$

Case 1. $\dfrac{W}{D\rho} < M - N.$

According to Equations (1) to (13), the annual total relevant cost, $\mathrm{TRC}(T)$, can be expressed by

$$\mathrm{TRC}(T) = \begin{cases} \mathrm{TRC}_1(T) & \left(T \leqq \dfrac{W}{D\rho}\right) \\ \mathrm{TRC}_2(T) & \left(\dfrac{W}{D\rho} \leqq T < M - N\right) \\ \mathrm{TRC}_3(T) & (M - N \leqq T < M) \\ \mathrm{TRC}_4(T) & \left(M \leqq T < \dfrac{PM}{D}\right) \\ \mathrm{TRC}_5(T) & \left(\dfrac{PM}{D} \leqq T\right), \end{cases} \tag{14}$$

where

$$\mathrm{TRC}_1(T) = \frac{A}{T} + \frac{DTh\rho}{2} + 0 - \frac{sI_eD(2M-2N-T)}{2}, \tag{15}$$

$$\mathrm{TRC}_2(T) = \frac{A}{T} + \frac{Pk(DT\rho - W)^2}{2DT(P-D)} + Wh - \frac{PW^2h}{2DT(P-D)} + 0 - \frac{sI_eD(2M-2N-T)}{2}, \tag{16}$$

$$\mathrm{TRC}_3(T) = \frac{A}{T} + \frac{Pk(DT\rho - W)^2}{2DT(P-D)} + Wh - \frac{PW^2h}{2DT(P-D)} + 0 - \frac{sI_eD(M-N)^2}{2T}, \tag{17}$$

$$\mathrm{TRC}_4(T) = \frac{A}{T} + \frac{Pk(DT\rho - W)^2}{2DT(P-D)} + Wh - \frac{PW^2h}{2DT(P-D)} + \frac{cI_kD(T-M)^2}{2T} - \frac{sI_eD(M-N)^2}{2T} \tag{18}$$

and

$$\mathrm{TRC}_5(T) = \frac{A}{T} + \frac{Pk(DT\rho - W)^2}{2DT(P-D)} + Wh - \frac{PW^2h}{2DT(P-D)} + \frac{cI_k\rho(DT^2 - PM^2)}{2T} - \frac{sI_eD(M-N)^2}{2T}. \tag{19}$$

Since

$$\mathrm{TRC}_1\left(\frac{W}{D\rho}\right) = \mathrm{TRC}_2\left(\frac{W}{D\rho}\right),$$
$$\mathrm{TRC}_2(M-N) = \mathrm{TRC}_3(M-N),$$
$$\mathrm{TRC}_3(M) = \mathrm{TRC}_4(M)$$

and

$$\mathrm{TRC}_4\left(\frac{PM}{D}\right) = \mathrm{TRC}_5\left(\frac{PM}{D}\right),$$

the function $\mathrm{TRC}(T)$ is continuous when $T > 0$.

Case 2. $M-N \leqq \dfrac{W}{D\rho} < M$.
According to Equations (1) to (13), the annual total relevant cost, $\mathrm{TRC}(T)$, can be expressed by

$$\mathrm{TRC}(T)=\begin{cases}\mathrm{TRC}_1(T) & (0<T<M-N)\\ \mathrm{TRC}_6(T) & \left(M-N\leqq T<\dfrac{W}{D\rho}\right)\\ \mathrm{TRC}_3(T) & \left(\dfrac{W}{D\rho}\leqq T<M\right)\\ \mathrm{TRC}_4(T) & \left(M\leqq T<\dfrac{PM}{D}\right)\\ \mathrm{TRC}_5(T) & \left(\dfrac{PM}{D}\leqq T\right),\end{cases} \tag{20}$$

where

$$\mathrm{TRC}_6(T)=\frac{A}{T}+\frac{DTh\rho}{2}+0+\frac{sI_eD(M-N)^2}{2T}. \tag{21}$$

Since

$$\mathrm{TRC}_1(M-N)=\mathrm{TRC}_6(M-N),$$
$$\mathrm{TRC}_6\left(\frac{W}{D\rho}\right)=\mathrm{TRC}_3\left(\frac{W}{D\rho}\right),$$
$$\mathrm{TRC}_3(M)=\mathrm{TRC}_4(M)$$

and

$$\mathrm{TRC}_4\left(\frac{PM}{D}\right)=\mathrm{TRC}_5\left(\frac{PM}{D}\right),$$

the function $\mathrm{TRC}(T)$ is continuous when $T>0$.

Case 3. $M \leqq \dfrac{W}{D\rho} < \dfrac{PM}{D}$.
According to Equations (1) to (13), the annual total relevant cost, $\mathrm{TRC}(T)$, can be expressed by

$$\mathrm{TRC}(T)=\begin{cases}\mathrm{TRC}_1(T) & (0<T<M-N)\\ \mathrm{TRC}_6(T) & (M-N\leqq T<M)\\ \mathrm{TRC}_7(T) & \left(M\leqq T<\dfrac{W}{D\rho}\right)\\ \mathrm{TRC}_4(T) & \left(\dfrac{W}{D\rho}\leqq T<\dfrac{PM}{D}\right)\\ \mathrm{TRC}_5(T) & \left(\dfrac{PM}{D}\leqq T\right),\end{cases} \tag{22}$$

where

$$\mathrm{TRC}_7(T)=\frac{A}{T}+\frac{DTh\rho}{2}+\frac{cI_kD(T-M)^2}{2T}-\frac{sI_eD(M-N)^2}{2T}. \tag{23}$$

Since

$$\mathrm{TRC}_1(M-N)=\mathrm{TRC}_6(M-N),$$
$$\mathrm{TRC}_6(M)=\mathrm{TRC}_7(M),$$
$$\mathrm{TRC}_7\left(\frac{W}{D\rho}\right)=\mathrm{TRC}_4\left(\frac{W}{D\rho}\right)$$

and

$$\mathrm{TRC}_4\left(\frac{PM}{D}\right)=\mathrm{TRC}_5\left(\frac{PM}{D}\right),$$

the function $\mathrm{TRC}(T)$ is continuous when $T>0$.

Case 4. $\dfrac{PM}{D} \leqq \dfrac{W}{D\rho}$.
According to Equations (1) to (13), the annual total relevant cost, $\mathrm{TRC}(T)$, can be expressed by

$$\mathrm{TRC}(T)=\begin{cases}\mathrm{TRC}_1(T) & (0<T<M-N)\\ \mathrm{TRC}_6(T) & (M-N\leqq T<M)\\ \mathrm{TRC}_7(T) & (M\leqq T<\frac{PM}{D})\\ \mathrm{TRC}_8(T) & \left(\frac{PM}{D}\leqq T<\frac{W}{D\rho}\right)\\ \mathrm{TRC}_5(T) & \left(\frac{W}{D\rho}\leqq T\right),\end{cases} \tag{24}$$

where

$$\mathrm{TRC}_8(T)=\frac{A}{T}+\frac{DTh\rho}{2}+\frac{cI_k\rho(DT^2-PM^2)}{2T}-\frac{sI_eD(M-N)^2}{2T}. \tag{25}$$

Since

$$\mathrm{TRC}_1(M-N)=\mathrm{TRC}_6(M-N),$$
$$\mathrm{TRC}_6(M)=\mathrm{TRC}_7(M),$$
$$\mathrm{TRC}_7\left(\frac{PM}{D}\right)=\mathrm{TRC}_8\left(\frac{PM}{D}\right)$$

and

$$\mathrm{TRC}_8\left(\frac{W}{D\rho}\right)=\mathrm{TRC}_5\left(\frac{W}{D\rho}\right),$$

the function $\mathrm{TRC}(T)$ is continuous when $T>0$.

In summary, all of the functions $\mathrm{TRC}_i(T)$ $(i=1,\cdots,8)$ are defined on $T>0$.

3 The Convexity of $\mathrm{TRC}_i(T)$ $(i=1,\cdots,8)$

Equations (15) to (19), (21), (23) and (25) yield the first-order and second-order derivatives as follows:

$$\mathrm{TRC}_1'(T) = -\frac{A}{T^2} + \frac{Dh\rho}{2} + \frac{sI_eD}{2}, \tag{26}$$

$$\mathrm{TRC}_1''(T) = \frac{2A}{T^3} \geqq 0, \tag{27}$$

$$\mathrm{TRC}_2'(T) = -\frac{A}{T^2} - \frac{W^2(k-h)}{2D\rho T^2} + \frac{Dk\rho}{2} + \frac{sI_eD}{2}, \tag{28}$$

$$\mathrm{TRC}_2''(T) = \frac{2A + \frac{W^2(k-h)}{D\rho}}{T^3} \geqq 0, \tag{29}$$

$$\mathrm{TRC}_3'(T) = -\frac{A}{T^2} - \frac{W^2(k-h)}{2D\rho T^2} + \frac{sI_eD(M-N)^2}{2T^2} + \frac{Dk\rho}{2}, \tag{30}$$

$$\mathrm{TRC}_3''(T) = \frac{2A + \frac{W^2(k-h)}{D\rho} - sI_eD(M-N)^2}{T^3}, \tag{31}$$

$$\mathrm{TRC}_4'(T) = -\frac{A}{T^2} - \frac{W^2(k-h)}{2D\rho T^2} - \frac{cI_kDM^2}{2T^2} + \frac{sI_eD(M-N)^2}{2T^2} + \frac{Dk\rho}{2} + \frac{cI_kD}{2}, \tag{32}$$

$$\mathrm{TRC}_4''(T) = \frac{2A + \frac{W^2(k-h)}{D\rho} + cI_kDM^2 - sI_eD(M-N)^2}{T^3}, \tag{33}$$

$$\mathrm{TRC}_5'(T) = -\frac{A}{T^2} - \frac{W^2(k-h)}{2D\rho T^2} + \frac{cI_kM^2(P-D)}{2T^2} + \frac{sI_eD(M-N)^2}{2T^2} + \frac{Dk\rho}{2} + \frac{cI_kD\rho}{2}, \tag{34}$$

$$\mathrm{TRC}_5''(T) = \frac{2A + \frac{W^2(k-h)}{D\rho} - cI_kM^2(P-D) - sI_eD(M-N)^2}{T^3}, \tag{35}$$

$$\mathrm{TRC}_6'(T) = -\frac{A}{T^2} + \frac{sI_eD(M-N)^2}{2T^2} + \frac{Dh\rho}{2}, \tag{36}$$

$$\mathrm{TRC}_6''(T) = \frac{2A - sIeD(M-N)^2}{T^3}, \tag{37}$$

$$\mathrm{TRC}_7'(T) = -\frac{A}{T^2} - \frac{cI_kDM^2}{2T^2} + \frac{sI_eD(M-N)^2}{2T^2} + \frac{Dh\rho}{2} + \frac{cI_kD}{2}, \tag{38}$$

$$\mathrm{TRC}_7''(T) = \frac{2A + cI_kDM^2 - sI_eD(M-N)^2}{T^3}, \tag{39}$$

$$\mathrm{TRC}_8'(T) = -\frac{A}{T^2} + \frac{cI_kM^2(P-D)}{2T^2} + \frac{sI_eD(M-N)^2}{2T^2} + \frac{Dh\rho}{2} + \frac{cI_kD\rho}{2} \tag{40}$$

and

$$\mathrm{TRC}_8''(T) = \frac{2A - cI_kM^2(P-D) - sI_eD(M-N)^2}{T^3}. \tag{41}$$

If we let

$$G_3 = 2A + \frac{W^2(k-h)}{D\rho} - sI_eD(M-N)^2, \tag{42}$$

$$G_4 = 2A + \frac{W^2(k-h)}{D\rho} + cI_kDM^2 - sI_eD(M-N)^2, \tag{43}$$

$$G_5 = 2A + \frac{W^2(k-h)}{D\rho} - cI_kM^2(P-D) - sI_eD(M-N)^2, \tag{44}$$

$$G_6 = 2A - sI_eD(M-N)^2, \tag{45}$$

$$G_7 = 2A + cI_kDM^2 - sI_eD(M-N)^2 \tag{46}$$

and

$$G_8 = 2A - cI_kM^2(P-D) - sI_eD(M-N)^2, \tag{47}$$

then Equations (42) to (47) imply that

$$G_4 > G_3 > G_5 > G_8 \tag{48}$$

and

$$G_4 > G_7 > G_6 > G_8. \tag{49}$$

Furthermore, Equations (26) to (41) lead us to the following results: Upon solving these equations, we find that

$$\mathrm{TRC}_1'(T) = 0 \qquad (i=1,\cdots,8), \tag{50}$$

$$T_1^* = \sqrt{\frac{2A}{Dh\rho + sI_eD}}, \tag{51}$$

$$T_2^* = \sqrt{\frac{2A + \frac{W^2(k-h)}{D\rho}}{Dk\rho + sI_eD}}, \tag{52}$$

$$T_3^* = \sqrt{\frac{2A + \frac{W^2(k-h)}{D\rho} - sI_eD(M-N)^2}{Dk\rho}} \qquad (G_3 > 0), \tag{53}$$

$$T_4^* = \sqrt{\frac{2A + \frac{W^2(k-h)}{D\rho} + cI_kDM^2 - sI_eD(M-N)^2}{Dk\rho + cI_kD}}, \quad (G_4 > 0), \quad (54)$$

$$T_5^* = \sqrt{\frac{2A + \frac{W^2(k-h)}{D\rho} - cI_kM^2(P-D) - sI_eD(M-N)^2}{Dk\rho + cI_kD\rho}}, \quad (G_5 > 0), \quad (55)$$

$$T_6^* = \sqrt{\frac{2A - sI_eD(M-N)^2}{Dh\rho}} \qquad (G_6 > 0), \quad (56)$$

$$T_7^* = \sqrt{\frac{2A + cI_kDM^2 - sI_eD(M-N)^2}{Dh\rho + cI_kD}} \qquad (G_7 > 0) \quad (57)$$

and

$$T_8^* = \sqrt{\frac{2A - cI_kM^2(P-D) - sI_eD(M-N)^2}{Dh\rho + cI_kD\rho}} \qquad (G_8 > 0). \quad (58)$$

Based upon the above observations, we now state the following lemma.

Lemma 1. *Each of the following assertions holds true*:

1. $\mathrm{TRC}_i(T)$ *is convex on* $T > 0$ *if* $i = 1, 2$.
2. $\mathrm{TRC}_i(T)$ *is convex on* $T > 0$ *if* $i = 3, \cdots, 8$ *and* $G_i > 0$ $(i = 3, \cdots, 8)$. *Otherwise the function* $\mathrm{TRC}_i(T)$ *is increasing on* $T > 0$.

If T_i^* *exists, then* $\mathrm{TRC}_i(T)$ *is convex on* $T > 0$ *and*

$$\mathrm{TRC}_i'(T) \begin{cases} < 0 & (0 < T < T_i^*) \\ = 0 & (T = T_i^*) \\ > 0 & (T > T_i^*). \end{cases} \quad (59)$$

Equation (59) *implies that the function* $\mathrm{TRC}_i(T)$ *is decreasing on* $(0, T_i^*]$ *and increasing on* $[T_i^*, \infty)$ *for all* $i = 1, \cdots, 8$.

4 The Number Δ_{ij}

We consider the following cases:

Case 1. $\dfrac{W}{D\rho} < M - N$.
Equations (26), (28), (30), (32) and (34) yield

$$\mathrm{TRC}_1'\left(\frac{W}{D\rho}\right) = \mathrm{TRC}_2'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{12}}{2\left(\frac{W}{D\rho}\right)^2}, \quad (60)$$

$$\mathrm{TRC}_2'(M-N) = \mathrm{TRC}_3'(M-N) = \frac{\Delta_{23}}{2(M-N)^2}, \quad (61)$$

$$\mathrm{TRC}_3'(M) = \mathrm{TRC}_4'(M) = \frac{\Delta_{34}}{2M^2} \quad (62)$$

and

$$\mathrm{TRC}_4'\left(\frac{PM}{D}\right) = \mathrm{TRC}_5'\left(\frac{PM}{D}\right) = \frac{\Delta_{45}}{2\left(\frac{PM}{D}\right)^2}, \quad (63)$$

where

$$\Delta_{12} = -2A + \frac{hW^2}{D\rho} + \frac{sI_eW^2}{D\rho}, \quad (64)$$

$$\Delta_{23} = -2A - \frac{W^2(k-h)}{D\rho} + sI_eD(M-N)^2 + Dk\rho(M-N)^2, \quad (65)$$

$$\Delta_{34} = -2A - \frac{W^2(k-h)}{D\rho} + sI_eD(M-N)^2 + Dk\rho M^2 \quad (66)$$

and

$$\Delta_{45} = -2A - \frac{W^2(k-h)}{D\rho} + sI_eD(M-N)^2 + \frac{cI_kM^2(P^2-D^2)}{D} + \frac{PkM^2(P-D)}{D}. \quad (67)$$

Equations (64) to (67) imply that

$$\Delta_{45} > \Delta_{34} > \Delta_{23} > \Delta_{12}. \quad (68)$$

Case 2. $M - N \leqq \dfrac{W}{D\rho} < M$.
Equations (26), (36), (30), (32) and (34) yield

$$\mathrm{TRC}_1'(M-N) = \mathrm{TRC}_6'(M-N) = \frac{\Delta_{16}}{2(M-N)^2}, \quad (69)$$

$$\mathrm{TRC}_6'\left(\frac{W}{D\rho}\right) = \mathrm{TRC}_3'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{63}}{2\left(\frac{W}{D\rho}\right)^2}, \quad (70)$$

$$\mathrm{TRC}_3'(M) = \mathrm{TRC}_4'(M) = \frac{\Delta_{34}}{2M^2} \quad (71)$$

and

$$\mathrm{TRC}_4'\left(\frac{PM}{D}\right) = \mathrm{TRC}_5'\left(\frac{PM}{D}\right) = \frac{\Delta_{45}}{2\left(\frac{W}{D\rho}\right)^2}, \quad (72)$$

where

$$\Delta_{16} = -2A + sI_eD(M-N)^2 + Dh\rho(M-N)^2 \quad (73)$$

and

$$\Delta_{63} = -2A + sI_eD(M-N)^2 + \frac{hW^2}{D\rho}. \quad (74)$$

Equations (66), (67), (73) and (74) imply that

$$\Delta_{45} > \Delta_{34} > \Delta_{63} \geqq \Delta_{16}. \quad (75)$$

Case 3. $M \leqq \dfrac{W}{D\rho} < \dfrac{PM}{D}$.
Equations (26), (32), (34), (36) and (38) yield

$$\mathrm{TRC}_1'(M-N) = \mathrm{TRC}_6'(M-N) = \frac{\Delta_{16}}{2(M-N)^2}, \tag{76}$$

$$\mathrm{TRC}_6'(M) = \mathrm{TRC}_7'(M) = \frac{\Delta_{67}}{2M^2}, \tag{77}$$

$$\mathrm{TRC}_7'\left(\frac{W}{D\rho}\right) = \mathrm{TRC}_4'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{74}}{2\left(\dfrac{W}{D\rho}\right)^2} \tag{78}$$

and

$$\mathrm{TRC}_4'\left(\frac{PM}{D}\right) = \mathrm{TRC}_5'\left(\frac{PM}{D}\right) = \frac{\Delta_{45}}{2\left(\dfrac{PM}{D}\right)^2}, \tag{79}$$

where

$$\Delta_{67} = -2A + sI_eD(M-N)^2 + Dh\rho M^2 \tag{80}$$

and

$$\Delta_{74} = -2A + sI_eD(M-N)^2 - cI_kDM^2 + \frac{hW^2}{D\rho} + \frac{cI_kW^2}{D\rho}. \tag{81}$$

Equations (67), (73), (80) and (81) imply that

$$\Delta_{45} > \Delta_{74} > \Delta_{67} \geqq \Delta_{16}. \tag{82}$$

Case 4. $\dfrac{PM}{D} \leqq \dfrac{W}{D\rho}$.
Equations (26), (34), (36), (38) and (40) yield

$$\mathrm{TRC}_1'(M-N) = \mathrm{TRC}_6'(M-N) = \frac{\Delta_{16}}{2(M-N)^2}, \tag{83}$$

$$\mathrm{TRC}_6'(M) = \mathrm{TRC}_7'(M) = \frac{\Delta_{67}}{2M^2}, \tag{84}$$

$$\mathrm{TRC}_7'\left(\frac{PM}{D}\right) = \mathrm{TRC}_8'\left(\frac{PM}{D}\right) = \frac{\Delta_{78}}{2\left(\dfrac{PM}{D}\right)^2} \tag{85}$$

and

$$\mathrm{TRC}_8'\left(\frac{W}{D\rho}\right) = \mathrm{TRC}_5'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{85}}{2\left(\dfrac{W}{D\rho}\right)^2}, \tag{86}$$

where

$$\Delta_{78} = -2A + sI_eD(M-N)^2 + \frac{cI_kM^2(P^2-D^2)}{D} + \frac{PkM^2(P-D)}{D} \tag{87}$$

and

$$\Delta_{85} = -2A + sI_eD(M-N)^2 + cI_kM^2(P-D) + \frac{hW^2}{D\rho} + \frac{cI_kW^2}{D\rho^2}. \tag{88}$$

Equations (73), (80), (87) and (88) imply that

$$\Delta_{85} > \Delta_{67} \geqq \Delta_{16} \tag{89}$$

and

$$\Delta_{78} > \Delta_{67} \geqq \Delta_{16}. \tag{90}$$

Based upon the above-mentioned arguments, we derive the following results.

Lemma 2. *Each of the following assertions holds true*:

1. *If* $\Delta_{23} \leqq 0$, *then*
   (a) $G_3 > 0$;
   (b) T_3^* *exists;*
   (c) $\mathrm{TRC}_3(T)$ *is convex on* $T > 0$.
2. *If* $\Delta_{34} \leqq 0$, *then*
   (a) $G_3 > 0$ *and* $G_4 > 0$;
   (b) T_3^* *and* T_4^* *exist;*
   (c) $\mathrm{TRC}_3(T)$ *and* $\mathrm{TRC}_4(T)$ *are convex on* $T > 0$.
3. *If* $\Delta_{45} \leqq 0$, *then*
   (a) $G_4 > 0$ *and* $G_5 > 0$;
   (b) T_4^* *and* T_5^* *exist;*
   (c) $\mathrm{TRC}_4(T)$ *and* $\mathrm{TRC}_5(T)$ *are convex on* $T > 0$.
4. *If* $\Delta_{16} \leqq 0$, *then*
   (a) $G_6 > 0$;
   (b) T_6^* *exists;*
   (c) $\mathrm{TRC}_6(T)$ *is convex on* $T > 0$.
5. *If* $\Delta_{63} \leqq 0$, *then*
   (a) $G_3 > 0$ *and* $G_6 > 0$;
   (b) T_3^* *and* T_6^* *exist;*
   (c) $\mathrm{TRC}_3(T)$ *and* $\mathrm{TRC}_6(T)$ *are convex on* $T > 0$.
6. *If* $\Delta_{67} \leqq 0$, *then*
   (a) $G_6 > 0$ *and* $G_7 > 0$;
   (b) T_6^* *and* T_7^* *exist;*
   (c) $\mathrm{TRC}_6(T)$ *and* $\mathrm{TRC}_7(T)$ *are convex on* $T > 0$.
7. *If* $\Delta_{74} \leqq 0$, *then*
   (a) $G_4 > 0$ *and* $G_7 > 0$;
   (b) T_4^* *and* T_7^* *exist;*
   (c) $\mathrm{TRC}_4(T)$ *and* $\mathrm{TRC}_7(T)$ *are convex on* $T > 0$.
8. *If* $\Delta_{78} \leqq 0$, *then*
   (a) $G_7 > 0$ *and* $G_8 > 0$;
   (b) T_7^* *and* T_8^* *exist;*
   (c) $\mathrm{TRC}_7(T)$ *and* $\mathrm{TRC}_8(T)$ *are convex on* $T > 0$.
9. *If* $\Delta_{85} \leqq 0$, *then*
   (a) $G_5 > 0$ *and* $G_8 > 0$;
   (b) T_5^* *and* T_8^* *exist;*
   (c) $\mathrm{TRC}_5(T)$ *and* $\mathrm{TRC}_8(T)$ *are convex on* $T > 0$.

Proof. Our item-wise demonstration of Lemma 2 is given belo.

1. (a) If $\Delta_{23} < 0$, then

$$2A + \frac{W^2(k-h)}{D\rho} \geqq sI_eD(M-N)^2 + Dk\rho(M-N)^2 \tag{91}$$

and

$$G_3 = 2A + \frac{W^2(k-h)}{D\rho} - sI_eD(M-N)^2 \geqq Dk\rho(M-N)^2 > 0. \tag{92}$$

(b) Equation (53) implies that T_3^* exists.
(c) Equation (31) and Lemma 1 imply that $TRC_3(T)$ is convex on $T > 0$.

2. (a) If $\Delta_{34} \leqq 0$, then

$$2A + \frac{W^2(k-h)}{D\rho} \geqq sI_eD(M-N)^2 + Dk\rho M^2, \tag{93}$$

$$G_3 = 2A + \frac{W^2(k-h)}{D\rho} - sI_eD(M-N)^2 \geqq Dk\rho M^2 > 0 \tag{94}$$

and

$$G_4 = 2A + \frac{W^2(k-h)}{D\rho} + cI_kDM^2 - sI_eD(M-N)^2 \geqq Dk\rho M^2 + cI_kDM^2 > 0. \tag{95}$$

(b) Equations (53) and (54) imply that T_3^* and T_4^* exist.
(c) Equations (31) and (33), together with Lemma 1, imply that $TRC_3(T)$ and $TRC_4(T)$ are convex on $T > 0$.

3. (a) If $\Delta_{45} \leqq 0$, then

$$2A + \frac{W^2(k-h)}{D\rho} \geqq sI_eD(M-N)^2 + \frac{cI_kM^2(P^2-D^2)}{D} + \frac{PkM^2(P-D)}{D}, \tag{96}$$

$$G_4 = 2A + \frac{W^2(k-h)}{D\rho} + cI_kDM^2 - sI_eD(M-N)^2 \geqq \frac{PkM^2(P-D)}{D} + \frac{cI_kP^2M^2}{D} > 0 \tag{97}$$

and

$$G_5 = 2A + \frac{W^2(k-h)}{D\rho} - cI_kDM^2(P-D) - sI_eD(M-N)^2 \geqq \frac{PkM^2(P-D)}{D} + \frac{cI_kPM^2(P-D)}{D} > 0. \tag{98}$$

(b) Equations (54) and (55) imply that T_4^* and T_5^* exist.
(c) Equations (33) and (35), together with Lemma 1, imply that $TRC_4(T)$ and $TRC_5(T)$ are convex on $T > 0$.

4. (a) $\Delta_{16} \leqq 0$, then

$$2A \leqq sI_eD(M-N)^2 + Dh\rho(M-N)^2 \tag{99}$$

and

$$G_6 = 2A - sI_eD(M-N)^2 \geqq Dh\rho(M-N)^2 > 0. \tag{100}$$

(b) Equation (56) implies that T_6^* exists.
(c) Equation (37) and Lemma 1 imply that $TRC_6(T)$ is convex on $T > 0$.

5. (a) If $\Delta_{63} \leqq 0$, then

$$2A \geqq sI_eD(M-N)^2 + \frac{hW^2}{D\rho}, \tag{101}$$

$$G_6 = 2A - sI_eD(M-N)^2 \geqq \frac{hW^2}{D\rho} > 0 \tag{102}$$

and

$$G_3 = 2A + \frac{W^2(k-h)}{D\rho} - sI_eD(M-N)^2 \geqq \frac{kW^2}{D\rho} > 0. \tag{103}$$

(b) Equations (53) and (56) imply that T_3^* and T_6^* exist.
(c) Equations (31) and (37), together with Lemma 1, imply that $TRC_3(T)$ and $TRC_6(T)$ are convex on $T > 0$.

6. (a) If $\Delta_{67} \leqq 0$, then

$$2A \leqq sI_eD(M-N)^2 + Dh\rho M^2, \tag{104}$$

$$G_6 = 2A - sI_eD(M-N)^2 \geqq Dh\rho M^2 > 0 \tag{105}$$

and

$$G_7 = 2A + cI_kDM^2 - sI_eD(M-N)^2 \geqq Dh\rho M^2 + cI_kDM^2 > 0. \tag{106}$$

(b) Equations (56) and (57) imply that T_6^* and T_7^* exist.
(c) Equations (37) and (39), together with Lemma 1, imply that $TRC_6(T)$ and $TRC_7(T)$ are convex on $T > 0$.

7. (a) If $\Delta_{74} \leqq 0$, then

$$2A + cI_kDM^2 > sI_eD(M-N)^2 + \frac{hW^2}{D\rho} + \frac{cI_kW^2}{D\rho^2}, \tag{107}$$

$$G_7 = 2A + cI_kDM^2 - sI_eD(M-N)^2 \geqq \frac{hW^2}{D\rho} + \frac{cI_kW^2}{D\rho^2} > 0 \tag{108}$$

and

$$G_4 = 2A + \frac{W^2(k-h)}{D\rho} + cI_kDM^2 - sI_eD(M-N)^2 \geqq \frac{kW^2}{D\rho} + \frac{cI_kW^2}{D\rho^2} > 0. \tag{109}$$

(b) Equations (54) and (57) imply that T_4^* and T_7^* exist.

(c) Equations (33) and (39), together with Lemma 1, imply that $TRC_4(T)$ and $TRC_7(T)$ are concave on $T > 0$.

8. (a) If $\Delta_{78} \leqq 0$, then

$$2A \geqq sI_eD(M-N)^2 + \frac{cI_kM^2(P^2-D^2)}{D} + \frac{PkM^2(P-D)}{D}, \tag{110}$$

$$G_7 = 2A + cI_kDM^2 - sI_eD(M-N)^2 \geqq \frac{PkM^2(P-D)}{D} + \frac{cI_kP^2M^2}{D} > 0, \tag{111}$$

and

$$G_8 = 2A - cI_kDM^2(P-D) - sI_eD(M-N)^2 \geqq \frac{PkM^2(P-D)}{D} + \frac{cI_kPM^2(P-D)}{D} > 0. \tag{112}$$

(b) Equations (57) and (58) imply that T_7^* and T_8^* exist.

(c) Equations (39) and (41), together with Lemma 1, imply that $TRC_7(T)$ and $TRC_8(T)$ are convex on $T > 0$.

9. (a) If $\Delta_{85} \leqq 0$, then

$$2A \geqq sI_eD(M-N)^2 + cI_kM^2(P-D) + \frac{hW^2}{D\rho} + \frac{cI_kW^2}{D\rho^2}, \tag{113}$$

$$G_8 = 2A - cI_kM^2(P-D) - sI_eD(M-N)^2 \geqq \frac{hW^2}{D\rho} + \frac{cI_kW^2}{D\rho^2} > 0 \tag{114}$$

and

$$G_5 = 2A + \frac{W^2(k-h)}{D\rho} - cI_kM^2(P-D) - sI_eD(M-N)^2 \geqq \frac{hW^2}{D\rho} + \frac{cI_kW^2}{D\rho^2} > 0. \tag{115}$$

(b) Equations (55) and (58) imply that T_5^* and T_8^* exist.

(c) Equations (35) and (41), together with Lemma 1, imply that $TRC_5(T)$ and $TRC_8(T)$ are convex on $T > 0$.

5 The Determination of the Optimal Cycle Time T^* of $\mathrm{TRC}(T)$

Theorem 1. *Suppose that*

$$\frac{W}{D\rho} < M - N.$$

Then each of the following assertions holds true:

1. *If* $\Delta_{12} > 0$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_1^*)$ *and* $T^* = T_1^*$.
2. *If* $\Delta_{23} > 0 \geqq \Delta_{12}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_2^*)$ *and* $T^* = T_2^*$.
3. *If* $\Delta_{34} > 0 \geqq \Delta_{23}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_3^*)$ *and* $T^* = T_3^*$.
4. *If* $\Delta_{45} > 0 \geqq \Delta_{34}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_4^*)$ *and* $T^* = T_4^*$.
5. *If* $0 \geqq \Delta_{45}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_5^*)$ *and* $T^* = T_5^*$.

Proof. Our item-wise proof of Theorem 1 runs as follows.

1. When $\Delta_{12} > 0$, then

$$\Delta_{45} > \Delta_{34} > \Delta_{23} > \Delta_{12} > 0.$$

Lemma 1 and Equation (59) imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, T_1^*]$ and increasing on $\left[T_1^*, \frac{W}{D\rho}\right]$.

(b) $\mathrm{TRC}_2(T)$ is increasing on $\left[\frac{W}{D\rho}, M-N\right]$.

(c) $\mathrm{TRC}_3(T)$ is increasing on $[M-N, M]$.

(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$.

(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equation (14) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, \infty)$, so we have

$$T^* = T_1^* \quad \text{and} \quad \mathrm{TRC}(T^*) = \mathrm{TRC}_1(T_1^*).$$

2. When $\Delta_{23} > 0 > \Delta_{12}$, then

$$\Delta_{45} > \Delta_{34} > \Delta_{23} > 0 > \Delta_{12}.$$

Lemma 1 and Equation (59) imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $\left(0, \frac{W}{D\rho}\right]$;
(b) $\mathrm{TRC}_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_2^*\right]$ and increasing on $[T_2^*, M-N]$;
(c) $\mathrm{TRC}_3(T)$ is increasing on $[M-N, M]$;
(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (14) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_2^*]$ and increasing on $[T_2^*, \infty)$, so we get

$$T^* = T_2^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_2(T_2^*).$$

3. When $\Delta_{34} > 0 > \Delta_{23}$, then

$$\Delta_{45} > \Delta_{34} > 0 > \Delta_{23} > \Delta_{12}.$$

Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $\left(0, \frac{W}{D\rho}\right]$;
(b) $\mathrm{TRC}_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, M-N\right]$.
(c) $\mathrm{TRC}_3(T)$ is decreasing on $[M-N, T_3^*]$ and increasing on $[T_3^*, M]$.
(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$.
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (14) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_3^*]$ and increasing on $[T_3^*, \infty)$, so we have

$$T^* = T_3^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_3(T_3^*).$$

4. When $\Delta_{45} > 0 > \Delta_{34}$, then

$$\Delta_{45} > 0 > \Delta_{34} > \Delta_{23} > \Delta_{12},$$

so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $\left(0, \frac{W}{D\rho}\right]$;
(b) $\mathrm{TRC}_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, M-N\right]$;
(c) $\mathrm{TRC}_3(T)$ is decreasing on $[M-N, M]$.
(d) $\mathrm{TRC}_4(T)$ is decreasing on $(M, T_4^*]$ and increasing on $\left[T_4^*, \frac{PM}{D}\right]$.
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (14) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_4^*]$ and increasing on $[T_4^*, \infty)$, so we have

$$T^* = T_4^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_4(T_4^*).$$

5. When $0 > \Delta_{45}$, then

$$0 > \Delta_{45} > \Delta_{34} > \Delta_{23} > \Delta_{12},$$

so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $\left(0, \frac{W}{D\rho}\right]$;
(b) $\mathrm{TRC}_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, M-N\right]$;
(c) $\mathrm{TRC}_3(T)$ is decreasing on $[M-N, M]$.
(d) $\mathrm{TRC}_4(T)$ is decreasing on $\left[M, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is decreasing on $\left[\frac{PM}{D}, T_5^*\right]$ and increasing on $[T_5^*, \infty)$.

Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (14) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_5^*]$ and increasing on $[T_5^*, \infty)$, so we have

$$T^* = T_5^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_5(T_5^*).$$

Theorem 2. *Suppose that*

$$M - N \leqq \frac{W}{D\rho} < M.$$

Then each of the following assertions holds true:

1. *If* $\Delta_{16} > 0$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_1^*)$ *and* $T^* = T_1^*$.
2. *If* $\Delta_{63} > 0 \geqq \Delta_{16}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_6^*)$ *and* $T^* = T_6^*$.
3. *If* $\Delta_{34} > 0 \geqq \Delta_{63}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_3^*)$ *and* $T^* = T_3^*$.
4. *If* $\Delta_{45} > 0 \geqq \Delta_{34}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_4^*)$ *and* $T^* = T_4^*$.
5. *If* $0 \geqq \Delta_{45}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_5^*)$ *and* $T^* = T_5^*$.

Proof. Our item-wise demonstration of Theorem 2 is presented below.

1. When $\Delta_{16} > 0$, then

$$\Delta_{45} > \Delta_{34} > \Delta_{63} > \Delta_{16} > 0,$$

so Lemma 1 and Equation (59) imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is increasing on $\left[M-N, \frac{W}{D\rho}\right]$;

(c) $\mathrm{TRC}_3(T)$ is increasing on $\left[\dfrac{W}{D\rho}, M\right]$;

(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[M, \dfrac{PM}{D}\right]$;

(e)$\mathrm{TRC}_5(T)$ is increasing on $\left[\dfrac{PM}{D}, \infty\right)$.

Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (20) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, \infty)$, so we have

$$T^* = T_1^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_1(T_1^*).$$

2. When $\Delta_{63} > 0 > \Delta_{16}$, then

$$\Delta_{45} > \Delta_{34} > \Delta_{63} > 0 > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;

(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, T_6^*]$ and increasing on $\left[T_6^*, \dfrac{W}{D\rho}\right]$;

(c) $\mathrm{TRC}_3(T)$ is increasing on $\left[\dfrac{W}{D\rho}, M\right]$;

(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[M, \dfrac{PM}{D}\right]$;

(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\dfrac{PM}{D}, \infty\right)$.

Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (20) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_6^*]$ and increasing on $[T_6^*, \infty)$, so we have

$$T^* = T_6^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_6(T_6^*).$$

3. When $\Delta_{34} > 0 > \Delta_{63}$, then

$$\Delta_{45} > \Delta_{34} > 0 > \Delta_{63} > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;

(b) $\mathrm{TRC}_6(T)$ is decreasing on $\left[M-N, \dfrac{W}{D\rho}\right]$.

(c) $\mathrm{TRC}_3(T)$ is decreasing on $\left[\dfrac{W}{D\rho}, T_3^*\right]$ and increasing on $[T_3^*, M]$;

(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[M, \dfrac{PM}{D}\right]$.

(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\dfrac{PM}{D}, \infty\right)$.

Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (20) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_3^*]$ and increasing on $[T_3^*, \infty)$, so we have

$$T^* = T_3^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_3(T_3^*).$$

4. When $\Delta_{45} > 0 > \Delta_{34}$, then

$$\Delta_{45} > 0 > \Delta_{34} > \Delta_{63} > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;

(b) $\mathrm{TRC}_6(T)$ is decreasing on $\left[M-N, \dfrac{W}{D\rho}\right]$;

(c) $\mathrm{TRC}_3(T)$ is decreasing on $\left[\dfrac{W}{D\rho}, M\right]$;

(d) $\mathrm{TRC}_4(T)$ is decreasing on $[M, T_4^*]$ and increasing on $\left[T_4^*, \dfrac{PM}{D}\right]$;

(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\dfrac{PM}{D}, \infty\right)$.

Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (20) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_4^*]$ and increasing on $[T_4^*, \infty)$, so we have

$$T^* = T_4^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_4(T_4^*).$$

5. When $0 > \Delta_{45}$, then

$$0 > \Delta_{45} > \Delta_{34} > \Delta_{63} > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;

(b) $\mathrm{TRC}_6(T)$ is decreasing on $\left[M-N, \dfrac{W}{D\rho}\right]$;

(c) $\mathrm{TRC}_3(T)$ is decreasing on $\left[\dfrac{W}{D\rho}, M\right]$;

(d) $\mathrm{TRC}_4(T)$ is decreasing on $\left[M, \dfrac{PM}{D}\right]$;

(e) $\mathrm{TRC}_5(T)$ is decreasing on $\left[\dfrac{PM}{D}, T_5^*\right]$ and increasing on $[T_5^*, \infty)$.

Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (20) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_5^*]$ and increasing on $[T_5^*, \infty)$, so we get

$$T^* = T_5^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_5(T_5^*).$$

Theorem 3. *Suppose that*

$$M \leqq \frac{W}{D\rho} < \frac{PM}{D}.$$

Then each of the following assertions holds true:

1. *If* $\Delta_{16} > 0$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_1^*)$ *and* $T^* = T_1^*$.
2. *If* $\Delta_{67} > 0 \geqq \Delta_{16}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_6^*)$ *and* $T^* = T_6^*$.
3. *If* $\Delta_{74} > 0 \geqq \Delta_{67}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_7^*)$ *and* $T^* = T_7^*$.

4. *If* $\Delta_{45} > 0 \geqq \Delta_{74}$*, then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_4^*)$ *and* $T^* = T_4^*$.
5. *If* $0 \geqq \Delta_{45}$*, then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_5^*)$ *and* $T^* = T_5^*$.

Proof. Our item-wise proof of Theorem 3 is presented below.

1. When $\Delta_{16} > 0$, then
$$\Delta_{45} > \Delta_{74} > \Delta_{67} > \Delta_{16} > 0,$$
so Lemma 1 and Equation (59) imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is increasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is increasing on $\left[M, \frac{W}{D\rho}\right]$;
(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.
Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (22) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, \infty)$, so we have
$$T^* = T_1^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_1(T_1^*).$$
2. When $\Delta_{67} > 0 > \Delta_{16}$, then
$$\Delta_{45} > \Delta_{74} > \Delta_{67} > 0 > \Delta_{16},$$
so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, T_6^*]$ and increasing on $[T_6^*, M]$;
(c) $\mathrm{TRC}_7(T)$ is increasing on $\left[M, \frac{W}{D\rho}\right]$;
(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.
Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (22) and the above conclusions would have $\mathrm{TRC}(T)$ is decreasing on $(0, T_6^*]$ and increasing on $[T_6^*, \infty)$, so we have
$$T^* = T_6^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_6(T_6^*).$$
3. When $\Delta_{74} > 0 > \Delta_{67}$, then
$$\Delta_{45} > \Delta_{74} > 0 > \Delta_{67} > \Delta_{16},$$
so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is decreasing on $[M, T_7^*]$ and increasing on $\left[T_7^*, \frac{W}{D\rho}\right]$;
(d) $\mathrm{TRC}_4(T)$ is increasing on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.
Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (22) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_7^*]$ and increasing on $[T_7^*, \infty)$, so we have
$$T^* = T_7^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_7(T_7^*).$$
4. When $\Delta_{45} > 0 > \Delta_{74}$, then
$$\Delta_{45} > 0 > \Delta_{74} > \Delta_{67} > \Delta_{16},$$
so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is decreasing on $\left[M, \frac{W}{D\rho}\right]$;
(d) $\mathrm{TRC}_4(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_4^*\right]$ and increasing on $\left[T_4^*, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.
Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (22) and the above conclusions would show $\mathrm{TRC}(T)$ is decreasing on $(0, T_4^*]$ and increasing on $[T_4^*, \infty)$, so we have
$$T^* = T_4^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_4(T_4^*).$$
5. When $0 > \Delta_{45}$, then
$$0 > \Delta_{45} > \Delta_{74} > \Delta_{67} > \Delta_{16},$$
so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is decreasing on $\left[M, \frac{W}{D\rho}\right]$;
(d) $\mathrm{TRC}_4(T)$ is decreasing on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is decreasing on $\left[\frac{PM}{D}, T_5^*\right]$ and increasing on $[T_5^*, \infty)$.
Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (22) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_5^*]$ and increasing on $[T_5^*, \infty)$, so we have
$$T^* = T_5^* \qquad \text{and} \qquad \mathrm{TRC}(T^*) = \mathrm{TRC}_5(T_5^*).$$

Theorem 4. *Suppose that*

$$\frac{PM}{D} \leqq \frac{W}{D\rho}.$$

Then each of the following assertions holds true:

1. *If* $\Delta_{16} > 0$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_1^*)$ *and* $T^* = T_1^*$.
2. *If* $\Delta_{67} > 0 \geqq \Delta_{16}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_6^*)$ *and* $T^* = T_6^*$.
3. *If* $\Delta_{78} > 0 \geqq \Delta_{67}$ *and* $\Delta_{85} > 0 \geqq \Delta_{67}$, *then* $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}(T_7^*)\}$ *and* $T^* = T_7^*$.
4. *If* $\Delta_{85} > 0 \geqq \Delta_{78}$, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_8^*)$ *and* $T^* = T_8^*$.
5. *If* $\Delta_{78} > 0 \geqq \Delta_{85}$, *then* $\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}(T_7^*), \mathrm{TRC}(T_5^*)\}$ *and* $T^* = T_7^*$ *or* T_5^*.
6. *If* $0 \geqq \Delta_{85}$ *and* Δ_{78}, *then* $\mathrm{TRC}(T^*) = \mathrm{TRC}(T_5^*)$ *and* $T^* = T_5^*$.

Proof. Our item-wise demonstration of Theorem 4 is presented below.

1. When $\Delta_{16} > 0$, then

$$\Delta_{78} > \Delta_{67} > \Delta_{16} > 0$$

and

$$\Delta_{85} > \Delta_{67} > \Delta_{16} > 0,$$

so Lemma 1 and Equation (59) imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is increasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$;
(d) $\mathrm{TRC}_8(T)$ is increasing on $\left[\frac{PM}{D}, \frac{W}{D\rho}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{W}{D\rho}, \infty\right)$.
Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (24) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_1^*]$ and increasing on $[T_1^*, \infty)$, so we have

$$T^* = T_1^* \quad \text{and} \quad \mathrm{TRC}(T^*) = \mathrm{TRC}_1(T_1^*).$$

2. When $\Delta_{67} > 0 > \Delta_{16}$, then

$$\Delta_{78} > \Delta_{67} > 0 > \Delta_{16}$$

and

$$\Delta_{85} > \Delta_{67} > 0 > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, T_6^*]$ and increasing on $[T_6^*, M]$;
(c) $\mathrm{TRC}_7(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$;
(d) $\mathrm{TRC}_8(T)$ is increasing on $\left[\frac{PM}{D}, \frac{W}{D\rho}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{W}{D\rho}, \infty\right)$.
Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (24) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_6^*]$ and increasing on $[T_6^*, \infty)$, so we have

$$T^* = T_6^* \quad \text{and} \quad \mathrm{TRC}(T^*) = \mathrm{TRC}_6(T_6^*).$$

3. When $\Delta_{78} > 0 > \Delta_{67}$, then

$$\Delta_{78} > 0 > \Delta_{67} > \Delta_{16}$$

and

$$\Delta_{85} > 0 > \Delta_{67} > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is decreasing on $[M, T_7^*]$ and increasing on $\left[T_7^*, \frac{PM}{D}\right]$;
(d) $\mathrm{TRC}_8(T)$ is increasing on $\left[\frac{PM}{D}, \frac{W}{D\rho}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{W}{D\rho}, \infty\right)$.
Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (24) and the above conclusions would have $\mathrm{TRC}(T)$ is decreasing on $(0, T_7^*]$ and increasing on $[T_7^*, \infty)$, so $T^* = T_7^*$ and $\mathrm{TRC}(T^*) = \mathrm{TRC}_7(T_7^*)$.

4. When $\Delta_{85} > 0 > \Delta_{78}$, then

$$\Delta_{85} > 0 > \Delta_{78} > \Delta_{67} > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that
(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is decreasing on $\left[M, \frac{PM}{D}\right]$;
(d) $\mathrm{TRC}_8(T)$ is decreasing on $\left[\frac{PM}{D}, T_8^*\right]$ and increasing on $\left[T_8^*, \frac{PM}{D}\right]$;
(e) $\mathrm{TRC}_5(T)$ is increasing on $\left[\frac{W}{D\rho}, \infty\right)$.
Since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (24) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_8^*]$ and increasing on $[T_8^*, \infty)$, so we have $T^* = T_8^*$ and $\mathrm{TRC}(T^*) = \mathrm{TRC}_8(T_8^*)$. When $\Delta_{78} > 0 > \Delta_{85}$, then

$$\Delta_{78} > 0 > \Delta_{85}\Delta_{67} > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, M]$.
(c) $\mathrm{TRC}_7(T)$ is decreasing on $[M, T_7^*]$ and increasing on $\left[T_7^*, \frac{PM}{D}\right]$;
(d) $\mathrm{TRC}_8(T)$ is increasing on $\left[\frac{PM}{D}, T_8^*\right]$ and decreasing on $\left[T_8^*, \frac{W}{D\rho}\right]$;
(e) $\mathrm{TRC}_5(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_5^*\right]$ and increasing on $[T_5^*, \infty)$.

Because $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (24) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_7^*]$ and increasing on $[T_7^*, T_8^*]$, decreasing on $[T_8^*, T_5^*]$ and increasing on $[T_5^*, \infty)$, so we get

$$\mathrm{TRC}(T^*) = \min\{\mathrm{TRC}_7(T_7^*), \mathrm{TRC}_5(T_5^*)\},$$

which yields $T^* = T_5^*$ or T_7^*.

5. When $0 > \Delta_{85}$, then

$$0 > \Delta_{78} > \Delta_{67} > \Delta_{16}$$

and

$$0 > \Delta_{85} > \Delta_{67} > \Delta_{16},$$

so Lemmas 1 and 2, together with Equation (59), imply that

(a) $\mathrm{TRC}_1(T)$ is decreasing on $(0, M-N]$;
(b) $\mathrm{TRC}_6(T)$ is decreasing on $[M-N, M]$;
(c) $\mathrm{TRC}_7(T)$ is decreasing on $\left[M, \frac{PM}{D}\right]$;
(d) $\mathrm{TRC}_8(T)$ is decreasing on $\left[\frac{PM}{D}, \frac{W}{D\rho}\right]$;
(e) $\mathrm{TRC}_5(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_5^*\right]$ and increasing on $[T_5^*, \infty)$.

Finally, since $\mathrm{TRC}(T)$ is continuous on $T > 0$, Equations (24) and the above conclusions would show that $\mathrm{TRC}(T)$ is decreasing on $(0, T_5^*]$ and increasing on $[T_5^*, \infty)$, so we have

$$T^* = T_5^* \quad \text{and} \quad \mathrm{TRC}(T^*) = \mathrm{TRC}_5(T_5^*).$$

6 The Sensitivity Analyses

We first set $D = 1000$ units/year, $P = 3000$ units/year, $A = \$400$/order, $c = \$10$/unit, $s = \$30$/unit, $h = \$40$/unit/year, $k = \$80$/unit/year, $i_k = 0.08\%$/year, $i_e = 0.04\%$/year, $M = 0.2$ year, $N = 0.1$ year and $W = 400$ units. We then increase/decrease the parameters by 25% and 50% at the same time to execute the sensitivity analyses. Based upon the computational results as shown in Figure 8 and Figure 9, we can get the following result in Table 1.

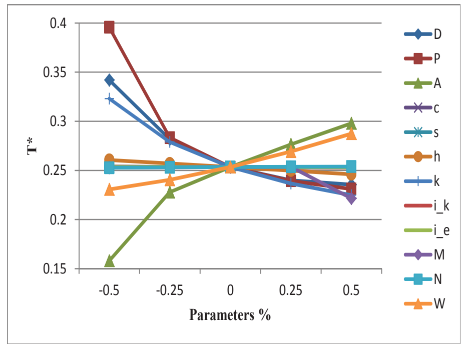

Fig. 8: The sensitivity analysis for T^*

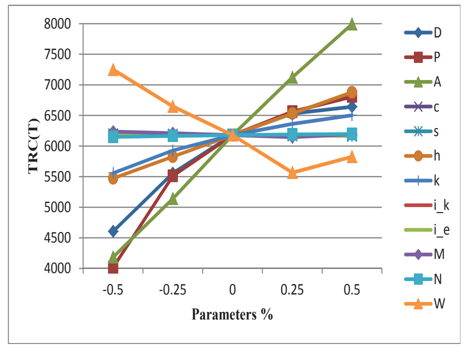

Fig. 9: The sensitivity analysis for T^*

Table 1: Comparison of relative parameters impact to T^* and $\mathrm{TRC}(T)$ in the sensitive analyses.

Impact	T^*	$\mathrm{TRC}(T)$
Positive & Major	A	D, P, A, h, k
Positive & Minor	W	c, I_k, I_e, N
Negative & Minor	c, s, h, I_k, I_e, M, N	s
Negative & Major	D, P, k	M, W

7 Perspective

The EPQ model in this paper is based upon two levels of trade-credit with finite replenishment rate and considers alternate due date of payment different from the payment terms used by Teng and Goyal [6], in which finite replenishment rate and limited storage capacity are used together in order to reflect the real situation. Our results are stated and proved as two lemmas (see Lemmas 1 and 2) and four theorems (see Theorems 1 to 4). Comparison of the relative parameters' impact to T^* and $\mathrm{TRC}(T)$ are identified in the sensitivity analyses so that the practitioners can make their managerial decisions with higher precision. Consequently, the earlier work of Yen *et al.* [18] can be treated as a special case of our present investigation.

References

[1] F. W. Harris, Factory, *The Magazine of Management* **10** (1913), 135–136.

[2] E. W. Taft, *Iron Age* **101** (1918), 1410–1412.

[3] R. V. Hartley, *Operations Research*: *A Managerial Emphasis*, Goodyear Publishing Company, Pacific Palisades, 1976.

[4] S. K. Goyal, *Journal of the Operational Research Society* **36** (1985), 335–338.

[5] Y.-F. Huang, *Journal of the Operational Research Society* **54** (2003), 1011–1015.

[6] J.-T. Teng and S. K. Goyal, *Journal of the Operational Research Society* **58** (2007), 1252–1255.

[7] Y.-F. Huang, *Applied Mathematical Modelling* **30** (2006), 418–436.

[8] Y.-F. Huang, *European Journal of Operational Research* **176** (2007), 1577–1591.

[9] K.-J. Chung and T.-S. Huang, *International Journal of Production Economics* **106** (2007), 127–145.

[10] K.-J. Chung, *International Journal of Production Economics* **114** (2008), 308–312.

[11] Y.-F. Huang, *International Journal of Production Economics* **112** (2008), 655–664.

[12] J.-J. Liao, *International Journal of Production Economics* **113** (2008), 852–861.

[13] H. Soni and N. H. Shah, *European Journal of Operational Research* **184** (2008), 91–100.

[14] J.-J. Liao and K.-J. Chung, *Journal of the Operational Research* **52** (2009), 46–57.

[15] J. Min, Y.-W. Zhou and J. Zhao, *Applied Mathematical Modelling* **34** (2010), 3273–3285.

[16] V. B. Karen and S.-J. Tan, *Expert Systems with Applications* **37** (2010), 5514–5522.

[17] K.-J. Chung, *Expert Systems with Applications* **38** (2011), 13482–13486.

[18] G.-F. Yen, K.-J. Chung and T.-C. Chen, *International Journal of Systems Science* **43** (2012), 2144–2159.

[19] K.-J. Chung, *International Journal of Systems Science* **44** (2013), 1675–1691.

[20] C. K. Maggi, S. K. Goyal and S. K. Goel, *European Journal of Operational Research* **190** (2008), 130–135.

[21] J.-T. Teng, *International Journal of Production Economics* **119** (2009), 415–423.

[22] J.-T. Teng, *European Journal of Operational Research* **195** (2009), 358–363.

[23] A. Thangam and R. Uthayakumar, *Computers and Industrial Engineering* **57** (2009), 773–786.

[24] J.-T. Teng, J. Chen and S. K. Goyal, *Applied Mathematical Modelling* **33** (2009), 4388–4396.

[25] L.-H. Chen and F.-S. King, *European Journal of Operational Research* **205** (2010), 47–58.

[26] K.-J. Chung, J.-J. Liao, P.-S. Ting, S.-D. Lin and H. M. Srivastava, *Applied Mathematics and Computation* **268** (2015), 322–333.

[27] K.-J. Chung, J.-J. Liao, P.-S. Ting, S.-D. Lin and H. M. Srivastava, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas* (*RACSAM*) **112** (2018), 509–538.

[28] K.-J. Chung, S.-D. Lin and H. M. Srivastava, *Applied Mathematics and Computation* **219** (2012), 141–156.

[29] K.-J. Chung, S.-D. Lin and H. M. Srivastava, *Applied Mathematical Modelling* **37** (2013), 10036–10052.

[30] J.-J. Liao, K.-N. Huang, K.-J. Chung, P.-S. Ting, S.-D. Lin and H. M. Srivastava, *Mathematical Methods in the Applied Sciences* **40** (2017), 2122–2139.

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