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On FS Normal Operators

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Abstract: In this paper, we introduce the definition of the FS normal operator, another special type of FS linear operators in the FS Hilbert space based on the FS inner product space definition, initiated by Faried, Ali and Sakr [1]. Moreover, one example supporting FS normal operators and another one that is against them are established. Furthermore, related results including the FS spectral theorems and many other results are investigated. Finally, the connection between the FS normal operators and the FS hermitian operators is introduced.

Keywords: Fuzzy set, FS linear operator, FS set, Soft set

1 Introduction

Zadeh [2] introduced an extension of ordinary sets to deal with uncertainty namely the theory of the fuzzy sets. The fuzzy set is determined by its characteristic function from the domain X to the interval $[0,1]$. After that, Molodtsov [3] proposed another extension of ordinary sets to solve complicated problems and overcome uncertainties, which is the soft set theory. It is very useful in solving difficult issues that can not be solved by ordinary methods in many fields of science such as: physics, decision-making, engineering, medicine, computer science, game theory, economics and many other areas. The soft set is a parameterized collection of subsets of universal set. Thus, Maji, Biswas and Roy [4] combined the fuzzy and soft concepts in one concept and named it FS set. Then, many researchers used the FS notion and introduced some new concepts such as FS point [5], FS normed spaces [6] and FS metric spaces [7]. Recently, Faried, Ali and Sakr [1] gave the definition of FS inner product on FS linear spaces along with introducing the properties and some other related results of them. After that, Faried, Ali and Sakr [8] introduced the FS Hilbert space definition along with establishing the properties and many other related results of it. In addition, Faried, Ali and Sakr [9] defined the FS linear operator in the FS Hilbert space along with its related theorems involving the spectral theory. Finally, Faried, Ali and Sakr defined the FS symmetric operator [10] and the FS hermitian operator [11] along with their related examples and theorems.

In this work, we define another particular FS linear operator as a special type, namely the FS normal operators, establish their related theorems involving the FS point spectrum, one example which is in favor of the FS normal operator and also another one which is not supporting it and show the connection between the FS normal operators and the FS hermitian operators.

2 Preliminaries

In this section, we state definitions, preliminaries and notations which are important in the following obtained results.

Definition 2.1.[4] Suppose that U is a universal set, E is a parameter set. Let $A \subseteq E$. Then, (G,A) is said to be an FS set over U, G is a map defined by $G : A \to \mathscr{F}(U)$, where $\mathscr{F}(U)$ is the collection of all fuzzy subsets of U. In case that all parameter sets are the same, then one can denote (G,A) by $FSS(U)_A = FSS(\tilde{U})$.

Definition 2.2.([1],[5]) $(G,A) \in FSS(\tilde{U})$ is said to be an FS point over the universal set U, denoted by $\tilde{u}_{f_{G(e)}}$, if we have for $e \in A$ and for $u \in U$:

$$f_{G(e)}(u) = \begin{cases} \alpha \ , if\ u = u_0 \in U\ and\ e = e_0 \in A, \\ 0 \ , if\ u \in U - \{u_0\}\ or\ e \in A - \{e_0\} \end{cases},$$

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where $\alpha \in (0,1]$ is the value of degree of membership. One can consider the FS point as the quadruple (u_0, e_0, G, α).

$\mathbb{R}(A)$ denotes the set of all FS real numbers and $\mathbb{C}(A)$ denotes the set of all FS complex numbers. Moreover, $\tilde{\theta} = (\tilde{0},\tilde{0},\tilde{0},\tilde{0})$ and $\tilde{j} = (\tilde{1},\tilde{1},\tilde{1},\tilde{1})$.

Theorem 2.1.[1] Suppose that $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ is an FS inner product space, then

$$|\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>}|^2 \tilde{\leq} \widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^1_{f_{1_{G(e_1)}}}>}\widetilde{<\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>},$$

for every $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} FSV(\tilde{U})$.

Definition 2.3.[1] Suppose that $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ is an FS inner product space and $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{H}$. $\tilde{v}^1_{f_{1_{G(e_1)}}}$ is said to be FS orthogonal to $\tilde{v}^2_{f_{2_{G(e_2)}}}$ (denoted by, $\tilde{v}^1_{f_{1_{G(e_1)}}} \tilde{\perp} \tilde{v}^2_{f_{2_{G(e_2)}}}$) if $\widetilde{<\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}>} \tilde{=} \tilde{0}$.

Theorem 2.2.[1] For FS orthogonal complement, we have the following properties:

1. $\{\tilde{\theta}\}^{\tilde{\perp}} \tilde{=} \tilde{U}$ and $\tilde{U}^{\tilde{\perp}} \tilde{=} \{\tilde{\theta}\}$.
2. $\tilde{\omega} \tilde{\cap} \tilde{\omega}^{\tilde{\perp}} \tilde{=} \{\tilde{\theta}\}$.
3. $\tilde{\omega}^{\tilde{\perp}\tilde{\perp}} \tilde{=} \tilde{\omega}$.
4. If $\tilde{\omega}_1 \tilde{\subseteq} \tilde{\omega}_2$, then $\tilde{\omega}_2^{\tilde{\perp}} \tilde{\subseteq} \tilde{\omega}_1^{\tilde{\perp}}$.

Definition 2.4.[8] Suppose that $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ is an FS inner product space. The FS complete space in the induced FS norm is, then, called an FS Hilbert space, denoted by $\tilde{H}$. Every FS Hilbert space is clearly an FS Banach space.

Theorem 2.3.[9] Suppose that $\tilde{H}$ is an FS inner product space and $\tilde{T} \tilde{\in} \tilde{\mathbb{L}}(\tilde{H})$. We, then, have

$$\tilde{\mathscr{R}}(\tilde{T})^{\tilde{\perp}} \tilde{=} \tilde{\mathscr{N}}(\tilde{T}^{\tilde{*}}),$$

where $\tilde{\mathscr{R}}(\tilde{T})$ is FS range of $\tilde{T}$ and $\tilde{\mathscr{N}}(\tilde{T}^{\tilde{*}})$ is FS kernel of $\tilde{T}^{\tilde{*}}$.

Theorem 2.4.[9] We have

$$\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{T}\|} \tilde{=} \widetilde{\|\tilde{T}\|}^2 \tilde{=} \widetilde{\|\tilde{T}\tilde{T}^{\tilde{*}}\|},$$

where $\tilde{H}$ is an FS Hilbert space and $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$.

Definition 2.5.[9] The eigenvalue $\tilde{\lambda} \tilde{\in} \mathbb{C}(A)$ is called an FS approximate eigenvalue of an FS linear operator $\tilde{T}$ if there exists an FS sequence of FS elements $\{\tilde{v}^n_{f_{n_{G(e_n)}}}\}$ in $\tilde{H}$ such that $\widetilde{\|\tilde{v}^n_{f_{n_{G(e_n)}}}\|} \tilde{=} \tilde{1}$ and $\tilde{T}\tilde{v}^n_{f_{n_{G(e_n)}}} - \tilde{\lambda}\tilde{v}^n_{f_{n_{G(e_n)}}} \to \tilde{0}$. The set of every those $\tilde{\lambda}$ is called the FS approximate point spectrum of an FS linear operator $\tilde{T}$, denoted by $\tilde{\sigma}_a(\tilde{T})$. We have

$$\tilde{\sigma}_a(\tilde{T}) \tilde{\subseteq} \tilde{\sigma}(\tilde{T}). \tag{1}$$

Definition 2.6.[9] Since, for $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$, where $\tilde{H}$ is an FS Hilbert space, $\tilde{\sigma}(\tilde{T})$ is a non-empty FS compact set (FS closed and FS bounded), we can define

$$\tilde{r}_{\tilde{\sigma}}(\tilde{T}) \tilde{=} \widetilde{\sup}\{|\tilde{\lambda}| : \tilde{\lambda} \tilde{\in} \tilde{\sigma}(\tilde{T})\}.$$

We call $\tilde{r}_{\tilde{\sigma}}(\tilde{T})$ the FS spectral radius of an FS linear operator $\tilde{T}$.
Also, we can show that:

$$\tilde{r}_{\tilde{\sigma}}(\tilde{T}) \tilde{=} \lim_{n\to\infty} \widetilde{\|\tilde{T}^n\|}^{\frac{1}{n}}.$$

Definition 2.7.[9] The set of every eigenvalues $\tilde{\lambda} \tilde{\in} \mathbb{C}(A)$ such that $\tilde{\lambda}\tilde{I} - \tilde{T}$ is FS injective, but its FS range is not FS dense in $\tilde{H}$, denoted by $\tilde{\sigma}_r(\tilde{T})$, is said to be the FS residual spectrum of an FS linear operator $\tilde{T}$, i.e.,

$$\tilde{\sigma}_r(\tilde{T}) \tilde{=} \tilde{\sigma}_{Com}(\tilde{T}) \setminus \tilde{\sigma}_p(\tilde{T}). \tag{2}$$

Definition 2.8.[9] The set of every eigenvalues $\tilde{\lambda} \tilde{\in} \mathbb{C}(A)$ such that $\tilde{\lambda}\tilde{I} - \tilde{T}$ is FS injective and has FS dense range in $\tilde{H}$, but is FS singular (i.e. has no FS inverse), denoted by $\tilde{\sigma}_c(\tilde{T})$, is said to be the FS continuous spectrum of an FS linear operator $\tilde{T}$. Note that

$$\tilde{\sigma}_p(\tilde{T}) \tilde{\cup} \tilde{\sigma}_c(\tilde{T}) \tilde{\subseteq} \tilde{\sigma}_a(\tilde{T}), \tag{3}$$

and

$$\tilde{\sigma}(\tilde{T}) \tilde{=} \tilde{\sigma}_p(\tilde{T}) \tilde{\cup} \tilde{\sigma}_c(\tilde{T}) \tilde{\cup} \tilde{\sigma}_r(\tilde{T}), \tag{4}$$

the terms on the right are mutually disjoint.

Definition 2.9.[9] Suppose that $\tilde{H}$ is an FS Hilbert space. If $\tilde{v}_{f_{G(e)}} \tilde{=} (\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}) \tilde{\in} \tilde{H}$. Then, one can define the FS operator $\tilde{\mathrm{R}}$ as follows:

$$\begin{aligned}\tilde{\mathrm{R}}\tilde{v}_{f_{G(e)}} &\tilde{=} \tilde{\mathrm{R}}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}) \\ &\tilde{=} (\tilde{0}, \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}).\end{aligned} \tag{5}$$

The FS operator $\tilde{\mathrm{R}}$ can be considered as an FS right shift operator.

Example 2.1.[9] Suppose that $\tilde{H} \tilde{=} \ell_2(A)$ is the space of every FS square-summable sequences. If $\tilde{v}_{f_{G(e)}} \tilde{=} (\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots) \tilde{\in} \ell_2(A)$. By applying the FS operator (5) from Definition (2.9), we can give the definition of the FS right shift (unilateral shift) operator $\tilde{U}$ in $\ell_2(A)$ as the following:

$$\begin{aligned}\tilde{U}\tilde{v}_{f_{G(e)}} &\tilde{=} \tilde{U}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots) \\ &\tilde{=} (\tilde{0}, \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots).\end{aligned} \tag{6}$$

In addition, we have:

$$\begin{aligned}\tilde{U}^{\tilde{*}}\tilde{v}_{f_{G(e)}} &\tilde{=} \tilde{U}^{\tilde{*}}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots) \\ &\tilde{=} (\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots).\end{aligned} \tag{7}$$

Equation (7) represents the FS adjoint of the FS unilateral shift operator, which is called an FS left shift operator.

Definition 2.10.[11] Suppose that $\tilde{H}$ is an FS Hilbert space. Then, $\tilde{T}\tilde{\in}\tilde{\mathbb{B}}(\tilde{H})$ is said to be an FS hermitian operator if

$$\tilde{T}\tilde{=}\tilde{T}^{\tilde{*}}. \tag{8}$$

3 Main Results

In this section, the definition of the FS normal operator in the FS Hilbert space is introduced. Moreover, some related theorems including its FS eigenvalues and its FS eigenvectors are established. Furthermore, one example which is supporting FS normal operators and also another one which is not supporting them are investigated.

Definition 3.1.(FS normal operator) Suppose that $\tilde{H}$ is an FS Hilbert space and $\tilde{T}\tilde{\in}\tilde{\mathbb{B}}(\tilde{H})$. Then, $\tilde{T}$ is said to be an FS normal operator if

$$\tilde{T}\tilde{T}^{\tilde{*}}\tilde{=}\tilde{T}^{\tilde{*}}\tilde{T}. \tag{9}$$

Remark 3.1. Any FS hermitian operator can be considered as an FS normal operator.

Proof. For, $\tilde{T}\tilde{T}^{\tilde{*}}\tilde{=}\tilde{T}\tilde{T}\tilde{=}\tilde{T}^{\tilde{*}}\tilde{T}$.

Example 3.1. The FS operator $\tilde{2}\tilde{i}\tilde{I}$ is FS normal.

Solution. For,

$$\begin{aligned}(\tilde{2}\tilde{i}\tilde{I})(\tilde{2}\tilde{i}\tilde{I})^{\tilde{*}}&\tilde{=}(\tilde{2}\tilde{i}\tilde{I})(-\tilde{2}\tilde{i}\tilde{I}^{\tilde{*}})\\&\tilde{=}(\tilde{2}\tilde{i}\tilde{I})(-\tilde{2}\tilde{i}\tilde{I})\\&\tilde{=}(-\tilde{2}\tilde{i}\tilde{I})(\tilde{2}\tilde{i}\tilde{I})\\&\tilde{=}(\tilde{2}\tilde{i}\tilde{I})^{\tilde{*}}(\tilde{2}\tilde{i}\tilde{I}).\end{aligned}$$

Example 3.2. The FS right shift operator on FS square-summable sequences space $\ell_2(A)$ is not FS normal.

Solution. Let $\tilde{T}$ be the FS right shift operator on $\ell_2(A)$. From (6) and (7) in Example (2.1), we have:

$$\begin{aligned}\tilde{T}\tilde{v}_{f_{G(e)}}&\tilde{=}\tilde{T}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots)\\&\tilde{=}(\tilde{0},\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots),\end{aligned}$$

and

$$\begin{aligned}\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}&\tilde{=}\tilde{T}^{\tilde{*}}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots)\\&\tilde{=}(\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots),\end{aligned}$$

for every $\tilde{v}_{f_{G(e)}}\tilde{=}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots)\tilde{\in}\ell_2(A)$.

Therefore,

$$\begin{aligned}\tilde{T}\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}&\tilde{=}\tilde{T}(\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}})\\&\tilde{=}\tilde{T}(\tilde{T}^{\tilde{*}}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots))\\&\tilde{=}\tilde{T}(\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots)\\&\tilde{=}(\tilde{0},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots),\end{aligned} \tag{10}$$

and

$$\begin{aligned}\tilde{T}^{\tilde{*}}\tilde{T}\tilde{v}_{f_{G(e)}}&\tilde{=}\tilde{T}^{\tilde{*}}(\tilde{T}\tilde{v}_{f_{G(e)}})\\&\tilde{=}\tilde{T}^{\tilde{*}}(\tilde{T}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots))\\&\tilde{=}\tilde{T}^{\tilde{*}}(\tilde{0},\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots)\\&\tilde{=}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots),\end{aligned} \tag{11}$$

for every $\tilde{v}_{f_{G(e)}}\tilde{=}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots)\tilde{\in}\ell_2(A)$.

Therefore, from (10) and (11), we obtain that:

$$\begin{aligned}(\tilde{0},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots)&\tilde{=}\tilde{T}\tilde{T}^{\tilde{*}}\\&\tilde{\neq}\tilde{T}^{\tilde{*}}\tilde{T}\tilde{=}(\tilde{v}^1_{f_{1_{G(e_1)}}},\tilde{v}^2_{f_{2_{G(e_2)}}},\tilde{v}^3_{f_{3_{G(e_3)}}},\ldots).\end{aligned}$$

Hence, $\tilde{T}$ is not FS normal operator.

Theorem 3.1. Suppose that $\tilde{H}$ is an FS Hilbert space and $\tilde{T}\tilde{\in}\tilde{\mathbb{B}}(\tilde{H})$. Then, $\tilde{T}$ is FS normal operator if and only if $\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}\|}\tilde{=}\widetilde{\|\tilde{T}\tilde{v}_{f_{G(e)}}\|}$, for every $\tilde{v}_{f_{G(e)}}\tilde{\in}\tilde{H}$.

Proof. Let $\tilde{T}$ be an FS normal operator. Then, using (9) from Definition (3.1), we get:

$$\begin{aligned}\widetilde{\|\tilde{T}\tilde{v}_{f_{G(e)}}\|}^2&\tilde{=}<\widetilde{\tilde{T}\tilde{v}_{f_{G(e)}},\tilde{T}\tilde{v}_{f_{G(e)}}}>\\&\tilde{=}<\widetilde{\tilde{v}_{f_{G(e)}},\tilde{T}^{\tilde{*}}\tilde{T}\tilde{v}_{f_{G(e)}}}>\\&\tilde{=}<\widetilde{\tilde{v}_{f_{G(e)}},\tilde{T}\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}}>\\&\tilde{=}<\widetilde{\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}},\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}}>\\&\tilde{=}\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}\|}^2,\end{aligned}$$

for every $\tilde{v}_{f_{G(e)}}\tilde{\in}\tilde{H}$.

Hence, we get that $\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}\|}\tilde{=}\widetilde{\|\tilde{T}\tilde{v}_{f_{G(e)}}\|}$.

Conversely, let $\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}\|}\tilde{=}\widetilde{\|\tilde{T}\tilde{v}_{f_{G(e)}}\|}$, for every $\tilde{v}_{f_{G(e)}}\tilde{\in}\tilde{H}$. Then, we obtain that:

$$\begin{aligned}<\widetilde{\tilde{T}\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>&\tilde{=}<\widetilde{\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}},\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}}>\\&\tilde{=}\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{v}_{f_{G(e)}}\|}^2\\&\tilde{=}\widetilde{\|\tilde{T}\tilde{v}_{f_{G(e)}}\|}^2\\&\tilde{=}<\widetilde{\tilde{T}\tilde{v}_{f_{G(e)}},\tilde{T}\tilde{v}_{f_{G(e)}}}>\\&\tilde{=}<\widetilde{\tilde{T}^{\tilde{*}}\tilde{T}\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>,\end{aligned}$$

for every $\tilde{v}_{f_{G(e)}}\tilde{\in}\tilde{H}$.

Then, $<\widetilde{(\tilde{T}\tilde{T}^{\tilde{*}}-\tilde{T}^{\tilde{*}}\tilde{T})\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>\tilde{=}\tilde{0}$, for every $\tilde{v}_{f_{G(e)}}\tilde{\in}\tilde{H}$. Thus, $(\tilde{T}\tilde{T}^{\tilde{*}}-\tilde{T}^{\tilde{*}}\tilde{T})\tilde{v}_{f_{G(e)}}\tilde{=}\tilde{\theta}$, for every

$\tilde{v}_{f_{G(e)}} \tilde{\in} \tilde{H}$. Therefore, $\tilde{T}\tilde{T}^{\tilde{*}} - \tilde{T}^{\tilde{*}}\tilde{T} \tilde{=} \tilde{\mathbf{0}}$ (FS zero operator). Hence, we get that $\tilde{T}\tilde{T}^{\tilde{*}} \tilde{=} \tilde{T}^{\tilde{*}}\tilde{T}$, i.e. $\tilde{T}$ is FS normal operator.

Theorem 3.2. If $\tilde{T}$ is an FS normal operator in FS Hilbert space, then $\tilde{r}_{\tilde{\sigma}}(\tilde{T}) \tilde{=} \widetilde{\|\tilde{T}\|}$, where $\tilde{r}_{\tilde{\sigma}}(\tilde{T})$ is the FS spectral radius of $\tilde{T}$, stated in Definition (2.6).

Proof. First, by replacing $\tilde{v}_{f_{G(e)}}$ by $\tilde{T}\tilde{v}_{f_{G(e)}}$ in Theorem (3.1), we have:
$\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{T}\tilde{v}_{f_{G(e)}}\|} \tilde{=} \widetilde{\|\tilde{T}^2\tilde{v}_{f_{G(e)}}\|}$, for every $\tilde{v}_{f_{G(e)}} \tilde{\in} \tilde{H}$. So, we have $\widetilde{\|\tilde{T}^2\|} \tilde{=} \widetilde{\|\tilde{T}^{\tilde{*}}\tilde{T}\|}$. Therefore, with help of Theorem (2.4), we obtain that:

$$\begin{aligned}\widetilde{\|\tilde{T}^2\|} &\tilde{=} \widetilde{\|\tilde{T}^{\tilde{*}}\tilde{T}\|} \\ &\tilde{=} \widetilde{\|\tilde{T}\|}^2.\end{aligned} \tag{12}$$

Since $\tilde{T}$ is FS normal operator, $\tilde{T}$ and $\tilde{T}^{\tilde{*}}$ commutes, and $(\tilde{T}^n)^{\tilde{*}} \tilde{=} (\tilde{T}^{\tilde{*}})^n$, then

$$\begin{aligned}(\tilde{T}^n)(\tilde{T}^n)^{\tilde{*}} &\tilde{=} \tilde{T}^n(\tilde{T}^{\tilde{*}})^n \\ &\tilde{=} (\tilde{T}\tilde{T}^{\tilde{*}})^n \\ &\tilde{=} (\tilde{T}^{\tilde{*}}\tilde{T})^n \\ &\tilde{=} (\tilde{T}^{\tilde{*}})^n\tilde{T}^n \\ &\tilde{=} (\tilde{T}^n)^{\tilde{*}}\tilde{T}^n.\end{aligned}$$

Hence, $\tilde{T}^n$ is FS normal operator.
Therefore, by mathematical induction, from (12), we get that $\widetilde{\|\tilde{T}^m\|} \tilde{=} \widetilde{\|\tilde{T}\|}^m$ for every m of the form 2^k, $k = 1, 2, 3, \cdots$.
Now, we have:

$$\begin{aligned}\tilde{r}_{\tilde{\sigma}}(\tilde{T}) &\tilde{=} \lim_n \widetilde{\|\tilde{T}^n\|}^{\frac{1}{n}} \\ &\tilde{=} \lim_n \widetilde{\|\tilde{T}^{2^n}\|}^{\frac{1}{2^n}} \\ &\tilde{=} \lim_n (\widetilde{\|\tilde{T}^{2^n}\|})^{\frac{1}{2^n}} \\ &\tilde{=} \lim_n \widetilde{\|\tilde{T}\|} \\ &\tilde{=} \widetilde{\|\tilde{T}\|}.\end{aligned}$$

Remark 3.2. Since any FS hermitian operator is an FS normal operator, then we get that $\tilde{r}_{\tilde{\sigma}}(\tilde{T}) \tilde{=} \widetilde{\|\tilde{T}\|}$, for any FS hermitian operator $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$.

Theorem 3.3. Suppose that $\tilde{T}$ is an FS normal operator in the FS Hilbert space $\tilde{H}$. If $\tilde{\lambda} \tilde{\in} \tilde{\sigma}_p(\tilde{T})$, then we have $\overline{\tilde{\lambda}} \tilde{\in} \tilde{\sigma}_p(\tilde{T}^{\tilde{*}})$.

Proof. We prove that $(\tilde{T} - \tilde{\lambda}\tilde{I})$ is FS normal as follows:

$$\begin{aligned}(\tilde{T} - \tilde{\lambda}\tilde{I})(\tilde{T} - \tilde{\lambda}\tilde{I})^{\tilde{*}} &\tilde{=} (\tilde{T} - \tilde{\lambda}\tilde{I})(\tilde{T}^{\tilde{*}} - \overline{\tilde{\lambda}}\tilde{I}) \\ &\tilde{=} \tilde{T}\tilde{T}^{\tilde{*}} - \tilde{\lambda}\tilde{T}^{\tilde{*}} - \overline{\tilde{\lambda}}\tilde{T} + |\tilde{\lambda}|^2\tilde{I} \\ &\tilde{=} \tilde{T}^{\tilde{*}}\tilde{T} - \overline{\tilde{\lambda}}\tilde{T} - \tilde{\lambda}\tilde{T}^{\tilde{*}} + |\tilde{\lambda}|^2\tilde{I} \\ &\tilde{=} (\tilde{T}^{\tilde{*}} - \overline{\tilde{\lambda}}\tilde{I})(\tilde{T} - \tilde{\lambda}\tilde{I}) \\ &\tilde{=} (\tilde{T} - \tilde{\lambda}\tilde{I})^{\tilde{*}}(\tilde{T} - \tilde{\lambda}\tilde{I}).\end{aligned}$$

Then, $(\tilde{T} - \tilde{\lambda}\tilde{I})$ is FS normal operator, and using Theorem (3.1), we get that
$\widetilde{\|(\tilde{T} - \tilde{\lambda}\tilde{I})\tilde{v}_{f_{G(e)}}\|} \tilde{=} \widetilde{\|(\tilde{T}^{\tilde{*}} - \overline{\tilde{\lambda}}\tilde{I})\tilde{v}_{f_{G(e)}}\|}$. Now, let $\tilde{\lambda} \tilde{\in} \tilde{\sigma}_p(\tilde{T})$, then $\tilde{T}\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\lambda}\tilde{v}_{f_{G(e)}}$, $\tilde{v}_{f_{G(e)}} \tilde{\neq} \tilde{\theta}$. That is to say that $(\tilde{T} - \tilde{\lambda}\tilde{I})\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{0}$, $\tilde{v}_{f_{G(e)}} \tilde{\neq} \tilde{\theta}$. Therefore, $(\tilde{T}^{\tilde{*}} - \overline{\tilde{\lambda}}\tilde{I})\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{0}$. Hence, $\overline{\tilde{\lambda}}$ is an FS eigenvalue of $\tilde{T}^{\tilde{*}}$, i.e., $\overline{\tilde{\lambda}} \tilde{\in} \tilde{\sigma}_p(\tilde{T}^{\tilde{*}})$.

Theorem 3.4. Suppose that $\tilde{T}$ is an FS normal operator in the FS Hilbert space $\tilde{H}$, then the FS eigenvectors corresponding to different FS eigenvalues are FS orthogonal.

Proof. Suppose that $\tilde{T}$ is an FS normal operator, and $\tilde{T}\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\lambda}\tilde{v}_{f_{G(e)}}$ and $\tilde{T}\tilde{u}_{f_{G(e)}} \tilde{=} \tilde{\mu}\tilde{u}_{f_{G(e)}}$; $\tilde{\lambda} \tilde{\neq} \tilde{\mu}$, $\tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} \tilde{\neq} \tilde{\theta}$. Then, by using Theorem (3.3), we obtain that $\tilde{T}^{\tilde{*}}\tilde{u}_{f_{G(e)}} \tilde{=} \overline{\tilde{\mu}}\tilde{u}_{f_{G(e)}}$, and thus we have:

$$\begin{aligned}\tilde{\lambda} \widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} >} &\tilde{=} \widetilde{< \tilde{\lambda}\tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} >} \\ &\tilde{=} \widetilde{< \tilde{T}\tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} >} \\ &\tilde{=} \widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{T}^{\tilde{*}}\tilde{u}_{f_{G(e)}} >} \\ &\tilde{=} \widetilde{< \tilde{v}_{f_{G(e)}}, \overline{\tilde{\mu}}\tilde{u}_{f_{G(e)}} >} \\ &\tilde{=} \tilde{\mu} \widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} >}.\end{aligned}$$

Therefore, $(\tilde{\lambda} - \tilde{\mu}) \widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} >} \tilde{=} \tilde{0}$. But $\tilde{\lambda} \tilde{\neq} \tilde{\mu}$, i.e., $\tilde{\lambda} - \tilde{\mu} \tilde{\neq} \tilde{0}$, then $\widetilde{< \tilde{v}_{f_{G(e)}}, \tilde{u}_{f_{G(e)}} >} \tilde{=} \tilde{0}$. Thus, $\tilde{v}_{f_{G(e)}}$ and $\tilde{u}_{f_{G(e)}}$ are FS orthogonal.

Theorem 3.5. Let $\tilde{H}$ be an FS Hilbert space and $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$ be FS normal operator. Then, $\tilde{\sigma}_r(\tilde{T}) \tilde{=} \phi$.

Proof. Let $\tilde{\lambda}$ be any FS scalar such that $\tilde{\lambda} \tilde{\in} \tilde{\sigma}_{Com}(\tilde{T})$. Then, the FS range of $\tilde{\lambda}\tilde{I} - \tilde{T}$ is not FS dense in $\tilde{H}$, i.e., $\overline{\tilde{\mathscr{R}}(\tilde{\lambda}\tilde{I} - \tilde{T})} \tilde{\neq} \tilde{H}$, and thus, by using (1) from Theorem (2.2), we have $\overline{\tilde{\mathscr{R}}(\tilde{\lambda}\tilde{I} - \tilde{T})}^{\tilde{\perp}} \tilde{\neq} \{\tilde{\theta}\}$. But, from Theorem (2.3), we obtain:

$$\begin{aligned}\{\tilde{\theta}\} &\tilde{\neq} \overline{\tilde{\mathscr{R}}(\tilde{\lambda}\tilde{I} - \tilde{T})}^{\tilde{\perp}} \\ &\tilde{\subset} \tilde{\mathscr{R}}(\tilde{\lambda}\tilde{I} - \tilde{T})^{\tilde{\perp}} \\ &\tilde{=} \tilde{\mathscr{N}}(\overline{\tilde{\lambda}}\tilde{I} - \tilde{T}^{\tilde{*}}).\end{aligned}$$

Therefore, $\tilde{\mathcal{N}}(\overline{\tilde{\lambda}}\tilde{I}-\tilde{T}^{\tilde{*}})\tilde{\neq}\{\tilde{\theta}\}$. Since $\tilde{T}$ is FS normal operator, then $(\tilde{\lambda}\tilde{I}-\tilde{T})$ is also FS normal operator, and $\tilde{\mathcal{N}}(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{=}\tilde{\mathcal{N}}(\overline{\tilde{\lambda}}\tilde{I}-\tilde{T}^{\tilde{*}})$ from Theorem (3.3). Then, $\tilde{\mathcal{N}}(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{\neq}\{\tilde{\theta}\}$, and so there is some non-zero FS elements $\tilde{v}_{f_{G(e)}}\tilde{\in}\tilde{H}$ such that $(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{v}_{f_{G(e)}}\tilde{=}\tilde{0}$, that is to say that $\tilde{T}\tilde{v}_{f_{G(e)}}\tilde{=}\tilde{\lambda}\tilde{v}_{f_{G(e)}}$, i.e., $\tilde{\lambda}\tilde{\in}\tilde{\sigma}_p(\tilde{T})$. Hence, for any FS scalar with $\overline{\tilde{\mathscr{R}}(\tilde{\lambda}\tilde{I}-\tilde{T})}\tilde{\neq}\tilde{H}$ (i.e., $\tilde{\lambda}\tilde{\in}\tilde{\sigma}_{Com}(\tilde{T})$), we have $\tilde{\lambda}\tilde{\in}\tilde{\sigma}_p(\tilde{T})$. Therefore, by using Equation (2) from Definition (2.7) of FS residual spectrum, we obtain that $\tilde{\sigma}_r(\tilde{T})\tilde{=}\phi$.

Corollary 3.1. Suppose that $\tilde{T}$ is an FS normal operator in the FS Hilbert space $\tilde{H}$, then we have $\tilde{\sigma}(\tilde{T})\tilde{=}\tilde{\sigma}_a(\tilde{T})$.

Proof. Since $\tilde{T}$ is FS normal operator, then, by using the above Theorem (3.5), we have $\tilde{\sigma}_r(\tilde{T})\tilde{=}\phi$. Now, recall Equations (3) and (4) from Definition (2.8) of FS continuous spectrum which are $\tilde{\sigma}_p(\tilde{T})\tilde{\cup}\tilde{\sigma}_c(\tilde{T})\tilde{\subset}\tilde{\sigma}_a(\tilde{T})$ and $\tilde{\sigma}(\tilde{T})\tilde{=}\tilde{\sigma}_p(\tilde{T})\tilde{\cup}\tilde{\sigma}_c(\tilde{T})\tilde{\cup}\tilde{\sigma}_r(\tilde{T})$. Therefore, $\tilde{\sigma}(\tilde{T})\tilde{\subseteq}\tilde{\sigma}_a(\tilde{T})\tilde{\cup}\tilde{\sigma}_r(\tilde{T})\tilde{=}\tilde{\sigma}_a(\tilde{T})$, that is to say that $\tilde{\sigma}(\tilde{T})\tilde{\subset}\tilde{\sigma}_a(\tilde{T})$. But, from Equation (1) in Definition (2.5), we obtain that $\tilde{\sigma}_a(\tilde{T})\tilde{\subset}\tilde{\sigma}(\tilde{T})$. Thus, $\tilde{\sigma}(\tilde{T})\tilde{=}\tilde{\sigma}_a(\tilde{T})$.

Theorem 3.6. Suppose that $\tilde{T}$ is an FS hermitian operator in the FS Hilbert space $\tilde{H}$, then $\tilde{\sigma}(\tilde{T})\tilde{=}\tilde{\sigma}_a(\tilde{T})\tilde{\subset}\mathbb{R}(A)$.

Proof. First, since we know from Remark (3.1) that any FS hermitian operator is an FS normal operator, then, by using Corollary (3.1), we get that $\tilde{\sigma}(\tilde{T})\tilde{=}\tilde{\sigma}_a(\tilde{T})$.
Now, we prove that $\tilde{\sigma}(\tilde{T})\tilde{\subset}\mathbb{R}(A)$. We have for any $\tilde{\lambda}\tilde{\in}\mathbb{C}(A)$ with $Im\tilde{\lambda}\tilde{\neq}\tilde{0}$ and for $\tilde{v}_{f_{G(e)}}\tilde{\neq}\tilde{\theta}$, that

$$\begin{aligned}
&\tilde{0}\tilde{<}|\tilde{\lambda}-\overline{\tilde{\lambda}}|\|\widetilde{\tilde{v}_{f_{G(e)}}}\|^2\tilde{=}|\tilde{\lambda}-\overline{\tilde{\lambda}}|<\widetilde{\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>\\
&\tilde{=}|\tilde{\lambda}<\widetilde{\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>-\overline{\tilde{\lambda}}<\widetilde{\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>|\\
&\tilde{=}|\tilde{\lambda}<\widetilde{\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>-<\widetilde{\tilde{T}\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>\\
&-\overline{\tilde{\lambda}}<\widetilde{\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>+<\widetilde{\tilde{T}\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>|\\
&\tilde{=}|<\widetilde{(\tilde{\lambda}-\tilde{T})\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>-<\widetilde{(\overline{\tilde{\lambda}}-\tilde{T})\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>|\\
&\tilde{=}|<\widetilde{(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>-<\widetilde{\tilde{v}_{f_{G(e)}},(\tilde{\lambda}\tilde{I}-\tilde{T}^{\tilde{*}})\tilde{v}_{f_{G(e)}}}>|\\
&\tilde{=}|<\widetilde{(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>-<\widetilde{\tilde{v}_{f_{G(e)}},(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{v}_{f_{G(e)}}}>|\\
&\tilde{\leq}|<\widetilde{(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{v}_{f_{G(e)}},\tilde{v}_{f_{G(e)}}}>|\\
&+|-<\widetilde{\tilde{v}_{f_{G(e)}},(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{v}_{f_{G(e)}}}>|.
\end{aligned}$$

Then, by using FS Cauchy-Schwartz Inequality in Theorem (2.1), we have:

$$\tilde{0}\tilde{<}|\tilde{\lambda}-\overline{\tilde{\lambda}}|\|\widetilde{\tilde{v}_{f_{G(e)}}}\|^2\tilde{\leq}\tilde{2}\|\widetilde{(\tilde{\lambda}\tilde{I}-\tilde{T})\tilde{v}_{f_{G(e)}}}\|\|\widetilde{\tilde{v}_{f_{G(e)}}}\|. \tag{13}$$

If $\tilde{\lambda}\tilde{\in}\tilde{\sigma}(\tilde{T})\tilde{=}\tilde{\sigma}_a(\tilde{T})$, thus, by using Definition (2.5) of FS approximate point spectrum, we obtain that there exists an FS sequence of FS elements $\{\tilde{v}^n_{f_{n_{G(e_n)}}}\}$ in $\tilde{H}$ such that $\|\widetilde{\tilde{v}^n_{f_{n_{G(e_n)}}}}\|\tilde{=}\tilde{1}$ and $\|(\tilde{\lambda}\tilde{I}-\widetilde{\tilde{T})\tilde{v}^n_{f_{n_{G(e_n)}}}}\|\to\tilde{0}$. Therefore, from (13), we get:

$$\tilde{0}\tilde{<}|\tilde{\lambda}-\overline{\tilde{\lambda}}|\tilde{\leq}\tilde{2}\|(\tilde{\lambda}\tilde{I}-\widetilde{\tilde{T})\tilde{v}^n_{f_{n_{G(e_n)}}}}\|\to\tilde{0}.$$

Thus, $|\tilde{\lambda}-\overline{\tilde{\lambda}}|\tilde{=}\tilde{0}$, i.e., $\tilde{\lambda}-\overline{\tilde{\lambda}}\tilde{=}\tilde{0}$. That is to say that $\tilde{\lambda}\tilde{=}\overline{\tilde{\lambda}}$, i.e., $\tilde{\lambda}$ is FS real scalar. Hence, $\tilde{\sigma}(\tilde{T})\tilde{\subset}\mathbb{R}(A)$ and this completes the proof.

4 Conclusions

The soft or fuzzy versions of some topics such as normed space, metric space, or Hilbert space has been introduced by many researchers. While, combining the fuzzy and the soft concepts together in one concept gives more generalized, extended and more accurate results. Some mathematicians have studied a few of these generalized extended definitions. In our paper, one of the special types of the FS linear operators, namely the FS normal operator is defined. Furthermore, one example which is in favor of it and also another one that is against it, are investigated. In the end, related results involving the FS spectral theorems and the relations between the FS normal operator and the FS hermitian operator are introduced.

Conflict of interest

The authors declare that they have no conflict of interest.

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