The object of the present paper is to use the ideal properties to study ideal bitopological ordered space $(X, \tau_1, \tau_2, R, I)$ which is a generalization of the study of bitopological ordered spaces $(X, \tau_1, \tau_2, R)$ and bitopological space $(X, \tau_1, \tau_2)$. Every ideal bitopological ordered space $(X, \tau_1, \tau_2, R, I)$ can be regarded as a bitopological space $(X, \tau_1, \tau_2)$ if $R$ is the equality relation "$\Delta$" and every bitopological space $(X, \tau_1, \tau_2)$ can be regarded as a topological space $(X, \tau)$ if $\tau_1 = \tau_2 = \tau$. Also, every bitopological ordered space $(X, \tau_1, \tau_2, R)$ can be regarded as a topological ordered space $(X, \tau, R)$ if $\tau_1 = \tau_2$. The relationship between these axioms and the axioms in [6, 12] have been obtained. Moreover, we show that the properties of being $\mathcal{PT}_i$-ordered spaces, $i = 0, 1, 2$ are preserved under a bijective, P-open and order (reverse) embedding mappings (see Theorems 3.2, 3.6). Furthermore, it is proved that the property of being $\mathcal{PT}_i$-ordered spaces, $i = 0, 1, 2$ is hereditary property (see Theorems 3.4, 3.7).
2 Preliminaries

In this section, we collect the relevant definitions and results from bitopological ordered spaces, lower separation axioms and mappings.

Definition 2.1.[10] Let $(X, R)$ be a poset. A set $A \subseteq X$ is said to be

1. Increasing if for every $a \in A$ and $x \in X$ such that $aRx$, then $x \in A$.
2. Decreasing if for every $a \in A$ and $x \in X$ such that $xRa$, then $x \in A$.

Definition 2.2. A mapping $f : (X, R) \rightarrow (Y, R')$ is called

1. Increasing (decreasing) if $\forall x_1, x_2 \in X$ such that $x_1Rx_2 \Rightarrow f(x_1)R'f(x_2)$ [10].
2. Order embedding if $\forall x_1, x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_1)R'f(x_2)$ [13].
3. Order reverse embedding if $\forall x_1, x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_2)R'f(x_1)$ [11].

Definition 2.3. [4] Let $X$ be a non-empty set. A class $\tau$ of subsets of $X$ is called a topology on $X$ iff $\tau$ satisfies the following axioms.

1. $X, \emptyset \in \tau$.
2. An arbitrary union of the members of $\tau$ is in $\tau$.
3. The intersection of any two sets in $\tau$ is in $\tau$.

The members of $\tau$ are then called $\tau$-open sets, or simply open sets. The pair $(X, \tau)$ is called a topological space. A subset $A$ of a topological space $(X, \tau)$ is called a closed set if its complement $A'$ is an open set.

Definition 2.4.[7] A bitopological space (bts, for short) is a triple $(X, \tau_1, \tau_2)$, where $\tau_1$ and $\tau_2$ are arbitrary topologies for a set $X$.

Definition 2.5.[8, 11] A function $f : (X_1, \tau_1, \tau_2) \rightarrow (X_2, \eta_1, \eta_2)$ is called

1. p.continuous (respectively p.open, p.closed) if $f : (X_1, \tau_1) \rightarrow (X_2, \eta_1), i = 1, 2$ are continuous (respectively open, closed).
2. p.homeomorphism if $f : (X_1, \tau_1) \rightarrow (X_2, \eta_1), i = 1, 2$ are homeomorphisms.

Definition 2.6.[12] A bitopological ordered space (bto-space, for short) has the form $(X, \tau_1, \tau_2, R)$, where $(X, R)$ is a poset and $(X, \tau_1, \tau_2)$ is a bts.

Definition 2.7.[12] A bto-space $(X, \tau_1, \tau_2, R)$ is said to be

1. Lower pairwise $T_l(PT_1$, for short)-ordered space if for every $a, b \in X$ such that $aRb$, there exists an increasing $\tau_1$-nbd $U$ of $a$ such that $b \not\in U, i = 1$ or 2.
2. Upper pairwise $T_l(UPPT_1$, for short)-ordered space if for every $a, b \in X$ such that $aRb$, there exists a decreasing $\tau_2$-nbd $V$ of $b$ such that $a \not\in V, i = 1$ or 2.

Definition 2.8.[12] A bto-space $(X, \tau_1, \tau_2, R)$ is said to be $PT_1$-ordered space if it is $LPT_1$ or $UPPT_1$ ordered space.

Definition 2.9.[12] A bto-space $(X, \tau_1, \tau_2, R)$ is said to be pairwise $T_1(PT_1$, for short), if it is $LPT_1$ and $UPPT_1$-ordered space.

Definition 2.10.[12] A bto-space $(X, \tau_1, \tau_2, R)$ is said to be pairwise $T_2(PT_2$, for short), if for every $a, b \in X$ such that $aRb$, there exist an increasing $\tau_1$-nbd $U$ of $a$ and a decreasing $\tau_2$-nbd $V$ of $b$ such that $U \cap V = \emptyset$.

Definition 2.11.[5] A non-empty collection $\mathcal{I}$ of subsets of a set $X$ is called an ideal on $X$, if it satisfies the following conditions

1. $\emptyset \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$.
2. $\mathcal{I}$ is an ideal on $X$.

Definition 2.12.[2] Let $(X, R)$ be a poset and $\mathcal{I} \subseteq P(X)$ be an ideal on $X$. Then, a set $A \subseteq X$ is called:

1. $\mathcal{I}$-increasing set iff $aR \cap A' \in \mathcal{I} \forall a \in A$, where $aR = \{b : (a, b) \in R\}$.
2. $\mathcal{I}$-decreasing set iff $Ra \cap A' \in \mathcal{I} \forall a \in A$, where $Ra = \{b : (b, a) \in R\}$.

Theorem 2.1.[2] Let $f : (X, R, \mathcal{I}) \rightarrow (Y, R', f(\mathcal{I}))$ be a bijective function and order embedding. Then for every $\mathcal{I}$-increasing (decreasing) subset $A$ of $X$, $f(A)$ is $f(\mathcal{I})$-increasing (decreasing) subset of $Y$.

Corollary 2.1.[2] Let $f : (X, R, \mathcal{I}) \rightarrow (Y, R', f(\mathcal{I}))$ be a bijective function and order embedding. If $B \subseteq Y$ is $f(\mathcal{I})$-increasing (decreasing), then $f^{-1}(B)$ is $\mathcal{I}$-increasing (decreasing) subset of $X$.

Theorem 2.2.[2] Let $f : (X, R, \mathcal{I}) \rightarrow (Y, R', f(\mathcal{I}))$ be a bijective function and order reverse embedding. Then for every $\mathcal{I}$-increasing (decreasing) subset $A$ of $X$, $f(\mathcal{I})$ is $f^{-1}(\mathcal{I})$-decreasing (increasing) subset of $Y$.

Corollary 2.2.[2] Let $f : (X, R, \mathcal{I}) \rightarrow (Y, R', f(\mathcal{I}))$ be a bijective function and order reverse embedding. If $B \subseteq Y$ is $f(\mathcal{I})$-increasing (decreasing), then $f^{-1}(B)$ is $\mathcal{I}$-decreasing (increasing) subset of $X$.

3 $\mathcal{I}$-P-Separation axioms

The aim of this section is to introduce new separation axioms $\mathcal{I}PT_i$-ordered spaces, $i = 0, 1, 2$ on the space $(X, \tau_1, \tau_2, R, \mathcal{I})$ which based on the notion of $\mathcal{I}$-increasing (decreasing) sets [2]. In addition, the relationship between these axioms and the axioms in [12] are obtained. Moreover, it is proved that the property of being $\mathcal{I}PT_i$-ordered spaces, $i = 0, 1, 2$ is invariant under a bijective, $P$-open and order embedding mapping (order reverse embedding mapping). Furthermore, it is proved that the property of being $\mathcal{I}PT_i$-ordered spaces, $i = 0, 1, 2$ is hereditary property.

Definition 3.1. A space $(X, \tau_1, \tau_2, R, \mathcal{I})$ is called an ideal bitopological ordered space if $(X, \tau_1, \tau_2, R)$ is a bitopological ordered space and $\mathcal{I} \subseteq P(X)$ is an ideal on $X$.

Remark 3.1. Every ideal bitopological ordered space
(X, τ₁, τ₂, R, ℱ) can be regarded as a bitopological ordered space (X, τ₁, τ₂, R) if ℱ = {φ} and can be regarded as bitopological space (X, τ₁, τ₂) if ℱ = {φ}, R is the equality relation “Δ”.

**Definition 3.2.** An ideal bitopological ordered space (X, τ₁, τ₂, R, ℱ) is said to be

1. ℱ lower PT₁(ℱPT₁), for short)-ordered space if for every a, b ∈ X such that aRb, there exists an ℱ-increasing τ₁-open set U such that a ∈ U and b ⊈ U, i = 1 or 2.

2. ℱ upper PT₁(ℱPT₁) for short)-ordered space if for every a, b ∈ X such that aRb, there exists an ℱ-decreasing τ₁-open set V such that b ∈ V and a ⊈ V, i = 1 or 2.

**Definition 3.3.** (X, τ₁, τ₂, R, ℱ) is said to be ℱPT₀-ordered space if it is ℱLPT₁ or ℱUPT₁ ordered space.

**Example 3.1.** Let X = {1, 2, 3, 4}, R = Δ ∪ {(1, 4), (1, 3), (2, 3), (3, 1)}, ℱ = {φ, {1}, {3}, {1, 3}}, τ₁ = {X, φ, {1}, {1, 2}, {1, 4}, {1, 2, 4}}, τ₂ = {X, φ, {2, 3}, {1, 2, 3}, {2, 3, 4}}, then, (X, τ₁, τ₂, R, ℱ) is ℱUPT₀-ordered space and consequently it is ℱPT₀-ordered space.

**Example 3.2.** In Example 3, let ℱ = {φ, {3}, {4}}, {3, 4}, τ₁ = {X, φ, {3}, {1, 2}, {1, 2, 3}, {1, 2, 4}}, τ₂ = {X, φ, {3, 4}, {1, 3, 4}, {2, 3, 4}}, then, (X, τ₁, τ₂, R, ℱ) is ℱLPT₀-ordered space and consequently it is ℱPT₀-ordered space.

The following proposition gives the relationship between Definition 3.3 and Definition 2.8 [12].

**Proposition 3.1.** Let (X, τ₁, τ₂, R, ℱ) be an ideal bitopological ordered space. Then, PT₀-ordered spaces ⇒ ℱPT₀-ordered spaces.

**Proof.** The proof follows directly from the definitions of PT₁-ordered spaces and ℱPT₁-ordered spaces.

**Example 3.1** shows that (X, τ₁, τ₂, R, ℱ) is ℱPT₀-ordered space, but not PT₁-ordered space since, it is not UT₁-ordered space (as, 1R2, all decreasing τ₁-open sets which contain 2 also containing 1). Also, 3R2, all increasing τ₁-open sets which contain 3 also containing 2).

**Definition 3.4.** An ideal bitopological ordered space (X, τ₁, τ₂, R, ℱ) is said to be ℱPT₁-ordered space if it is ℱLPT₁ and ℱUPT₁-ordered space.

**Example 3.3.** Let τ₁ = {X, φ, {3}, {2, 3}, {3, 4}, {2, 3, 4}}, τ₂ = {X, φ, {1}, {1, 2}, {1, 3}, {1, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}} in Example 3.1. Then, (X, τ₁, τ₂, R, ℱ) is ℱPT₁-ordered space.

Let Y ⊆ X and R be a relation on X. Then, R_Y := R ∩ (Y × Y) is a relation on Y and is called the relation induced by R on Y. If a relation has any properties of reflexivity, transitivity, symmetry and anti-symmetry, then the properties are inherited by induced relations [9].

If (X, τ, ℱ) is an ideal topological space and A is a subset of X, then (A, τ_A, ℱ_A), where τ_A is the relative
topology on $A$ and $\mathcal{J}_A = \{ A \cap J : J \in \mathcal{J} \}$, is an ideal topological subspace [3].

**Theorem 3.3.** Let $(X, R, \mathcal{J})$ be an ideal ordered space. If $A \subseteq X, (A, R_A, \mathcal{J}_A)$ is an ideal ordered subspace of $(X, R, \mathcal{J})$ and $B$ is an $\mathcal{I}$-increasing (decreasing) set, then $B \cap A$ is an $\mathcal{J}_A$-increasing (decreasing) set.

**Proof.**
The proof for both parts are similar. So, we only present the proof for the part not in the parentheses. We want to prove $B \cap A$ is an $\mathcal{J}_A$-increasing set (i.e. if the complement of $B \cap A$ with respect to $A$ is $A \setminus (B \cap A)$, then $xR_A \cap [A \setminus (B \cap A)] \in \mathcal{J}_A \forall x \in B \cap A$).

So,

$$
\begin{align*}
\text{xR}_A \cap [A \setminus (B \cap A)] &= xR_A \cap [(X \setminus B) \cap A] \\
&= (xR \cap A) \cap [B \cap A] \\
&= xR \cap B \cap A.
\end{align*}
$$

Since $B$ is an $\mathcal{J}$-increasing set, so $xR \cap B \in \mathcal{J} \forall x \in B$. Consequently $xR \cap B \cap A \in \mathcal{J}_A \forall x \in B \cap A$, which follows that $B \cap A$ is an $\mathcal{J}_A$-increasing set.

The following theorem shows that the property of being $\mathcal{J}_T_1$-ordered space, $i = 0, 1$ is a hereditary property.

**Theorem 3.4.** Let $(X, \tau_1, \tau_2, R, \mathcal{J})$ be an $\mathcal{J}_T_1$-ordered space. Then every subspace of $\mathcal{J}_T_1$-ordered space is also $\mathcal{J}_A$ and $\mathcal{J}_T_1$-ordered space. ($i = 1, 0$).

**Proof.**
We give a proof in case ($i = 0$ and the case $i = 1$ is similar). Let $(X, \tau_1, \tau_2, R, \mathcal{J})$ be an $\mathcal{J}_T_1$-ordered space, $(A, \tau_1A, \tau_2A, R_A, \mathcal{J}_A)$ be any subspace of $(X, \tau_1, \tau_2, R, \mathcal{J})$ and $a, b \in A$ such that $aRb$, it follows that $a \in \mathcal{J}$. Since $(X, \tau_1, \tau_2, R, \mathcal{J})$ is an $\mathcal{J}_T_0$-ordered space, there exists an $\mathcal{J}$-increasing $\tau_1$-open set $U$ such that $a \in U$ and $b \notin U$ or there exists an $\mathcal{J}$-decreasing $\tau_1$-open set $V$ such that $b \in V$ and $a \notin V$, $i = 1$ or 2. By Theorem 3.3 there exists $(\mathcal{J}_A)$-increasing $\tau_1$-open set $G$ such that $a = A \cap G$ and $a \not\in A \cap G$ or there exists $(\mathcal{J}_A)$-decreasing $\tau_1$-open set $H$ such that $b \in H = A \cap G$ and $a \notin H = A \cap G$). Hence, $(A, \tau_1A, \tau_2A, R_A, \mathcal{J}_A)$ is an $\mathcal{J}_T_0$-ordered space.

**Definition 3.5.** An ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{J})$ is said to be an $\mathcal{J}_T_2$-ordered space iff for all $a, b \in X$ such that $aBb$, there exists an $\mathcal{J}$-increasing $\tau_1$-open set $O_a$ and an $\mathcal{J}$-decreasing $\tau_2$-open set $O_b$ such that $O_a \cap O_b \subseteq \mathcal{J}$.

**Example 3.4.** In Example 3.1 take $\mathcal{J} = \{ \phi, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\} \subseteq \mathcal{J} \subseteq \mathcal{P}(X)$, $\tau_1 = \{X, \phi, \{3\}, \{4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$, $\tau_2 = \{X, \phi, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. It is clear that $(X, \tau_1, \tau_2, R, \mathcal{J})$ is an $\mathcal{J}_T_2$-ordered space.

The following theorem studies the relationship between Definitions 3.4, 3.5 and Definition 2.10 [12].

**Theorem 3.5.** Let $(X, \tau_1, \tau_2, R, \mathcal{J})$ be an ideal bitopological ordered space. Then, $\mathcal{J}_T_2$-ordered spaces $\Rightarrow \mathcal{J}_T_1$-ordered spaces $\Rightarrow \mathcal{J}_T_0$-ordered space.

**Proof.**
The proof follows directly from the definitions of $\mathcal{J}_T_2$-ordered spaces, $\mathcal{J}_T_1$-ordered space and $\mathcal{J}_T_0$-ordered spaces.

Example 3.4 shows that $(X, \tau_1, \tau_2, R, \mathcal{J})$ is an $\mathcal{J}_T_1$-ordered space, but not $\mathcal{J}_T_2$-ordered space as, $\mathcal{J}_T_2$ and all increasing $\tau_2$-open sets which contain 1 are the sets $X, \{1, 2, 3, 4\}$, intersect the only decreasing $\tau_1$-open set $X$ which contains 2.

Example 3.3 shows that $(X, \tau_1, \tau_2, R, \mathcal{J})$ is an $\mathcal{J}_T_1$-ordered space, but not $\mathcal{J}_T_2$-ordered space as, $\mathcal{J}_T_2$ and all increasing $\tau_2$-open sets which contains 1 and not contain 2 are the sets $\{1, 4\}, \{1, 3, 4\}$ and all $\mathcal{J}$-decreasing $\tau_1$-open set which contains 2 are the sets $X, \{2, 3, 4\}$, while $\{1, 4\} \cap \{2, 3, 4\} = \emptyset \notin \mathcal{J}$. $\{1, 3, 4\} \cap \{2, 3, 4\} = \{3, 4\} \notin \mathcal{J}$, $\{1, 4\} \cap X = \{1, 4\} \notin \mathcal{J}$, then $\{1, 3, 4\} \cap X = \{1, 3, 4\} \notin \mathcal{J}$.

On account of Proposition 3.1, Theorems 3.1, 3.5 and [6,12], we have the following corollary.

**Corollary 3.1.** For an ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{J})$, we have the following implications:

- $\mathcal{J}_T_2$-ordered space $\Rightarrow \mathcal{J}_T_1$-ordered space $\Rightarrow \mathcal{J}_T_0$-ordered space
- $\mathcal{J}_T_2$-ordered space $\Rightarrow \mathcal{J}_T_1$-ordered space $\Rightarrow \mathcal{J}_T_0$-ordered space
- $\mathcal{J}_T_2$-ordered space $\Rightarrow \mathcal{J}_T_1$-ordered space $\Rightarrow \mathcal{J}_T_0$-ordered space

The following theorem shows that the property of being $\mathcal{J}_T_2$-ordered space is preserved by a bijective, $P$-open and order (reverse) embedding mapping.

**Theorem 3.6.** Let $(X, \tau, \tau^\prime, R, \mathcal{J})$ is a $\mathcal{J}_T_2$-ordered space, $f : (X, \tau_1, \tau_2, R, \mathcal{J}) \rightarrow (Y, \eta_1, \eta_2, R^\prime, f(\mathcal{J}))$ is a bijective, $P$-open and order (reverse) embedding mapping. Then, $(Y, \eta_1, \eta_2, R^\prime, f(\mathcal{J}))$ is $f(\mathcal{J})$-$PT_2$-ordered space.

**Proof.**
We give a proof in the case of order embedding mapping and the case of order reverse embedding mapping is similar.

Let $y_1, y_2 \in Y$ be such that $y_1 \leq y_2$. Then, there exist $x_1, x_2 \in X, f(x_1) = y_1, f(x_2) = y_2$ and $x_1 \leq x_2$. As $(X, \tau_1, \tau_2, R, \mathcal{J})$ is a $\mathcal{J}_T_2$-ordered space, then there exist an $\mathcal{J}$-increasing $\tau_1$-open set $U$ contains $x_1$ and an $\mathcal{J}$-decreasing $\tau_2$-open set $V$ contains $x_2$ such that $U \cap V \in \mathcal{J}$. Since $f$ is $P$-open and by Theorem 2.1, $f(U)$ is an $\mathcal{J}$-increasing $\eta_1$-open set contains $y_1 = f(x_1)$ and $f(V)$ is an $\mathcal{J}$-decreasing $\eta_2$-open set contains $y_2 = f(x_2)$. So, $f(U) \cap f(V) = f(U \cap V) \in f(\mathcal{J})$. Hence, $(Y, \eta_1, \eta_2, R^\prime, f(\mathcal{J}))$ is $f(\mathcal{J})$-$PT_2$-ordered space.
The following theorem shows that the property of being $\mathcal{S}PT_2$-ordered space is a hereditary property.

**Theorem 3.7.** Let $(X, \tau_1, \tau_2, R, \mathcal{S})$ be $\mathcal{S}PT_2$-ordered space. Then, every subspace of $\mathcal{S}PT_2$-ordered space is also $\mathcal{S}PT_2$-ordered space.

**Proof.**
Let $(X, \tau_1, \tau_2, R, \mathcal{S})$ be a $\mathcal{S}PT_2$-ordered space, $(A, \tau_{1A}, \tau_{2A}, R_A, \mathcal{S}_A)$ be any subspace of $(X, \tau_1, \tau_2, R, \mathcal{S})$ and $a, b \in A$ such that $a \neq b$. Since $(X, \tau_1, \tau_2, R, \mathcal{S})$ is $\mathcal{S}PT_2$-ordered, then there exists an $\mathcal{S}$-increasing $\tau_1$-open set $O_a$ and an $\mathcal{S}$-decreasing $\tau_2$-open set $O_b$ such that $O_a \cap O_b \in \mathcal{S}$. By Theorem 3.3 there exists an $\mathcal{S}_A$-increasing $\tau_{1A}$-open set $G$ such that $a \in G = A \cap O_a$ and an $\mathcal{S}_A$-decreasing $\tau_{2A}$-open set $H$ such that $b \in H = A \cap O_b$. Hence, $(A, \tau_{1A}, \tau_{2A}, R_A, \mathcal{S}_A)$ is $\mathcal{S}_A PT_2$-ordered.

**4 Conclusion**

In this paper, we use the notion of $\mathcal{S}$-increasing (decreasing) sets [2], which based on the notion of ideal $\mathcal{S}$, to generate new separation axioms $\mathcal{S}Pr_i, i = 0, 1, 2$, on ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{S})$. These types of separation axioms are a generalization of the previous one [6, 12]. Some properties of these separation have been obtained. In the future, we study the separation axioms $\mathcal{S}Pr_i, i = 3, 4, 5$ and $\mathcal{S}Pr_j$-ordered spaces, $j = 0, 1, 2, 3, 4$.

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**References**


