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Mapping Stability: Real Rational Maps of Degree Zero

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Abstract: In memory of our beloved Professor José Rodrigues Santos de Sousa Ramos (1948-2007), who João Cabral, one of the authors of this paper, had the honour of being his student between 2000 and 2006, we wrote this paper following the research by experimentation, using the new technologies to capture a new insight about a problem, as him so much love to do it. His passion was to create new relations between different fields of mathematics. He was a builder of bridges of knowledge, encouraging the birth of new ways to understand this science. One of the areas that Sousa Ramos researched was the iteration of maps and the description of its behaviour, using the symbolic dynamics. So, in this issue of this journal, honoring his memory, we use experimental results to find some stable regions of a specific family of real rational maps, the ones that he worked with João Cabral. In this paper we describe a parameter space (a,b) to the real rational maps $f_{a,b}(x) = (x^2-a)/(x^2-b)$, using some tools of dynamical systems, as the study of the critical point orbit and Lyapunov exponents. We give some results regarding the stability of these family of maps when we iterate it, specially the ones connected to the order 3 of iteration. We hope that our results would help to understand better the behaviour of these maps, preparing the ground to a more efficient use of the Kneading Theory on these family of maps, using symbolic dynamics.

Keywords: Real Rational Maps, Topological Entropy, Stability

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

Let $f_{a,b}(x) = (x^2-a)/(x^2-b)$ be a real rational map, with a and b real parameters with domain D_f. Since these maps are a family of degree zero, with Schwarzian derivative $S\left(f_{a,b}(x)\right) = -3/2x^2$ always negative, should be important to understand its behaviour under iteration, because all knowledge regarding the dynamics of these class of maps can help to extend the present theories in [6] and [7]. The family of $f_{a,b}$ is a large and distinct group of functions that we can organize relatively to the lines $b = a$, $b = -a$, $b = 0$ and $a = 0$, in a parameter space (a,b). At most, we can obtain eight different groups, each one with its own behaviour. We can see in figure 1 one example of the graphic for each family.

Following the notations, usually used in the study of dynamical systems, see [4], in this paper we will use the symbol $f^n_{a,b}$ to denote $f_{a,b} \circ f_{a,b} \circ \ldots \circ f_{a,b}$ (n times). For $n = 0$, we have $f^0 = id$. For a point $x \in D_f$ we define the

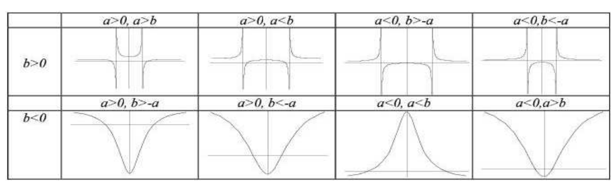

Fig. 1: The family of maps $f_{a,b}$

orbit of x, denoted by $Orb_f(x)$, as the set $\{f^n_{a,b}(x) : n = 0,1,2\ldots\}$. The point x is a fixed point for $f_{a,b}$ if $f_{a,b}(x) = x$. The point x is a fixed point for $f_{a,b}$ of period n if $f^n_{a,b}(x) = x$. The least positive n for each $f^n_{a,b}(x) = x$ is called the prime period of x. A point x is eventually periodic of period n if x is not periodic, but there exists $t > 0$ such that $f^{n+i}_{a,b}(x) = f^i_{a,b}(x)$, for all $i \geqslant t$. That is, $f^i_{a,b}(x)$ is periodic for $i \geqslant t$. If p is a point with

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period n we denote $W^s(p)$ as the stable set of p, that is, the set of values x that are forward asymptotic to p, $\lim_{i\to\infty} f^{in}_{a,b}(x) = p$. The set of points backwards asymptotic to p is called the unstable set of p and is denoted by $W^u(p)$. A point x is a critical point of $f_{a,b}$ if $f'_{a,b}(x) = 0$. The critical point is non-degenerate if $f''_{a,b}(x) \neq 0$ and degenerate if $f''_{a,b}(x) = 0$. The point p is called hyperbolic if $\left|\left(f^n_{a,b}\right)'(p)\right| \neq 1$. If p is a hyperbolic fixed point with $\left|(f_{a,b})'(p)\right| < 1$ it is called an attractor and there is an open interval U, about p, such if $x \in U$, then $\lim_{n\to+\infty} f^n_{a,b}(x) = p$, and $U = W^s(p)$. If p is a hyperbolic fixed point with $\left|(f_{a,b})'(p)\right| > 1$ it is called a repeller point and there is an open interval U, about p, $x \neq p$, such if $x \in U$, then there exists $t > 0$ such that $f^t_{a,b}(x) \notin U$, and $U = W^u(p)$. If $x = c$ is the critical point of the map and $f^n_{a,b}(c) = c$, for some n, then we say that c is a supercritical point, then if $\left|(f_{a,b})'(c)\right| < 1$, then we have a super stability region of the map.

As we can read in almost all literature, regarding to dynamical systems, the role of the orbit of a critical point, on understanding the behaviour of a map is very important. See, for example, all the use given to this kind of orbits in [1], [4], [6] and [7]. It has an extreme importance in convert the dynamics of a system in a symbolic way, since this kind of orbits are the capstone of the foundations of Kneading Theory.

In continuous real maps, the orbit of any point will eventually fall in the orbit of the critical point, and this critical orbit will converge to a fixed point p, building a stable set around p; diverge from a point p, building an unstable set around p, or can ignore the presence of p if p is a saddle point. See [4] to a better description of this phenomena. Since $f_{a,b}$ is piecewise step continuous map there will be some differences, but the orbit of the critical point still has big importance, since we can identify stable and unstable sets only by its orbit, but it doesn't work alone. We must count also with the role played by the discontinuities and the forward orbit of ∞. We can see some of this interaction in [2] and [3].

Another tool used to study the behaviour of maps is the bifurcation diagram. See [4] and [1], to obtain the mathematical description of bifurcation diagram an some graphical examples to a relatively large group of maps. A bifurcation diagram shows the possible long-term values (equilibria/fixed points or periodic orbits) of a system as a function of a bifurcation parameter in the system. Through it, we can analyze, with a careful interpretation, for certain values of the parameters (a,b), the type of orbit that the critical point of the map $f_{a,b}$ will produce, and its order. This tool is very important when we depend mostly from experimental results, obtained by computational calculus, as we did on this paper. And to confirm some aspects related to the value of topological entropy we use the Lyapunov exponents, since they are a quantity that characterizes the rate of separation of infinitesimally close orbits, and using the Lyapunov spectrum we can obtain an estimate of the rate of entropy production of the map $f_{a,b}$, see [1].

The main goal of this paper is to show that we can associate a parameter space (a,b) to the real rational map $f_{a,b}$, extracting a lot of information about the periodicity of the orbit of the critical point and study the stability of certain regions, working this space as a chart, in the same relation that the Mandelbrot set has to Complex Rational Maps, see [4]. Using this parameter space (a,b) for $f_{a,b}$ we can identify the values of a and b, where in this map the critical value will have stable and unstable orbits. We believe that our work is an interesting contribution in order to understand better the dynamics of this class of maps.

2 The parameter space (a,b) for $f_{a,b}$

To study the behaviour, under iteration, of the map $f_{a,b}$, the most valuable points are the discontinuity points, the critical points and the fixed points. Since, under iteration, the images of the discontinuities go to ∞ and after that to the value 1, we can use the orbits of $x = 0$ and $x = 1$ as references to understand how this maps behave under iteration, see [2] and [3]. All the other values $x \neq 0$ and $x \neq 1$ have orbits that will fall, eventually, in the orbits $Orb_f(0)$ or $Orb_f(1)$. The orbit of the critical point plays, as on the continuous real rational maps, see [1], an important role in the dynamics of the map $f_{a,b}$. It is useful to remember that $x = 0$ is a non degenerate critical point, since $f''_{a,b}(0) = 2(a-b)/b^2 \neq 0$, $a \neq b$ and $b \neq 0$, but its behaviour, under iteration, is not completely understood.

Using Wolfram Mathematica 8.0 (WM80) we can explore the generic parameter space (a,b) of $f_{a,b}$, searching for the number of possible solutions of the equation

$$f^n_{a,b}(x) = x, \tag{1}$$

for each natural n. Due to the hard difficulty in calculating the exact values to all n, of this fixed point equation, we can use the iteration of $x = 0$, as a start point to simplify the solution of the implicit equations, to estimate regions where we can find some solutions of the generic equation.

Using the density plot function of WM80, we obtained the figures 2 and 3, iterating the values $x = 1$ and $x = 0$, respectively, until the order 50. The black regions are the ones where we can find combinations of a and b, where the equation 1 has a finite solution for each n. Although, this set is not completely accurate, due to some natural errors in numerical algorithms used by WM80, it can be used as a good approximation to the set that will play the role of Mandelbrot to our map $f_{a,b}$. The dark regions are numerical approximations to the regions where we can find stable orbits to the critical value $x = 0$ or $x = 1$. To control the long time that WM80 needs to calculated this figures we only worked with the values

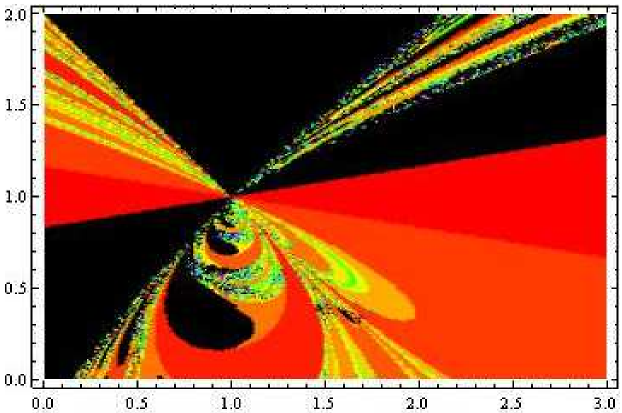

Fig. 2: Approximated regions where we can have solutions to $f^n_{a,b}(x) = x$, using iteration of $x = 1$, $0 < a < 3$, $0 < b < 2$.

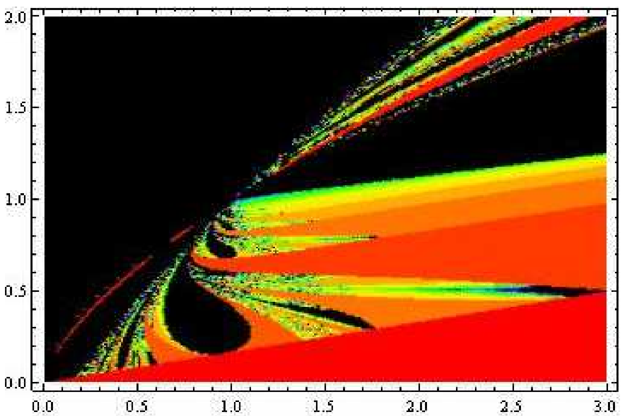

Fig. 3: Approximated regions where we can have solutions to $f^n_{a,b}(x) = x$, using iteration of $x = 0$, $0 < a < 3$, $0 < b < 2$.

```
Space[a_, b_] := Length[FixedPointList[
    (#^2 - a) / (#^2 - b) &, 1, 50,
    SameTest → (Abs[#] > 6 &)]];
DensityPlot[Space[a, b],
 {a, 0.00001, 3}, {b, 0.000001, 2},
 Mesh → False, AspectRatio → Automatic,
 Frame → True, Axes → True, PlotPoints → 75,
 ColorFunction → (If[# == 1, RGBColor[0, 0, 0],
     Hue[0.9 #]] &)]
```

Fig. 4: Program used to find stability regions using $x = 1$ as a seed.

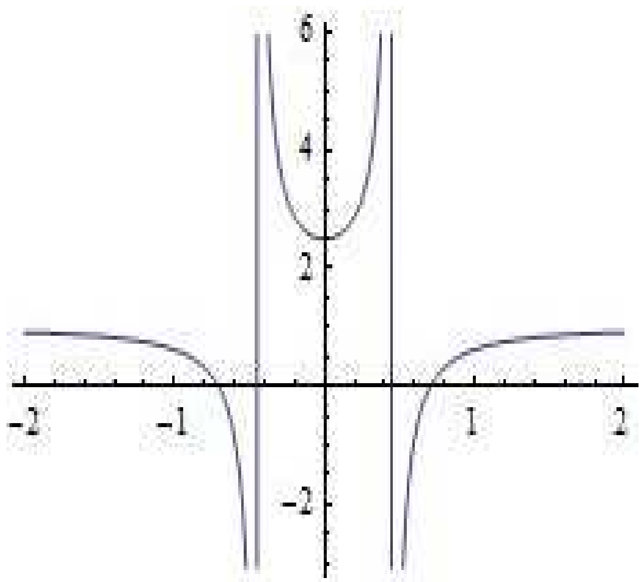

Fig. 5: Graph of function $f(x) = (x^2 - 0.5)/(x^2 - 0.2)$

$|x| < \theta$, $\theta = 6$, $0 < a < 3$ and $0 < b < 2$. We can improve the accuracy of this picture increasing the value of θ, and the order of iteration for each orbit of $x = 1$ and $x = 0$. In figure 4 we can see an example of the program used in WM80 to calculate these regions.

In our work we will dedicate our attention to the restriction of the parameter space (a,b), where $a > 0$ and $b < a$, in order to study the behaviour of an unique type of function, like the one in figure 5. With this restriction the parameter space (a,b) becomes the one present in figure 6.

As we can see in figure 6, exists some regularity in the black regions, that resembles the existent similarity in the Mandelbrot picture for complex quadratic rational maps, see [4]. These black regions are the ones, in parameter space (a,b), where we can find some type of solution of degree n, of the equation 1. We will call these regions "Bulbs", following the designation already know to the Bulbs of Mandelbrot sets. Using B-Spline functions, collecting some points, in WM80, we can approximate the border of these regions, obtaining the geometric result showed in figure 7.

Definition 1. *We define as Bulb of period n, the region of the parameter space (a,b) where we can find the solutions of the equation 1, for a fixed n.*

The biggest bulb present in figure 6, is one region where we can always found at least one solution to the equation 1, with $n = 3$. But, in the same figure we can also found bulbs for all the other periods. Selecting the proper values for (a,b) in the interior of each bulb it is easy to find the period of the other bulbs. In this case the proper values must be the ones that are located at the left

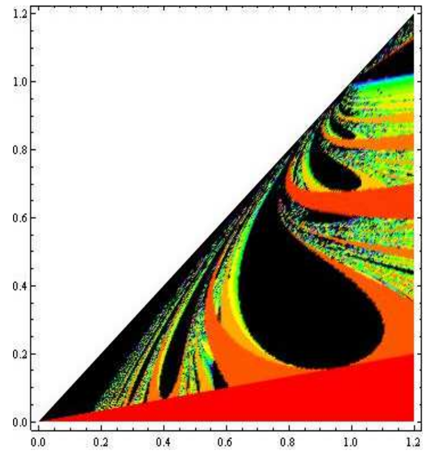

Fig. 6: Restriction of the space (a,b) to $0 < b < a = 1.2$

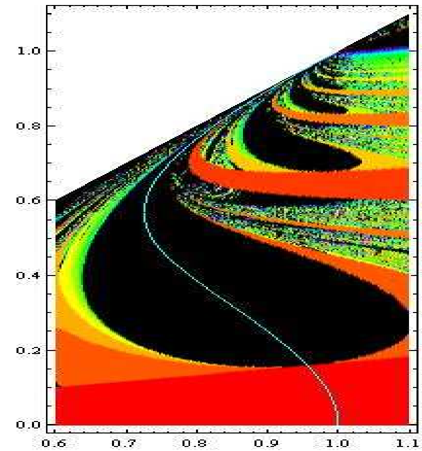

Fig. 8: Super stable line of period 3 crossing bulb of period 3.

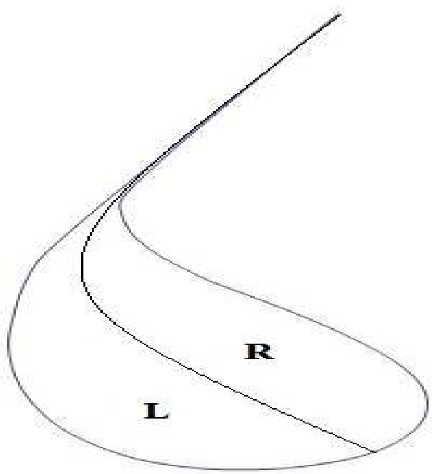

Fig. 7: The geometric configuration of a bulb

side of the bulb, the region of the bulb designated by L in figure 7, then we can iterate the critical value $x = 0$ and we will obtain orbits of the same order of the bulb.

Incorporated in each bulb, there is a special line, that divides the bulb in two regions, L and R. This line is the solution of the equation 1, for $x = 0$, and it is the line of super stability of the function for each n, that is, this line is the set of values (a,b) where the critical point will have a super stable orbit.

Lemma 1. *The Bulb of period $n = 3$ contains a set of values (a,b), solutions of the equation $f^3_{a,b}(0) = 0$, for $0 < b < a$.*

Proof. Since the bulb is the region where $f^3_{a,b}(x) = x$ has solutions, for any x, we can solve this equation implicitly with the help of WM80, trough numerical computations, and we have in particular for $x = 0$, a set of solutions that are in the interior of the bulb. This set of values is a continuous line.

Example 1. Using WM80, we can solve the equation $f^3_{a,b}(0) = 0$, and we obtain the figure 8, where we can see the super stable line of period 3 in the interior of the bulb of period 3, dividing it in two parts. With some help WM80, we can see that this line is continuous and differentiable, in all its extension.

Each bulb of the parameter space (a,b) gives us information about the behaviour of the rational maps of degree zero which can be created using the values a and b. In the next section we will summarize some characteristics of the bulbs, using as reference the bulb of period 3. Those characteristics can be easily extended to the other bulbs belonging to the restriction $0 < b < a$, since we are working with the same family of maps.

Notice that when (a,b) belongs to a bulb of period n, it only means that in this bulb we will find at least one stable orbit for the critical value of period n. Other orbits, of different periods can exist in the same bulb. Since the Bulbs are created by numerical computations, with the help of WM80, solving the equation $f^n_{a,b}(x) = x$, on a limited region (a,b), for fixed values a and b, it is easy to see that the result of Lemma 1 will occur in Bulbs of higher orders. So, we present the following result, giving a proof for it in the next section.

Proposition 1. *The Bulb of period $n \geqslant 3$ contains at least one set of values (a,b) where the equation 1 has a solution, and have other solutions of period $n.p$, with $p = 2,3,...$.*

3 The bulb of period 3 enigma

The map $f_{a,b}$ is a two parameter map, but if we apply a restriction to a and b, in order to transform it in one

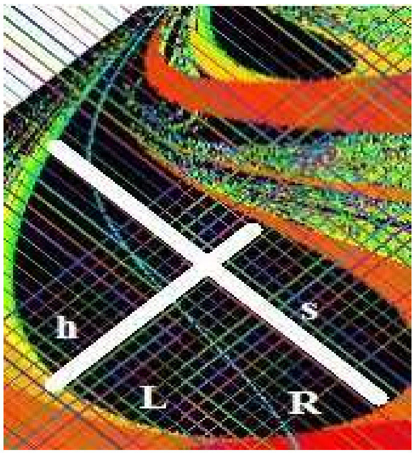

Fig. 9: Bulb of period 3 and the line h with positive slope and s with negative slope

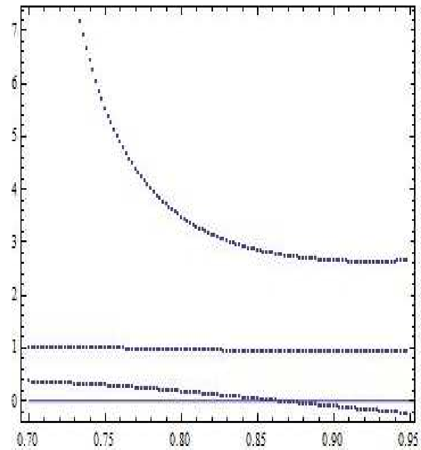

Fig. 10: Bifurcation diagram of $f_a(x)$, with $0.7 \lesssim a \lesssim 0.95$.

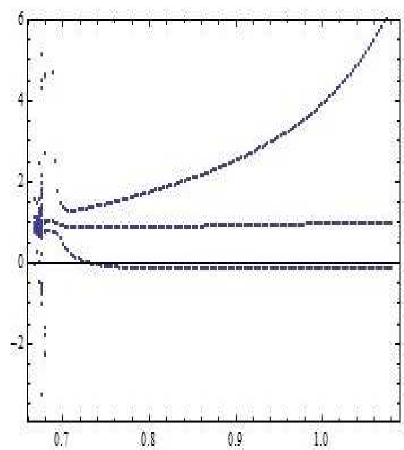

Fig. 11: Bifurcation diagram for $f_a(x)$ with $0.70 \lesssim a \lesssim 1.08$

parameter map, we can use the already known tools used in Dynamical Systems for piecewise continuous maps and for m-modal maps, to study the dynamics of $f_{a,b}$, see [1] and [7]. We can achieve this making $b = g(a)$, and $a \in]c,d[$ in a way that the pair $(a,g(a))$ is in the interior of a specific bulb of period n. The values c and d, are reals such the pair $(c,g(c))$ belongs to the left border of the region L, and the pair $(d,g(d))$ belongs to the right border of the region R, in the figure 7. The function g is continuous and C^1.

Let $n = 3$, and $b = g(a) = m.a + k$, $m > 0$, with $a \in]c,d[$ and $0 < b < a$. With these conditions we are working with straight lines g that belongs to the interior of the bulb of period 3. The line h in figure 9 is one example. Now $f_{a,b}$ is a map with one parameter, $f_a(x) = (x^2 - a)/(x^2 - g(a))$ and we can use the bifurcation diagram to understand better its dynamics.

If we select approximated values of $c + \theta$ and $d - \theta$, with $\theta \to 0$, we can generate h as in figure 9, $g(a) = -0.55 + a$, with $0.7 \lesssim a \lesssim 0.95$, and the bifurcation diagram $\{a, f_a(x)\}$, that represents the orbit of the critical point $x = 0$, will be the figure 10.

We can observe in figure 10 that, indeed, the period of the critical orbit is 3, and the super stability happens when $a \approx 0.86$, result also obtained visualizing the figure 9, since h crosses the super stable line of the bulb approximately on the point $(0.86, g(0.86))$.

After all the possible calculations, we observed that all the bifurcation diagrams of the map $f_a(x)$, depending on $g(a)$ with positive slope, $m > 0$, with $a \in]c,d[$, have the same characteristics: the orbit of the critical value is stable of period 3, and it starts at the left side border of the region L, from a previous border collision bifurcation, that origins an orbit with three different starts, $f(0)$, 1 and ∞, all with decreasing monotonicity, ending at the right side border of the region R, in a double period bifurcation. For this case we didn't find any value of a such $f_a(x)$ would become chaotic. And the results where the same for $m = 0$.

But, for $b = g(a) = m.a + k$, $m < 0$, when we were expecting to get the same results, a surprise emerged and something new happened. For the lines as s in figure 9, generated in the same conditions as the line h, only with negative slope instead of positive, we found a new and different behaviour for some values (a,b) in the interior of the region R.

For $g(a) = 1.275 - a$, for $0.70 \lesssim a \lesssim 1.08$ we have the bifurcation diagram for $f_a(x)$ represented in figure 11.

For $g(a) = 1.325 - a$, for $0.73 \lesssim a \lesssim 1.09$ we have the bifurcation diagram for $f_a(x)$ represented in figure 12.

For $g(a) = 1.35 - a$, for $0.77 \lesssim a \lesssim 1.12$ we have the bifurcation diagram for $f_a(x)$ represented in figure 13.

As happened on the bifurcation diagrams resulting from $g(a)$ with positive slope, the orbits also born in the left side border of region L, from a border collision

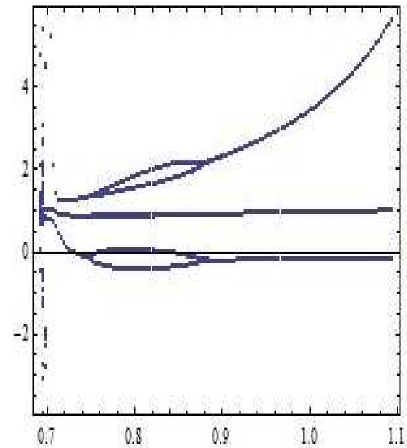

Fig. 12: Bifurcation diagram for $f_a(x)$ with $0.73 \lesssim a \lesssim 1.09$

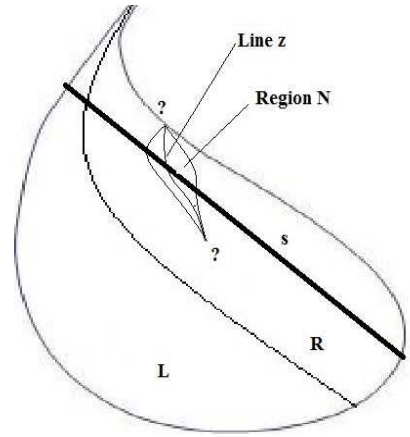

Fig. 14: The region N

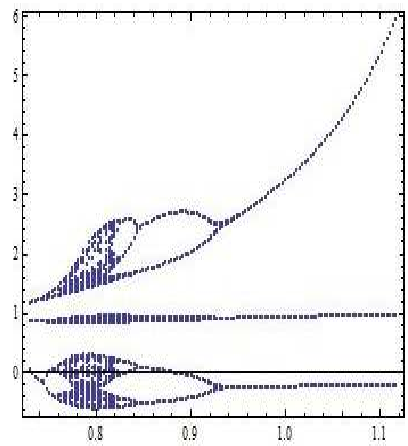

Fig. 13: Bifurcation diagram for $f_a(x)$ with $0.77 \lesssim a \lesssim 1.12$

bifurcation, with starts in $f(0), 1$ and ∞, also with decreasing monotonicity, but after the value a, which represents the value of super stability for the orbit of $x = 0$, and the value where the line s crosses the super stable line of the bulb of period 3, the monotonicity changes and we arrive to a region N, see figure 14, inside the region R, where the map $f_a(x)$ will have reverse period-doubling bifurcation sequences on its bifurcation diagram! To obtain some information about reverse bifurcation we can check [4] and [1]. Also, our experimental results points to the existence of all orbits of period $3p$, with $p = 2, 3, \ldots$ in this region N. This result will be easy to check, since it happens in a region of reverse period bifurcation sequences, if we could find the exact value of a, or some neighbourhood, where the map goes to chaos, the line z.

All our results, obtained with the help of WM80 points to the existence of this region N, and for the existence of a line z, in the interior of the region N, that are the set of points $(a, g(a))$ where the map $f_a(x)$ has maximum entropy in the bulb of period 3, when $g(a)$ has negative slope. Big questions arise to our mind, and the most important ones are: (1) Why the bifurcation diagram of the map when $g(a)$ had positive slope crossed this region N didn't detect the existence of reverse bifurcations? (2) Where is the starting/ending point of the line z, and which analytic equation has as solution the line z?

We registered this phenomena in all possible bulbs of our parameter space (a, b), since our machine could not work with periods bigger than period 15. Numerical calculus prove the existence of this region N, and the line z, for all $g(a)$ with negative slope, in the same relative position of all regions R, of all bulbs of odd order $3, 5, 7, 9, \ldots < 15$.

Now we can give a proof to proposition 1.

Proof.(of Proposition 1)

In the bulb of period $n \geqslant 3$, it is obvious that the equation 1 has at least one solution. It is the set of values (a, b) that solves the equation $f^n_{a,b}(0) = 0$. When $b = g(a) = m.a + k$, for $m < 0$, with $a \in]c, d[$ and $0 < b < a$, with $(c, g(c))$ and $(d, g(d))$ points of the left border of region L and the right border of R, respectively, will exist a region $N \subset R$, where double period bifurcations occurs, so the existence of all periodic $n.p$ solutions of $f^{np}_{a,b}(0) = 0$ is guaranteed.

4 Conclusions

In dynamical systems it is still an open problem the proof about what will be the simplest map that presents reverse bifurcations, see [7], [1] and [4]. The discovery of this region N in our bulb of period 3, for our map $f_{a,g(a)}$ is a great contribution in order to solve this problem, since the

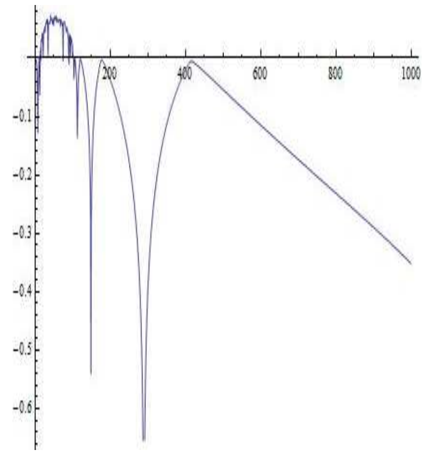

Fig. 15: Lyapunov exponents for $f_{a,g(a)}$ with $g(a) = 1.35 - a$, for $0.77 < a < 1.12$

region N exists in all other bulbs of period n of our map, and it is easy to estimate its position computationally.

The discovery of the region N, on the bulb of period 3, where the map $f_{a,b}$ has infinite complexity, gave us a way to calculate the maximum value of entropy on this bulb, since now we can use Lyapunov Exponents. From Katok results, see [5], the Lyapunov exponents for smooth transformations can be computed by

$$\gamma(x) = \frac{1}{m}\sum_{k=1}^{m} \ln\left|f'_{a,b}(x_{k-1})\right| \qquad (2)$$

and the envelope of the upper limit of $\gamma(x)$ is an estimative of the topological entropy value h_{top}. As example, calculating this estimative to $f_{a,g(a)}$ with $g(a) = 1.35 - a$, for $0.77 \lesssim a \lesssim 1.12$, we obtain the figure 15. On this way, we can compute the maximum value for the topological entropy on the interval $0.77 \lesssim a \lesssim 1.12$, and it is $h_{top} = 0.0736631$. The use of this tool allows us the study of the variation of the entropy of $f_{a,b}$, in each bulb of period n, and in the parameter space (a,b) in a global way.

With the help of the parameter space (a,b) it is easy to obtain the possible regions of stability of $f_{a,b}$ and so we can apply more efficiently the techniques of Kneading Theory, studying the behaviour of this class of maps in a symbolic way, as Sousa Ramos loved to do and taught to all his students.

The future of this work passes by an improvement of the accuracy of the parameter space (a,b) and then apply the techniques used in [2] in order to hunt for admissible Kneading pairs.

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References

[1] Alsedà, L. & Llibre, J. & Misiurewicz, M. (2000) " Combinatorial Dynamics and Entropy in Dimension One", Second edition, Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific. ISBN 981-02-4053-8.
[2] Cabral, J. (2013). "Hunting Admissible Kneading pairs of a Real Rational Map". Asia Online Journals: Asian Journal of Fuzzy and Applied Mathematics, 1(4), 120-126. ISSN 2321-564X.
[3] Cabral, J. (2013). "Symbolic Dynamics for Real Rational Maps". Asia Online Journals: Asian Journal of Fuzzy and Applied Mathematics , 1(4), 69-75. ISSN 2321-564X.
[4] Devaney, R. (1989) "An Introduction to chaotic dynamical Systems". 2nd edition, Perseus Books. ISBN 0-201-13046-7.
[5] Katok, A. (1980) "Lyapunov exponents, entropy and periodic orbits for diffeomorphisms". Publications Mathématiques de l'IHÉS, 51, p. 137-173
[6] Milnor, J. & Thurston, W. (1988) "On Iterated Maps of the Interval", In J.C. Alexander (ed.) Proceedings Univ. Maryland 1986-1987. Lect. Notes in Math. 1342, pp. 465-563, Springer-Verlag, berlin-New York.
[7] Ramos, J. S. (1989) "Hiperbolicidade e Bifurcação de Sistemas Simbólicos", PhD. Thesis, Universidade de Lisboa, available at http://purl.pt/7368

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