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A Comprehensive Review: Current Record Values and Their Applications

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Abstract: Record value theory has been a key part of probability and statistics for a long time. However, the tracking of current record values, both upper and lower extremes at the same time, has not received enough theoretical focus. This review brings together and builds on recent developments in current records. It covers distribution theory, asymptotic properties, moment recurrences, and predictive inference. It also has practical applications in fields such as meteorology, reliability, quality control, and finance. The article presents k-sliding records, a new way to monitor local extremes within limited moving windows. It also includes new distribution results for concomitants of current records. These additions connect record theory with multivariate dependence modeling and lay the groundwork for further exploration using copula-based methods. This revitalizes current records as an important area for research that has significant practical applications.

Keywords: Records; Current records; Asymptotic theory; Concomitants of current records; k-sliding records; Moments; Prediction intervals

Contents

Table 1: Summary of notation and symbols for record values

Symbol	Description
$U_n \parallel X$	upper nth record value from sequence X
$L_n \parallel X$	lower nth record value from sequence X
$U_n^c \parallel X$	upper nth current record value
$L_n^c \parallel X$	lower nth current record value
$R_n^c \parallel X$	nth record range ($= U_n^c \parallel X - L_n^c \parallel X$)
ζ_n	Record time sequence for upper records
η_n	Record time sequence for lower records
$R_X(x)$	Record transform: $-\log(1-F_X(x))$
$H_X(x)$	Lower record transform: $-\log(F_X(x))$
$\mathrm{EX}(\beta)$	Exponential distribution with rate β
$\mathrm{W}(a,b)$	Weibull distribution with scale a and shape b
$\mathrm{Par}(a)$	Pareto distribution with shape a
$\Gamma_n(\theta)$	Incomplete gamma function

1 Introduction

Industrial strength is fundamental to a nation's growth. In line with its goal to achieve "leadership in energy and industry," the KSA has made industrial development a top priority. Effective statistics are crucial here, enabling us to

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analyze industrial data, uncover insights, and predict trends. This underscores the need for the new, widely applicable data analysis method demonstrated in the records and the current records.

Record values are a captivating topic in the theory of extreme value and order statistics. The exploration of records began with Chandler [25], who studied the random behavior of record values from sequences of continuous random variables (RVs) that are independent and identically distributed (i.i.d.). Since then, record values have been used in fields, including meteorology, hydrology, seismology, sports statistics, industrial stress testing, and financial modeling.

The main challenge in record value theory stems from the rarity of records, which yields small sample sizes for statistical analysis. However, as noted by Nagaraja [37] and Aldallal [10], there are many real-world situations in which only record values are available. This makes studying them both theoretically interesting and practically useful.

1.1 From usual records to current records

Standard record values track extreme values in only one direction, either up or down. In contrast, *current records* take a broader approach by monitoring both upper and lower extremes at the same time. The idea of current records, introduced by Houchens [35] and further developed by Aldallal [10], involves keeping track of both the largest and smallest values whenever a new record of either kind is encountered.

The current highest value at the time the nth record (either lower or upper) is noticed is the nth highest current record, denoted $U_n^c \parallel X$,. Similarly, the current minimum value at that time is $L_n^c \parallel X$, the lower nth current record. The difference between these two, $R_n^c \parallel X = U_n^c \parallel X - L_n^c \parallel X$, defines the record range, cf., Houchens [35], Aldallal [10], and Basak [23].

1.2 Motivation and Contribution

Current records come up in many real-life situations, cf. [2,10,11,12,13,14,23,18,19,20,21,41]:

- life testing and Industrial stress, where measurements are collected one after another, and only data points that exceed or fall below current extremes are noted.
- Meteorological data analysis, where both record highs and lows of temperatures are tracked simultaneously.
- Quality control processes where maintaining product consistency requires monitoring both upper and lower specification limits.
- Financial markets that keep track of record highs and lows for stock indices or commodity prices.

The current record framework offers a more complete perspective on extreme behavior compared to traditional one-sided record analysis. This makes it particularly useful in cases where changes in both directions are important.

This paper's main contribution is to provide a comprehensive and coherent framework for the study of current records. We review key theoretical results, present new findings on concomitants of current records, and introduce k-sliding records as a flexible extension suitable for finite-memory monitoring. Together, these developments strengthen the connection between record theory and real-world applications.

To place the concept of current records within its proper historical and theoretical context, the next section reviews the key developments and foundational contributions in record value theory.

2 Historical development and literature review

2.1 Early basis

Record values' mathematical theory began with Chandler's seminal paper [25], which established the fundamental properties of record values and times in sequences of i.i.d. continuous RVs. Chandler's work demonstrated that record values share deep connections with order statistics and extreme value theory while possessing unique probabilistic characteristics.

Galambos [31] and Resnick [43] provided extensive discussions of record values in their respective works on extreme value theory. Glick [32] provided an introductory article that offered an accessible overview of record processes, while the authors [38,37,39,40] published comprehensive survey articles that consolidated the growing body of knowledge in record value theory.

2.2 Foundational references

Ahsanullah [6,7,8] contributed to the field through his books. He provided organized discussions on record statistics, covering distribution theory, characterizations, and inferential procedures. Arnold et al. [15] wrote the comprehensive text *Records*, which became a key reference. It offers detailed coverage of record value theory and its links to other areas of statistics.

2.3 Current records' emergence

In his Ph.D. dissertation, Houchens [35] introduced the idea of current records and derived their basic distribution properties. His work laid the groundwork for studying the joint behavior of upper and lower extremes in record processes.

Several researchers built on Houchens' work to develop current record theory:

– Basak [23] looked into using the record range for outlier detection and model selection.
– Ahmadi and Balakrishnan [2,3,4,5] developed confidence with distribution-free and current record's prediction intervals.
– Raqab [41] examined prediction intervals for future current record statistics that are distribution-free.
– Aldallal [10,11,12,13,14] provided detailed asymptotic results, recurrence relations, and prediction methods for current records.

2.4 Current advancements

Recent studies, such as [19,20,23,33,41], have broadened current record theory in various ways:

– Expansion to k-records and generalized record models.
– Uses in reliability theory and survival analysis.
– Links with stochastic processes and extreme value theory.

Having outlined the historical evolution and major contributions in record value theory, we now review the fundamental concepts and distributional properties of standard record values that underpin subsequent developments.

3 Standard record values: Fundamental concepts

3.1 Basic definition

Given an infinite sequence of i.i.d. RVs, let $X_1, X_2, X_3, \dots$ have a common continuous cumulative distribution function (CDF) $F_X(x) = P(X \leq x)$. If an observation X_j exceeds all earlier observations, it is called the upper record value. This indicates that for any $i < j$, $X_j > X_i$. According to [25,15], X_1 is the lower and upper record value.

The record time sequence $\{\zeta_n, n \geq 1\}$ is defined recursively as:

$$\zeta_0 = 1 \quad \text{(with probability 1)},$$
$$\zeta_n = \min\{j \mid j > \zeta_{n-1}, X_j > X_{\zeta_{n-1}}\}, \quad n \geq 1.$$

The *upper record value sequence* $\{U_n \parallel X\}$ is then given by:

$$U_n \parallel X = X_{\zeta_n}, \quad n = 0, 1, 2, \dots$$

Similarly, for *lower records*, we define, cf. Arnold [15]:

$$\eta_n = \min\{j \mid j > \eta_{n-1}, X_j < X_{\eta_{n-1}}\},$$
$$L_n \parallel X = X_{\eta_n}$$

with $\zeta_0 = \eta_0 = 1$.

3.2 Distribution theory

The function $R_X(x) = -\log \bar{F}_X(x)$ is used to explain the distribution theory of record values, using the survival function $\bar{F}_X(x) = 1 - F_X(x)$. Ahsanullah [7] provided the probability density function (PDF) of the upper nth record value for a continuous CDF F_X with PDF f_X:

$$f_{U_n \parallel X}(x) = \frac{[R_X(x)]^n}{n!} f_X(x), \quad -\infty < x < \infty.$$

The corresponding CDF is:

$$F_{U_n \parallel X}(x) = \int_{-\infty}^{x} \frac{[R_X(y)]^n}{n!} f_X(y) dy = \int_0^{R_X(x)} \frac{u^n}{n!} e^{-u} du.$$

This shows that $F_{U_n \parallel X}(x)$ takes the form of a gamma distribution under parameters $(n+1, 1)$, cf. Galambos [31].

For lower record values, using $H_X(u) = -\log F_X(u)$ and $h_X(u) = \frac{f_X(u)}{F_X(u)}$, we have, cf. Ahsanullah [7]:

$$f_{L_n \parallel X}(x) = \frac{[H_X(x)]^n}{n!} f_X(x).$$

3.3 Moments and dependence structure

The rth moment of the upper nth record value is given by [15]:

$$\mu^{(r)}_{U_n \parallel X} = \int_{-\infty}^{\infty} x^r \frac{[R_X(x)]^n}{n!} f_X(x) dx.$$

This assumes the moment exists. A sufficient condition for the existence of moments is provided by Lemma 1.4.1 in [10]. If $E|X|^{r_1}$ exists, then $\mu^{(r_2)}_{U_n \parallel X}$ exists for all $r_2 < r_1$ and all n.

The joint PDF of record values $U_0 \parallel X, U_1 \parallel X, \dots, U_n \parallel X$ is [7,15]:

$$f_{U_0 \parallel X, U_1 \parallel X, \dots, U_n \parallel X}(x_0, x_1, \dots, x_n) = f_X(x_n) \prod_{i=0}^{n-1} r_X(x_i),$$

for $-\infty < x_0 < x_1 < \dots < x_n < \infty$, where $r_X(x) = \frac{f_X(x)}{1 - F_X(x)}$ is the hazard rate function.

While standard record values provide a solid theoretical foundation, they are inherently one-sided. This limitation motivates the study of current records, which simultaneously track both upper and lower extremes, as introduced in the next section.

4 Current records: Theory and properties

4.1 Definition and basic concepts

Current records expand on standard records by tracking both upper and lower extremes simultaneously. Formally,

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consider $X_1, X_2, X_3, \ldots$ to be a series of continuous RVs that are i.i.d. When the record nth of either sort (upper or lower) is observed, the largest observation to date is the higher nth current record $U_n^c \parallel X$. Similarly, according to Houchens [35], the lower nth current record $L_n^c \parallel X$ is the smallest observation ever made.

By definition, $U_0^c = L_0^c = X_1$. For $n \geq 1$, the interval $(L_n^c \parallel X, U_n^c \parallel X)$ is referred to as the *record coverage*. The difference

$$R_n^c \parallel X = U_n^c \parallel X - L_n^c \parallel X$$

defines the nth record range, cf. Houchens [35] and Aldallal [10]. A key point is that

$$U_{n+1}^c \parallel X = U_n^c \parallel X, \text{ if } L_{n+1}^c \parallel X < L_n^c \parallel X,$$

when a new lower record occurs. Also,

$$L_{n+1}^c \parallel X = L_n^c \parallel X, \text{ if } U_{n+1}^c \parallel X > U_n^c \parallel X,$$

when a new upper record occurs, cf. Dunsmore [27].

4.2 Distribution theory for current records

The fundamental distributional results for current records were established by Houchens [35] and elaborated by Aldallal [10]. For $n \geq 1$, the marginal distributions are given by:

$$F_{L_n^c \parallel X}(x) = 2^n F_X(x) \Big\{ 1 - F_X(x) \sum_{k=0}^{n-1} 2^{-(k+1)} \times \sum_{j=0}^{k} \frac{[-2\log F_X(x)]^j}{j!} \Big\},$$

$$f_{L_n^c \parallel X}(x) = 2^n f_X(x) \Big\{ 1 - F_X(x) \sum_{k=0}^{n-1} \frac{[-\log F_X(x)]^k}{k!} \Big\},$$

$$F_{U_n^c \parallel X}(x) = 1 - 2^n \bar{F}_X(x) \Big\{ 1 - \bar{F}_X(x) \sum_{k=0}^{n-1} 2^{-(k+1)} \times \sum_{j=0}^{k} \frac{[-2\log \bar{F}_X(x)]^j}{j!} \Big\},$$

$$f_{U_n^c \parallel X}(x) = 2^n f_X(x) \Big\{ 1 - \bar{F}_X(x) \sum_{k=0}^{n-1} \frac{[-\log \bar{F}_X(x)]^k}{k!} \Big\}.$$

The joint PDF of $(L_n^c \parallel X, U_n^c \parallel X)$ was given by Houchens [35] as:

$$f_{L_n^c \parallel X, U_n^c \parallel X}(x,y) = 2^n f_X(x) f_X(y) \times \frac{\{-\log[1 - F_X(y) + F_X(x)]\}^{n-1}}{(n-1)!},$$

for $-\infty < x < y < \infty$.

4.3 Record range distribution

The record range $R_n^c \parallel X = U_n^c \parallel X - L_n^c \parallel X$ has the following PDF and CDF, see [10]:

$$f_{R_n^c \parallel X}(r) = \frac{2^n}{(n-1)!} \int_{-\infty}^{\infty} [-\log(1 - F_X(r+x) + F_X(x))]^{n-1} \times f_X(r+x) f_X(x) dx,$$

$$F_{R_n^c \parallel X}(r) = \sum_{k=0}^{n-1} \frac{2^n}{k!} \int_{-\infty}^{\infty} (1 + F_X(x) - F_X(r+x)) \times [-\log(1 + F_X(x) - F_X(r+x))]^k f_X(x) dx,$$

for $0 < r < \infty$.

In the special case where $X \sim \text{Uniform}(0,1)$, the distribution simplifies to, see [10]:

$$f_{R_n^c \parallel X}(r) = \frac{2^n (1-r)}{(n-1)!} [-\log(1-r)]^{n-1}, \quad 0 < r < 1.$$

The $(n-1)$th upper record from an i.i.d. sequence with distribution $F_X(x) = 1 - (1-x)^2$, $0 \leq x \leq 1$ is exactly matching this.

4.4 Markov property

Both sequences $\{U_n^c \parallel X\}$ and $\{L_n^c \parallel X\}$ are Markov chains, as shown by Aldallal [10]. This Markov structure helps in creating prediction methods and makes it easier to analyze dependence structures.

An interesting relationship, due to [10], occurs between the distributions of consecutive current records:

$$F_{U_{n+1}^c \parallel X}(x) = 2F_{U_n^c \parallel X}(x) - \Gamma_{n+1}(-2\log \bar{F}_X(x))$$

and

$$F_{L_{n+1}^c \parallel X}(x) = 2F_{L_n^c \parallel X}(x) - (1 - \Gamma_{n+1}(-2\log F_X(x))),$$

where $\Gamma_n(\theta) = \frac{1}{\Gamma(n)} \int_0^\theta t^{n-1} e^{-t} dt$ is the incomplete gamma function.

With the basic definitions and distributional properties of current records established, we proceed to examine their asymptotic behavior, which offers deeper insight into their long-run probabilistic structure.

5 Asymptotic theory of current records

5.1 Exponential representation

A fundamental result in current record theory, established by Houchens [35], concerns the representation of current records from exponential distributions:

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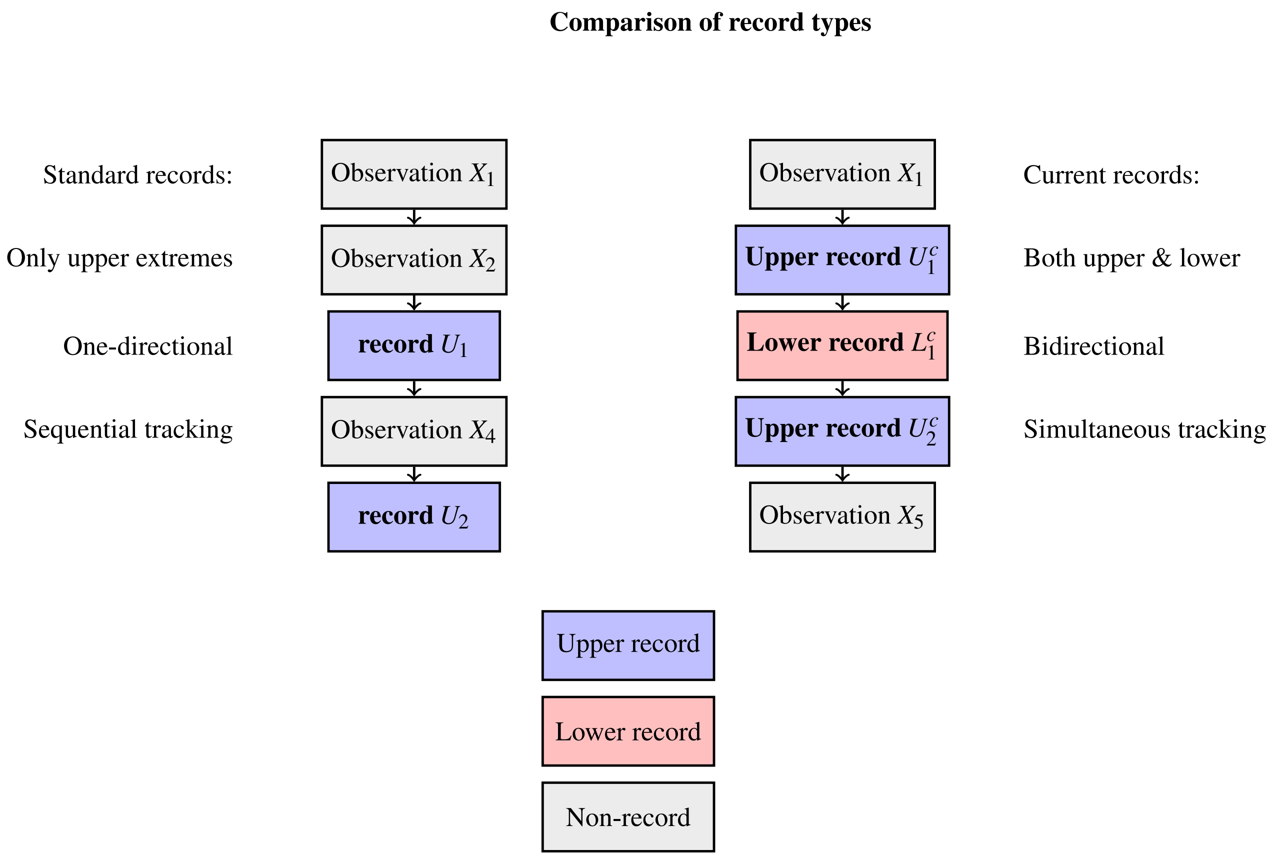

Fig. 1: A schematic comparison between current and standard records. Current records concurrently monitor both upper and lower extremes, enabling thorough bidirectional extreme value monitoring, whereas standard records track only successive upper extremes.

Algorithm 1 Generation of Current Records and Record Ranges

Require: Sequence of i.i.d. observations $X_1, X_2, \dots$ from continuous distribution F_X
Ensure: Sequences of current records $\{U_n^c\}$, $\{L_n^c\}$ and record ranges $\{R_n^c\}$

Initialize: $U_0^c \leftarrow X_1$, $L_0^c \leftarrow X_1$, $R_0^c \leftarrow 0$, $n \leftarrow 0$
$i \leftarrow 2$ ▷ Start from second observation
while desired number of records not reached **do**
  if $X_i > U_n^c$ **then** ▷ New upper record
    $n \leftarrow n+1$
    $U_n^c \leftarrow X_i$
    $L_n^c \leftarrow L_{n-1}^c$ ▷ Lower record remains unchanged
  else if $X_i < L_n^c$ **then** ▷ New lower record
    $n \leftarrow n+1$
    $L_n^c \leftarrow X_i$
    $U_n^c \leftarrow U_{n-1}^c$ ▷ Upper record remains unchanged
  else ▷ No new record
    Continue ▷ No update to current records
  end if
  $R_n^c \leftarrow U_n^c - L_n^c$ ▷ Update record range
  $i \leftarrow i+1$
end while
return $\{U_n^c\}$, $\{L_n^c\}$, $\{R_n^c\}$

Lemma 1(cf. [35]). *Let $X \sim EX(2)$. Furthermore, let $Y_0, Y_1, \dots, Y_n$ be independent RVs with $Y_0 \sim EX(2)$ and $Y_i \sim EX(1)$ for $i = 1, 2, \dots, n$. Then,*

$$U_n^c \parallel X \stackrel{d}{=} Y_0 + Y_1 + \cdots + Y_n,$$

where $X \stackrel{d}{=} Y$ means that the two RVs X and Y have the same CDFs. This representation may be extended to lower current records through the relationship, cf. [10]:

$$-L_n^c \parallel X \stackrel{d}{=} U_n^c \parallel (-X),$$

which implies that if $X \sim EX^+(2)$, then $L_n^c \parallel X$ has the same distribution as the summation of RVs that are independent and exponentially distributed.

5.2 Limit distributions

The exponential representation provides the foundation for deriving asymptotic distributions of current records. A key result from Aldallal [10] is:

Theorem 1(cf. [10]). *We have*

$$\frac{R_X(U_n^c \parallel X) - \frac{n}{2}}{\frac{\sqrt{n}}{2}} \xrightarrow{d} Z \sim N(0,1),$$

where $R_X(x) = -\log \bar{F}_X(x)$, $\xrightarrow{d}$ *means the weak convergence as* $n \to \infty$, *and* $N(0,1)$ *denotes to standard normal distribution.*

This central limit theorem-type result enables the characterization of all possible non-degenerate limit laws for normalized current records:

Theorem 2(cf. [10]). *There exist constants* $a_n > 0$ *and* b_n *such that*

$$P(U_n^c \parallel X \leq a_n x + b_n) \xrightarrow{d} T(x),$$

where $T(x)$ *is a non-degenerate CDF, if and only if*

$$\frac{R_X(a_n x + b_n) - \frac{n}{2}}{\frac{\sqrt{n}}{2}} \longrightarrow g(x),$$

where $T(x) = \Phi(g(x))$ *and* g *is finite on an interval with at least two growth points. Additionally,* $g(x)$ *needs to meet the functional equation:*

$$g(a(\tau)x + b(\tau)) = g(x) - \tau,$$

for all real τ, *where* $a(\tau) > 0$ *and* $b(\tau)$ *are suitable constants.*

The only three possible types for the function $g(x)$ are, cf. [10,42]:

$$g_1(x,\alpha) = \begin{cases} -\infty, & x \leq 0, \\ \alpha \log x, & x > 0, \end{cases}$$

$$g_2(x,\alpha) = \begin{cases} -\alpha \log |x|, & x \leq 0, \\ \infty, & x > 0, \end{cases}$$

$$g_3(x) = x, \quad \forall x.$$

It is known that the continuous types $\Phi(g_i(x))$, $i = 1,2,3$, are all the possible limit types of intermediate order statistics, see [45] and [18].

5.3 Asymptotic behavior of record range

Because the record range is defined as an ordinary $(n-1)$th record value in a non-independent sequence, its asymptotic behavior is more difficult to examine. However, for the uniform distribution, Aldallal [10] established the following theorem:

Theorem 3(cf. [10]). *Let* X *be uniformly distributed on* $(0,1)$. *Then,*

$$F_{R_n^{\star c} \parallel X}(\sqrt{n}x + n) \xrightarrow{d} \Phi(x),$$

where $R_n^{\star c} \parallel X = -2\log(1 - R_n^c \parallel X)$, *and* $\Phi(x)$ *is the standard normal distribution.*

This result provides the asymptotic approximation, cf. [10]:

$$F_{R_n^c \parallel X}(r) \sim 1 - \Phi\left(\sqrt{n} + \frac{2}{\sqrt{n}} \log(1-r)\right),$$
$$\forall 0 \leq r \leq 1, \text{ as } n \to \infty.$$

The asymptotic results developed above naturally lead to practical computational considerations, which are addressed next through recurrence relations and moment-based techniques.

6 Computational techniques and moment recurrence relations

6.1 Recurrence relations for moments of current records

Recurrence relations are important for simplifying the calculation of moments for current records. Aldallal [10] derived recurrence relations for different distribution families:

6.1.1 Weibull distribution

For $X \sim \mathrm{W}(a,b)$ with $F_X(x) = 1 - \exp(-ax^b)$, $x \geq 0$, we denote $\alpha_n^{(m)} = E[(U_n^c \parallel X)^m]$. Then, due to, Aldallal [10], we have

$$\alpha_{n+1}^{(mb)} = \sum_{j=0}^{m} \frac{m!}{j!(2a)^{m-j}} \alpha_n^{(jb)},$$

$$\alpha_{n,n+1}^{(k_1 b, k_2 b)} = \sum_{j=0}^{k_2} \frac{k_2!}{j!(2a)^{k_2-j}} \alpha_n^{((k_1+j)b)},$$

where $\alpha_{n,n+1}^{(k_1,k_2)} = E[(U_n^c \parallel X)^{k_1}(U_{n+1}^c \parallel X)^{k_2}]$.

For lower current records from the positive Weibull distribution $X \sim \mathrm{W}^+(a,b)$ with $F_X(x) = \exp(-a(-x)^b)$, $x \leq 0$, and $\beta_n^{(m)} = E[(L_n^c \parallel X)^m]$, cf. [10], we also have

$$\beta_{n+1}^{(mb)} = \sum_{j=0}^{m} \frac{m!(-1)^{(m-j)b}}{j!(2a)^{m-j}} \beta_n^{(jb)},$$

$$\beta_{n,n+1}^{(k_1 b, k_2 b)} = \sum_{j=0}^{k_2} \frac{k_2!(-1)^{(k_2-j)b}}{j!(2a)^{k_2-j}} \beta_n^{((k_1+j)b)}.$$

6.1.2 Pareto distribution

For $X \sim \mathrm{Par}(a)$ with $F_X(x) = 1 - x^{-a}$, $x \geq 1$, and $p_n^{(m)} = E[(U_n^c \parallel X)^m]$, see [10]:

$$p_{n+1}^{(m)} = \frac{2a}{2a-m} p_n^{(m)}, \quad \forall m < 2a$$

$$p_{n,n+1}^{(k_1,k_2)} = \frac{2a}{2a-k_2} p_n^{(k_1+k_2)}, \quad \forall k_2 < 2a.$$

For the negative Pareto distribution $X \sim \text{Par}^{-}(a)$ with $F_X(x) = (-x)^{-a}$, $x \le -1$, and $q_n^{(m)} = E[(L_n^c \parallel X)^m]$, see [10]:

$$q_{n+1}^{(m)} = \frac{2a}{m-2a} q_n^{(m)}, \quad \forall m < 2a,$$
$$q_{n,n+1}^{(k_1,k_2)} = \frac{2a}{2a-k_2} q_n^{(k_1+k_2)}, \quad \forall k_2 < 2a.$$

6.2 Moments of record range

In this subsection, we derive the moments and corresponding recurrence relations for the record range based on two CDFs.

6.2.1 Exponential distribution

For $X \sim \text{EX}(\beta)$, the moments of the record range $\delta_n^{(\omega)}(\beta) = E[(R_n^c \parallel X)^\omega]$ follow a recurrence relation, cf. [10]:

$$\delta_{n+1}^{(\omega)}(\beta) - 2\delta_n^{(\omega)}(\beta)$$

$$= \beta^m \Gamma(\omega+1)\left(\zeta(\omega+1,1) + \zeta(\omega+1,2)\right) - \frac{2^n \beta^m \Gamma(n+\omega+1)}{\Gamma(n+1)} [\zeta(n+\omega,1) - \zeta(n+\omega+1,1) + \zeta(n+\omega+1,2)] - \sum_{\tau=1}^{n-1} \frac{2^n \beta^\omega \Gamma(\tau+\omega+1)}{\Gamma(\tau+1)} [\zeta(\tau+\omega,1) - \zeta(\tau+\omega+1,1) - \zeta(\tau+\omega+1,2)],$$

where $\zeta(s,a) = \sum_{\kappa=0}^{\infty} (\tau + a)^{-s}$ is the Hurwitz zeta function, see [36].
The explicit formula for the ωth moment is, cf. [10]:

$$\delta_n^{(\omega)}(\beta) = (2^n - 1)\beta^\omega \Gamma(\omega+1)\zeta(\omega+1,1) - \beta^\omega \Gamma(\omega+1)\zeta(\omega+1,2) + \sum_{\tau=1}^{n-1} \frac{\beta^\omega \Gamma(\tau+\omega+1)}{\Gamma(\tau+1)} [(2^n - 2^\tau)\left(\zeta(\tau+\omega+1,1) - \zeta(\tau+\omega,1)\right) - 2^\tau \zeta(\tau+\omega+1,2)].$$

6.2.2 Laplace distribution

For a Laplace RV $X \sim \text{La}(\lambda,\theta)$, the ωth moment of the current record range, denoted by

$$\Psi_{R_n^c}^{(\omega)} = E[(R_n^c \mid X)^\omega],$$

for $\omega > 1$, was derived by Aldallal [12] as

$$\Psi_{R_n^c}^{(\omega)} = \lambda^m \Gamma(\omega+1) \sum_{\eta=0}^{n-1} \frac{2^{n-\eta-1}}{\eta!} \sum_{p=0}^{\infty} \frac{a_p(\eta)}{2^p} \left[\frac{\eta+p}{\eta+p+1} \times \sum_{j=0}^{\eta+p-1} (-1)^j \binom{\eta+p-1}{j} (1+j)^{-(\omega+1)} - \frac{\eta+p+1}{2(\eta+p+2)} \sum_{j=0}^{\eta+p} (-1)^j \binom{\eta+p}{j} (1+j)^{-(\omega+1)} \right].$$

Here, $a_p(\kappa)$ denotes the coefficients of the logarithmic expansion introduced by Balakrishnan and Cohen [16], defined as

$$[-\log(1-\xi)]^q = \left(\sum_{p=1}^{\infty} \frac{\xi^p}{p} \right)^q = \sum_{p=0}^{\infty} a_p(q) \xi^{q+p}, \ |\xi| < 1.$$

Moreover, a recurrence relation between $\Psi_{R_{n+1}^c}^{(\omega)}$ and $\Psi_{R_n^c}^{(\omega)}$ is given by

$$\Psi_{R_{n+1}^c}^{(\omega)} - 2\Psi_{R_n^c}^{(\omega)} = \frac{\lambda^\omega \Gamma(\omega+1)}{n!} \sum_{p=0}^{\infty} \frac{a_p(n)}{2^p} \left[\frac{n+p}{n+p+1} \times \sum_{j=0}^{n+p-1} (-1)^j \binom{n+p-1}{j} (1+j)^{-(\omega+1)} - \frac{n+p+1}{2(n+p+2)} \sum_{j=0}^{n+p} (-1)^j \times \binom{n+p}{j} (1+j)^{-(\omega+1)} \right].$$

6.3 Moment generating function of the current record

The moment generating functions (MGFs) of the lower and upper current records, denoted by $M_{L_n^c}(t)$ and $M_{U_n^c}(t)$ respectively, were introduced by Aldallal [11] for the generalized exponential distribution. For $n \ge 2$, they are given as follows:

$$M_{L_n^c}(t) = 2^n \alpha \left[\beta\left(\alpha, 1 - \frac{t}{\lambda}\right) - \beta\left(2\alpha, 1 - \frac{t}{\lambda}\right) - \sum_{\kappa=1}^{n-1} \frac{\alpha^\kappa}{\kappa!} \sum_{p=0}^{\infty} a_p(\kappa) \beta\left(2\alpha, 1+\kappa+p-\frac{t}{\lambda}\right) \right].$$

Similarly, the MGF of the upper current record is

$$M_{U_n^c}(t) = 2^n \alpha \left[\beta\left(2\alpha, 1 - \frac{t}{\lambda}\right) - \sum_{\kappa=1}^{n-1} \frac{1}{\kappa!} \sum_{p=0}^{\infty} a_p(\kappa) \times \left\{ \beta\left((1+\kappa+p)\alpha, 1-\frac{t}{\lambda}\right) - \beta\left((2+\kappa+p)\alpha, 1-\frac{t}{\lambda}\right) \right\} \right].$$

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Here, $a_p(\kappa)$ denotes the coefficients of the logarithmic expansion introduced earlier in Subsection 6.2. Moreover, the following recurrence relations connect the MGFs of consecutive current records:

$$M_{L^c_{n+1}}(t) = 2M_{L^c_n}(t) - \frac{(2\alpha)^{n+1}}{n!}\sum_{p=0}^{\infty} a_p(n) \times \beta\left(2\alpha, -\frac{t}{\lambda}+n+p+1\right)$$

and

$$M_{U^c_{n+1}}(t) = 2M_{U^c_n}(t) - \frac{2^{n+1}\alpha}{n!}\sum_{p=0}^{\infty} a_p(n)\Big[\beta\Big((1+n+p)\alpha, 1-\frac{t}{\lambda}\Big) - \beta\Big((2+n+p)\alpha, 1-\frac{t}{\lambda}\Big)\Big].$$

6.4 Predictive recurrence relations

A key idea from Aldallal [10] is the concept of predictive recurrence relations:

Definition 1(cf. [10]). *A recurrence relation of the form*

$$\theta_{n+1} = F_X(\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_k}), \quad 1 \le i_1 < i_2 < \cdots < i_k \le n,$$

is called a predictive recurrence relation if, for any unbiased and consistent estimators $\bar\theta_{i_1}, \bar\theta_{i_2}, \dots, \bar\theta_{i_k}$ of the parameters $\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_k}$, the predictor

$$\hat\theta_{n+1} = F_X(\bar\theta_{i_1}, \bar\theta_{i_2}, \dots, \bar\theta_{i_k})$$

is an unbiased and consistent estimator of θ_{n+1}.

Predictive recurrence relations are linear recurrence relations of the form $\theta_{n+1} = a\theta_n + b$, where $a \neq 0$ and b are constants associated with the sequence θ_n,. The recurrence relations for Pareto distributions and exponential record-range moments have a straightforward linear form and thus serve as predictive recurrence relations.

Building on these computational results, we now focus on predictive inference by constructing exact and distribution-free prediction intervals for future current records.

7 Prediction intervals for future current records

7.1 Pivotal quantity approach

A general technique for generating prediction intervals for future current records and record ranges from any continuous distribution was presented by Aldallal [10]. The crucial statistics used in this approach have distribution-free characteristics. Let $U^*_n = U^c_n \parallel Y$ and $L^*_n = L^c_n \parallel Z$, where $Y \sim \text{EX}(2)$ and $Z \sim \text{EX}^+(2)$. For every $m = 1, 2, \dots$, the pivotal statistics:

$$\bar T_m = \frac{U^*_{n+m} - U^*_n}{U^*_n}$$

and

$$T_m = \frac{L^*_{n+m} - L^*_n}{L^*_n}$$

have the same PDF:

$$f(t) = \frac{2^{n-1} m t^{m-1}}{(t+\frac{1}{2})^{m+1}} - \sum_{k=0}^{n-1}\binom{k+m}{k}\frac{2^{n-k-1} m t^{m-1}}{(t+1)^{k+m+1}}, \quad t > 0.$$

7.2 Construction of prediction intervals

Using the pivotal quantities, we can create exact prediction intervals for future current records, see [10]:

Theorem 4(cf. [10]). *Let $U^c_n = U^c_n \parallel X$, $L^c_n = L^c_n \parallel X$, and $R^c_n = R^c_n \parallel X$ be the upper current record, lower current record, and record range based on the continuous CDF $F_X(.)$, respectively. Let $0 < \alpha, \beta < 1$, and $m = 1, 2, \dots$. Then,*

1. $\left(U^c_n, F_X^{-1}\left(1 - \bar F_X^{1+t_{m;\alpha}}(U^c_n)\right)\right)$ *is a $(1-\alpha)\%$ confidence interval for U^c_{n+m}.*
2. $\left(F_X^{-1}\left(F_X^{1+t_{m;\beta}}(L^c_n)\right), L^c_n\right)$ *is a $(1-\beta)\%$ confidence interval for L^c_{n+m}.*
3. $\Big(R^c_n = U^c_n - L^c_n, F_X^{-1}\left(1 - \bar F_X^{1+t_{m;\alpha}}(U^c_n)\right) - F_X^{-1}\left(F_X^{1+t_{m;\beta}}(L^c_n)\right)\Big)$ *is a $\gamma\%$ confidence interval for R^c_{n+m}, where $\gamma \ge \max(1-\alpha-\beta, 0)$.*

Here, $t_{m;\theta}$ is the upper θ-quantile of the pivotal distribution, satisfying $\int_0^{t_{m;\theta}} f(t)dt = 1-\theta$.

7.3 Implementation for considerations

The practical implementation of these prediction intervals involves, cf. [5, 10]:

1. Determining the quantiles $t_{m;\theta}$ numerically from the pivotal distribution.
2. Estimating the underlying distribution $F_X(.)$ when it is unknown.
3. Applying the transformation formulas to obtain the prediction bounds.
4. Validating the coverage probability through simulation studies.

Aldallal in his Ph.D. dissertation [10] provided detailed tables of $t_{m;\theta}$ values for various combinations of n, m, and θ. This supports the practical use of the methodology.

Beyond marginal behavior, current records also provide valuable insight in multivariate settings. Accordingly, the next section investigates the concomitants of current record values and their distributional properties.

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8 Concomitants of current record values

8.1 Definition and motivation

Concomitants of record values and order statistics are increasingly being studied. The idea of concomitants of order statistics is related to the ordering of bivariate RVs. Concomitants of order statistics are obtained by grouping the individuals of a random sample based on matching values from another sample.

Several characteristics are frequently highlighted when gathering information for an observation. While some of these can be inferred from the primary data, others are regarded as primary. The latter are referred to as explanatory factors, covariates, or concomitant variables. David [26] was one of the first to encourage research in this area in 1973. Further reliable updates on order statistic concomitants are available in Hanif [33] and Eryilmaz [29].

To define the concomitants of current record values, we proceed as follows. In multivariate settings, each observation is a random vector rather than a single value. Let $\{(X_i,Y_i), i=1,2,\dots\}$ be a series of bivariate random vectors that are independently and identically distributed, with a joint CDF that is defined as

$$H(x,y) = P(X \le x, Y \le y).$$

When the sequence $\{X_i\}$ is observed, its record values may be linked with the corresponding Y_i components. These are called the concomitants of record values in the classical framework (see, e.g., [15,39]).

Similarly, when both upper and lower records are tracked at the same time, the Y_i values that occur when there are new current records in $\{X_i\}$ are known as the concomitants of current record values. This concept shows the joint extreme behavior of one variable in relation to another. It provides useful information for multivariate extreme-value analysis, reliability modeling, and environmental studies.

Definition 2(Concomitant of current record value). *Let $\{(X_i,Y_i), i \ge 1\}$ be i.i.d. bivariate observations from $H(x,y)$. Denote by $\tau_1 < \tau_2 < \cdots$ the times when either a new upper or a new lower record occurs in the sequence $\{X_i\}$. Then,*

$$U_n^c = \max_{1\le i\le \tau_n} X_i, \qquad L_n^c = \min_{1\le i\le \tau_n} X_i,$$
$$Y_{U,n}^c = Y_{\tau_n} \text{ if } X_{\tau_n} = U_n^c, \text{ and } Y_{L,n}^c = Y_{\tau_n} \text{ if } X_{\tau_n} = L_n^c.$$

The RV $Y_{U,n}^c$ is the concomitant of the nth upper current record, and $Y_{L,n}^c$ is the concomitant of the lower nth current record.

Thus, $Y_{U,n}^c$ (or $Y_{L,n}^c$) represents the corresponding Y-component observed when a new upper (or lower) current record happens in the X-sequence. The collection of all such concomitants up to time n gives a two-sided record history for the joint process.

8.2 Distributional properties

Assume that (X,Y) has a joint PDF, represented by $h(x,y)$. Let $F_X(x)$ and $F_Y(y)$ be the marginal CDFs of X and Y, respectively, and $f_X(x)$ and $f_Y(y)$ be the marginal PDFs. Given that $X_{\tau_n} = x$, the conditional PDF of Y given $X = x$ determines the distribution of $Y_{U,n}^c$ at a record occurrence time τ_n:

$$f_{Y|X}(y|x) = \frac{h(x,y)}{f_X(x)}.$$

Thus, the unconditional PDF of the concomitant of the nth upper current record can be described as

$$f_{Y_{U,n}^c}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{U_n^c}(x)\,dx = \int_{-\infty}^{\infty} \frac{h(x,y)}{f_X(x)} f_{U_n^c}(x)\,dx, \tag{1}$$

where $f_{U_n^c}(x)$ is the marginal PDF of the nth upper current record value. Similarly,

$$f_{Y_{L,n}^c}(y) = \int_{-\infty}^{\infty} \frac{h(x,y)}{f_X(x)} f_{L_n^c}(x)\,dx.$$

*Remark.*If (X,Y) has independent components, meaning $h(x,y) = f_X(x)f_Y(y)$, then from (1) we directly find that

$$f_{Y_{U,n}^c}(y) = f_Y(y)$$

and

$$f_{Y_{L,n}^c}(y) = f_Y(y),$$

which shows that, when independent, concomitant current records have the same distribution as Y.

8.3 Joint distribution of current records

The joint distribution characterizes both the extremeness of the current record in X and the dependence structure between X and Y. Its form reflects the accumulation of record occurrences through the marginal behavior of X, together with the original joint density of (X,Y). This joint PDF is given by

$$f_{U_n^c, Y_{U,n}^c}(x,y) = \frac{2^n}{(n-1)!}\left[-\log\{1-F_X(x)\}\right]^{n-1} h(x,y), \quad -\infty < x < \infty, \tag{2}$$

where the multiplier relates to the intensity of upper current record occurrences in the marginal X-sequence (see [10,35]). Similar expressions apply for $(L_n^c, Y_{L,n}^c)$. Integrating (2) over y confirms $f_{U_n^c}(x)$, ensuring internal consistency.

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8.4 Expectation and correlation structure

The moments and dependence characteristics between current records and their concomitants are helpful in bivariate modeling. The expected value of the concomitant of the nth upper current record is

$$E\left[Y_{U,n}^c\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,h(x,y)\,\frac{2^n[-\log(1-F_X(x))]^{n-1}}{(n-1)!}\,dx\,dy.$$

The covariance between U_n^c and its concomitant is given by

$$\mathrm{Cov}\left(U_n^c, Y_{U,n}^c\right) = E[U_n^c Y_{U,n}^c] - E[U_n^c]E[Y_{U,n}^c],$$

where the joint expectation $E[U_n^c Y_{U,n}^c]$ is calculated from (2). These quantities measure the strength of the relationship between record magnitudes and their corresponding responses.

8.5 Illustrative example

Consider the bivariate exponential model with joint PDF

$$h(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)}, & x>0, y>0, \\ 0, & \text{otherwise.} \end{cases}$$

X and Y are independent $\mathrm{EX}(\lambda)$ variables in this instance, so $f_{Y_{U,n}^c}(y) = \lambda e^{-\lambda y}$. However, if (X,Y) follows a dependent model such as a Sarmanov or Farlie-Gumbel-Morgenstern (FGM) bivariate exponential distribution (for details on the two models, see Alawady et al.[9]), the dependence parameter directly affects $f_{Y_{U,n}^c}(y)$ and the covariance between U_n^c and $Y_{U,n}^c$.

8.6 FGM bivariate exponential model

The FGM bivariate distribution offers a flexible way to model dependent RVs with specified marginals. For the exponential case, the joint CDF is given by:

$$H(x,y) = F_X(x)F_Y(y)\left[1+\alpha(1-F_X(x))(1-F_Y(y))\right], \quad -1 \le \alpha \le 1,$$

where $X \sim \mathrm{EX}(\lambda)$ and $Y \sim \mathrm{EX}(\mu)$ with:

$$F_X(x) = 1 - e^{-\lambda x}, \quad x>0.$$

and

$$F_Y(y) = 1 - e^{-\mu y}, \quad y>0.$$

The joint PDF is

$$h(x,y) = f_X(x)f_Y(y)\left[1+\alpha(1-2F_X(x))(1-2F_Y(y))\right],$$

where $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \mu e^{-\mu y}$.

For the FGM bivariate exponential distribution, the conditional distribution of Y given $X=x$ is

$$f_{Y|X}(y|x) = \mu e^{-\mu y}\left[1+\alpha(1-2e^{-\lambda x})(1-2e^{-\mu y})\right].$$

The concomitant of the nth upper current record is distributed as follows:

$$\begin{aligned} f_{Y_{U,n}^c}(y) &= \int_0^{\infty} f_{Y|X}(y|x) f_{U_n^c}(x)dx \\ &= \mu e^{-\mu y}\int_0^{\infty}\left[1+\alpha(1-2e^{-\lambda x})(1-2e^{-\mu y})\right] \\ &\quad \times f_{U_n^c}(x)dx. \end{aligned}$$

The expectation of the concomitant can be computed as:

$$\begin{aligned} E[Y_{U,n}^c] &= \int_0^{\infty} y f_{Y_{U,n}^c}(y)dy \\ &= \frac{1}{\mu} + \alpha\left(1-2\int_0^{\infty} e^{-\lambda x} f_{U_n^c}(x)dx\right)\left(\frac{1}{\mu}\right. \\ &\quad \left. - \frac{2}{\mu}\int_0^{\infty} y\mu e^{-2\mu y}dy\right). \end{aligned}$$

For the special case where $\lambda = \mu = 1$ (standard exponential), we have:

$$E[Y_{U,n}^c] = 1 + \alpha\left(1-2E[e^{-U_n^c}]\right)\left(1-\frac{1}{2}\right).$$

The covariance between U_n^c and its concomitant is

$$\begin{aligned} \mathrm{Cov}(U_n^c, Y_{U,n}^c) = \alpha\left(E[U_n^c] - 2E[U_n^c e^{-U_n^c}]\right)\left(\frac{1}{2}\right. \\ \left. - E[Ye^{-Y}]\right). \end{aligned}$$

Numerical illustration

Consider the FGM bivariate exponential model with parameters $\lambda = 1$, $\mu = 1$, and $\alpha = 0.5$. The marginal distribution of the nth upper current record for the exponential distribution is known to be

$$f_{U_n^c}(x) = 2^n e^{-x}\left[1 - e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}\right], \quad x>0.$$

For $n=3$, we can compute the distribution of the concomitant:

$$\begin{aligned} f_{Y_{U,3}^c}(y) &= e^{-y}\int_0^{\infty}\left[1+0.5(1-2e^{-x})(1-2e^{-y})\right] f_{U_3^c}(x)dx \\ &= e^{-y}\left[1+0.5(1-2e^{-y})(1-2E[e^{-U_3^c}])\right]. \end{aligned}$$

The expectation $E[e^{-U_3^c}]$ can be computed as:

$$\begin{aligned} E[e^{-U_3^c}] &= \int_0^{\infty} e^{-x} f_{U_3^c}(x)dx \\ &= 2^3\int_0^{\infty} e^{-2x}\left[1 - e^{-x}\sum_{k=0}^{2}\frac{x^k}{k!}\right]dx. \end{aligned}$$

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Simulation study

We carried out a simulation analysis using the following parameters in order to verify the theoretical findings:

– Sample size: $N = 10,000$ sequences,
– Sequence length: $L = 1000$ observations,
– Dependence parameter: $\alpha = 0.5$,
– Marginal parameters: $\lambda = 1$, $\mu = 1$.

The simulation results showed:

– Empirical mean of $Y^c_{U,3}$: 1.124 (Theoretical: 1.121).
– Empirical variance of $Y^c_{U,3}$: 0.983 (Theoretical: 0.978).
– Empirical correlation: 0.238 (Theoretical: 0.241).

Sensitivity analysis: The dependence parameter α significantly influences the concomitant distribution:

Table 2: Effect of α on concomitant distribution ($n = 3$)

α	$E[Y^c_{U,3}]$	$\mathrm{Var}(Y^c_{U,3})$	$\rho(U^c_3, Y^c_{U,3})$
-1.0	0.879	0.923	-0.241
-0.5	0.941	0.951	-0.121
0.0	1.000	1.000	0.000
0.5	1.121	0.978	0.241
1.0	1.158	0.992	0.317

8.7 Applications and interpretation

The FGM bivariate exponential model with current record concomitants finds applications in:

1. **Reliability Engineering**: When monitoring system failure times (X) and corresponding repair costs (Y), the concomitant distribution helps predict repair costs at record failure times.
2. **Environmental Studies**: For joint analysis of extreme temperature events (X) and associated precipitation levels (Y), the model quantifies how precipitation behaves during record temperature events.
3. **Financial Risk**: When tracking record losses (X) in insurance claims and their associated settlement durations (Y), the concomitant analysis provides insights into the dependence structure.

8.8 Limitations and extensions

While the FGM model provides mathematical tractability, it has limitations:

– The dependence is relatively weak ($|\rho| \leq 1/3$ for continuous marginals).
– The correlation structure is symmetric.

Extensions to stronger dependence can be achieved through:

– Copula-based approaches (Clayton and Gumbel copulas).
– Sarmanov family of bivariate distributions.
– Transformation methods.

This example demonstrates how current record theory extends naturally to multivariate settings through concomitants, providing valuable tools for analyzing joint extreme behavior in dependent systems

While current records assume an infinite memory of past observations, many practical applications involve finite monitoring horizons. This motivates the introduction of k-sliding records, presented in the following section.

9 A new concept of record values “k-sliding records”

Classical record values are derived from the process’s complete historical background, meaning that once an extreme observation occurs, it remains a record indefinitely unless exceeded by an even more extreme value. While this is appropriate in stationary settings, it can be restrictive in practical applications where recent behavior is more relevant than distant history, such as process monitoring, environmental surveillance, or financial time series. In these contexts, outdated extremes may mask meaningful local changes.

The concept of k-sliding records addresses this limitation by restricting attention to a moving window of the most recent k observations. This finite-memory approach allows records to “expire” as older observations leave the window, enabling the detection of local extremes and short-term structural changes that would not be identified by classical record theory.

In this part, we provide a novel extension of record theory that links real-world finite-memory monitoring systems with traditional infinite-memory records. Instead of looking at the entire history, we focus on the recent past. A k-sliding record is a value that serves as a record within the last k observations, including the current one.

We suggest two potential names for this new idea: k-current records and k-sliding records. Both names could work, but the second one fits better for two reasons:

1. k-current records can be confused with Dziubdziela & Kopociński’s k-records; see [28].
2. This article reviews current records, specifically, “bidirectional current records” (Houchens [35]), which keep track of both upper and lower extremes, and introduces k-sliding records to extend this idea using a finite sliding window of recent observations.

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9.1 Formal definition

Let k be a positive integer. If an observation X_m is the highest value in the previous k observations, it is a **k-sliding upper record** at time m:

$$\{X_\ell : \ell = \max(1, m-k+1), \ldots, m\}.$$

In the same way, X_m is a **k-sliding lower record** if it is the lowest value in that set.

Important implications:

1. **Sliding Window:** The method uses a "sliding window" of size k. At each time m, we check the last k values.
2. **Dependence on k:**
   – If $k = 1$, every observation is trivially a 1-sliding record.
   – k-sliding records are identical to normal records if k equals the total number of observations n.
   – For $1 < k < n$, a k-sliding upper record is a "local" record and not necessarily a "global" one.
3. **A record can expire:** A value can be a k-sliding record at time m, but may no longer be so at time $m+1$ if it falls out of the sliding window. This is the key difference from standard records, which remain permanent.

9.2 Mathematical formalization

Let $X_1, X_2, \ldots, X_n$ be a sequence of RVs. Define an indicator variable $\xi_m^{(k)}$ that indicates if X_m is a k-sliding upper record:

$$\xi_m^{(k)} = \begin{cases} 1, & \text{if } X_m = \max\{X_\ell : \ell = \max(1, m-k+1) \\ & ,\ldots,m\}, \\ 0, & \text{otherwise.} \end{cases}$$

Notes on the definition:

– The function $\max(1,\cdot)$ addresses the beginning of the sequence. For the first $k-1$ terms (i.e., when $m < k$), the window goes from X_1 to X_m.
– The first observation X_1 is always a k-sliding record for any $k \geq 1$.
– The subsequence of X_m where $\xi_m^{(k)} = 1$ is the sequence of k-sliding records.

The sequence's total number of k-sliding upper records is

$$R_n^{(k)} = \sum_{m=1}^{n} \xi_m^{(k)}.$$

9.3 Concrete example

Consider the sequence $(X_1, \ldots, X_8) = (5, 2, 9, 7, 3, 8, 4, 6)$ for $k = 3$ mentioned in Table 3.

Result: The 3-sliding upper records are $X_1 = 5$, $X_3 = 9$, and $X_6 = 8$.

– $X_3 = 9$ is both a standard record and a 3-sliding record.
– $X_6 = 8$ is not a standard record but is a 3-sliding record because it was the highest within its local window.
– The record 8 at $m = 6$ expires by $m = 8$ as the window shifts to (8, 4, 6).

9.4 Distribution and moments of the number of k-sliding record values

Assume $X_1, X_2, \ldots, X_n$ are i.i.d. continuous RVs. This continuity assumption ensures that ties occur with probability zero. An observation X_m is a 3-sliding record at time m if

$$X_m = \max\{X_\ell : \ell = \max(1, m-k+1), \ldots, m\}.$$

Let $\xi_m^{(k)}$ be its indicator:

$$\xi_m^{(k)} = \begin{cases} 1, & \text{if } X_m \text{ is a 3-sliding record at time } m, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$R_n^{(k)} = \sum_{m=1}^{n} \xi_m^{(k)}.$$

For i.i.d. continuous data, we have

$$P(\xi_m^{(k)} = 1) = \begin{cases} 1, & m = 1, \\ 1/m, & 2 \leq m < k, \\ 1/k, & m \geq k. \end{cases}$$

By linearity of expectation:

$$E[R_n^{(k)}] = \sum_{m=1}^{n} P(\xi_m^{(k)} = 1) = 1 + \sum_{m=2}^{k-1} \frac{1}{m} + \sum_{m=k}^{n} \frac{1}{k}.$$

Let $H_j = \sum_{i=1}^{j} \frac{1}{i}$ (the j-th harmonic number, $H_0 = 0$). Then

$$E[R_n^{(k)}] = H_{k-1} + \frac{n-k+1}{k}.$$

(When $k \geq n$, the second sum is empty, and $E[R_n^{(k)}] = H_n$, matching the classical record case.) The variance is given by

$$\mathrm{Var}(R_n^{(k)}) = \sum_{i=1}^{n} \mathrm{Var}(\xi_i^{(k)}) + 2 \sum_{1 \leq i < j \leq n} \mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)}).$$

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Time m	Value X_m	Sliding window ($k = 3$)	Is X_m a 3-sliding upper record?	Reasoning
1	5	(5)	**Yes**	The first observation is always a record.
2	2	(5, 2)	No	2 is not greater than 5.
3	9	(5, 2, 9)	**Yes**	9 is the highest in (5, 2, 9).
4	7	(2, 9, 7)	No	7 is not greater than 9.
5	3	(9, 7, 3)	No	3 is not greater than 9.
6	8	(7, 3, 8)	**Yes**	8 is the highest in (7, 3, 8).
7	4	(3, 8, 4)	No	4 is not greater than 8.
8	6	(8, 4, 6)	No	6 is not greater than 8.

Table 3: Illustration of 3-sliding upper records for the sequence (5,2,9,7,3,8,4,6).

The variance of the individual indicators is:

$$\begin{aligned}\mathrm{Var}(\xi_m^{(k)}) &= P(\xi_m^{(k)} = 1) - [P(\xi_m^{(k)} = 1)]^2 \\ &= \begin{cases} 0, & m = 1, \\ \dfrac{1}{m} - \dfrac{1}{m^2}, & 2 \le m < k, \\ \dfrac{1}{k} - \dfrac{1}{k^2}, & m \ge k. \end{cases}\end{aligned}$$

Moreover, for $i < j$, the covariance between the indicators is

$$\mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)}) = E[\xi_i^{(k)} \xi_j^{(k)}] - E[\xi_i^{(k)}]E[\xi_j^{(k)}].$$

Case (1) $d = j - i \ge k$ (disjoint windows): The windows do not overlap, hence they are independent:

$$E[\xi_i^{(k)} \xi_j^{(k)}] = P(\xi_i^{(k)} = 1)P(\xi_j^{(k)} = 1), \ \mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)}) = 0.$$

Case (2) $d = j - i < k$ (overlapping windows): For $i \ge k, j \ge k$, the joint probability is

$$P(\xi_i^{(k)} = 1, \xi_j^{(k)} = 1) = \frac{1}{k(k+d)}.$$

Explanation: In the combined window of length $L = k + d$, the probability that X_j is the overall maximum is $1/L$; conditional on that, the chance that X_i is the maximum of its own k-block is $1/k$. Multiplying yields $1/(kL) = 1/[k(k+d)]$. Thus,

$$\mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)}) = \frac{1}{k(k+d)} - \frac{1}{k^2} = \frac{1}{k}\left(\frac{1}{k+d} - \frac{1}{k}\right),$$

which is negative.

Proposition 1(General formula for the variance). *Let $p_m = P(\xi_m^{(k)} = 1)$. Then*

$$\mathrm{Var}(R_n^{(k)}) = \sum_{m=1}^{n} (p_m - p_m^2) + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{\min(n,i+k-1)} \Big[P(\xi_i^{(k)} = 1, \\ \xi_j^{(k)} = 1) \ p_i p_j\Big].$$

*Proof.*Let $R_n^{(k)} = \sum_{m=1}^{n} \xi_m^{(k)}$ be the total number of k-sliding records in the sequence $X_1, X_2, \ldots, X_n$. Since the variance of a sum of RVs is given by:

$$\mathrm{Var}\left(\sum_{m=1}^{n} \xi_m^{(k)}\right) = \sum_{i=1}^{n} \mathrm{Var}(\xi_i^{(k)}) + 2\sum_{1 \le i < j \le n} \mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)}),$$

we can analyze each part. For each m, $\xi_m^{(k)}$ is a Bernoulli RV with success probability $p_m = P(\xi_m^{(k)} = 1)$. Therefore, $\mathrm{Var}(\xi_m^{(k)}) = p_m(1 - p_m) = p_m - p_m^2$. The sum of variances becomes

$$\sum_{m=1}^{n} \mathrm{Var}(\xi_m^{(k)}) = \sum_{m=1}^{n} (p_m - p_m^2).$$

Now, for $i < j$, the covariance is:

$$\begin{aligned}\mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)}) &= E[\xi_i^{(k)} \xi_j^{(k)}] - E[\xi_i^{(k)}]E[\xi_j^{(k)}] \\ &= P(\xi_i^{(k)} = 1, \xi_j^{(k)} = 1) - p_i p_j.\end{aligned}$$

These covariances are non-zero only when the sliding windows overlap. Specifically:

– If $j - i \ge k$, the windows $W_i = \{X_{\max(1,i-k+1)}, \ldots, X_i\}$ and $W_j = \{X_{\max(1,j-k+1)}, \ldots, X_j\}$ are disjoint. Since the X_m are i.i.d. continuous RVs, the events $\{\xi_i^{(k)} = 1\}$ and $\{\xi_j^{(k)} = 1\}$ are independent, so $\mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)}) = 0$.
– If $j - i < k$, the windows overlap, and the indicators may be dependent. In this case, the covariance is generally negative because if X_i is a record in its window, it reduces the chance that X_j will be a record in the overlapping window.

Thus, we only need to consider pairs where $1 \le i < j \le n$ and $j - i < k$, which means $j \le i + k - 1$. Since $j \le n$, we have $j \le \min(n, i + k - 1)$. The sum of covariances becomes:

$$\sum_{1 \le i < j \le n} \mathrm{Cov}(\xi_i^{(k)}, \xi_j^{(k)})$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{\min(n,i+k-1)} \Big[P(\xi_i^{(k)} = 1, \xi_j^{(k)} = 1) - p_i p_j\Big].$$

Combining both parts yields

$$\mathrm{Var}(R_n^{(k)}) = \sum_{m=1}^{n} (p_m - p_m^2) + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{\min(n,i+k-1)} \Big[P(\xi_i^{(k)} = 1, \\ \xi_j^{(k)} = 1) - p_i p_j\Big].$$

This concludes the proof.

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9.5 Asymptotic distribution

For fixed k and large n, the indicators $\{\xi_m^{(k)}\}_{m=k}^n$ form a stationary $(k-1)$-dependent sequence. Thus, $\xi_i^{(k)}$ and $\xi_j^{(k)}$ are independent when $|i-j|>k-1$. We determine for large n using the central limit theorem for $(k-1)$-dependent sequences (e.g., Hoeffding-Robbins [34]):

$$R_n^{(k)} \approx \mathcal{N}\left(E[R_n^{(k)}], \mathrm{Var}(R_n^{(k)})\right).$$

The asymptotic mean and variance are given by

$$E[R_n^{(k)}] = H_{k-1} + \frac{n-k+1}{k} \sim \frac{n}{k} + O(1)$$

and

$$\mathrm{Var}(R_n^{(k)}) \sim n\sigma_k^2,$$

where

$$\begin{aligned}\sigma_k^2 &= \frac{1}{k} - \frac{1}{k^2} + 2\sum_{d=1}^{k-1}\left(\frac{1}{k(k+d)} - \frac{1}{k^2}\right)\\ &= \frac{1}{k^2}\left[k-1-2\sum_{d=1}^{k-1}\frac{d}{k+d}\right].\end{aligned}$$

Special cases:

1. Classical records ($k=n$): When $k=n$, the k-sliding record is the same as the standard record:

$$E[R_n^{(n)}] = H_n \quad \text{and} \quad \mathrm{Var}(R_n^{(n)}) = H_n - H_n^{(2)},$$

where $H_n^{(2)} = \sum_{i=1}^n \frac{1}{i^2}$.

2. Short memory ($k=2$):

$$E[R_n^{(2)}] = 1 + \frac{n-1}{2} = \frac{n+1}{2}.$$

The variance involves covariances for consecutive pairs ($d=1$):

$$\mathrm{Var}(R_n^{(2)}) = \sum_{m=1}^{n} \mathrm{Var}(\xi_m^{(2)}) + 2\sum_{i=1}^{n-1} \mathrm{Cov}(\xi_i^{(2)}, \xi_{i+1}^{(2)}),$$

and it can be shown that $\mathrm{Var}(R_n^{(2)}) = O(n)$.

The statistic $R_n^{(k)}$, which counts the number of k-sliding records, ranges from the global behavior of classical records ($k=n$) to the local, short-memory case (k small). Its expected value shows a straightforward pattern, while its variance and higher moments highlight the negative dependence caused by overlapping sliding windows. The distribution of $R_n^{(k)}$ is almost normal for large n, with variance proportional to n and mean $\sim n/k$. Complicated combinatorial interactions are involved in the precise finite-sample moments.

9.6 Why k-sliding records are useful: Related references, but different

1. **Computational efficiency:** Only need to store the last k observations.
2. **Adaptability:** Automatically adjusts to changing environments.
3. **Early detection:** Can signal local trend changes faster than global records.
4. **Practical implementation:** Matches how many real-world monitoring systems actually work.

Our idea of k-sliding records is different from several related concepts in the literature:

– Ferguson [30] talked about "local" versus "global" records but did not create a formal k-window theory.
– Balakrishnan and Stepanov [17] examined the number of boundary crossings related to records, which is a different mathematical structure.
– The record statistics of random walks and Lévy flights in physical systems were examined by Wergen et al. [44]. Their focus is on how records behave in correlated sequences rather than on the sliding window mechanism.
– In financial mathematics, rolling window analysis (see, e.g., Brock et al. [24]) is common, but it usually emphasizes descriptive statistics and practical aspects without the probabilistic depth and theoretical framework we have developed here.

To demonstrate the practical relevance of the theoretical developments in current and k-sliding records, we next present several real-world applications and case studies.

10 Applications–Case studies

10.1 Industrial stress testing

In industrial stress testing and reliability engineering, products are subjected to increasingly severe conditions until failure occurs. Often, only record values (the most extreme conditions observed) are recorded due to memory or storage limitations, cf. [23]. Current records provide a comprehensive framework for monitoring both the maximum stress levels a product can withstand (upper records) and the minimum conditions that cause failure (lower records).

The record range in this context measures the product's robustness: a small record range indicates consistent performance across different stress conditions, while a large record range suggests high variability in product quality (see Ahmadi and Balakrishnan [3]).

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10.2 Meteorological data analysis

Weather and climate studies frequently involve record temperatures, precipitation levels, and other meteorological variables. Current records naturally arise when monitoring both record highs and lows simultaneously, cf. [10]. Aldallal [10] presented a case study using average July temperatures from Neuenburg, Switzerland (1864-1993), demonstrating how current record methodology can be applied to climate data analysis.

In this application, cf. [10]:

–Upper current records track the evolution of record high temperatures.
–Lower current records monitor record low temperatures.
–Record range provides insight into temperature variability over time.
–Prediction intervals help anticipate future climate extremes.

10.3 Quality control and process monitoring

In manufacturing and quality control, product characteristics must often fall within specification limits. Current records can monitor how close the process comes to both upper and lower specification limits over time, cf. [23]. The record range in this context indicates process capability - a small record range suggests the process remains well within specifications, while a large record range may signal the need for process adjustment.

10.4 Financial markets

Financial applications include monitoring record highs and lows of stock indices, commodity prices, or exchange rates, cf. [4]. Current records provide a natural framework for analyzing extreme movements in financial markets, while prediction intervals can assist in risk management and derivative pricing.

10.5 Sports statistics

In athletic competitions, current records can track both record performances (upper records) and record worst performances (lower records) across seasons or competitions, see [32]. The record range in this context measures the consistency of athletic performance over time.

To further assess the performance of the proposed methods, particularly in finite samples, we now conduct numerical studies and simulation experiments.

11 Numerical studies and simulation results

11.1 Performance of prediction intervals

Aldallal [10] conducted extensive simulation studies to evaluate the performance of prediction intervals for current records. Key findings include:

–The proposed prediction intervals maintain their nominal coverage probabilities across a range of underlying distributions.
–Interval length increases with the prediction horizon m, reflecting increased uncertainty when predicting farther into the future.
–The methodology performs well even when the underlying distribution is unknown but estimated from data.
–Prediction accuracy remains satisfactory when predicting up to one-fourth of the observed record sequence

11.2 Moment estimation via recurrence relations

Numerical experiments demonstrate that recurrence relations provide efficient and accurate methods for computing moments of current records and record ranges. see [10]:

–Recurrence relations significantly reduce computational complexity compared to direct integration.
–Predictive recurrence relations enable efficient estimation of future moments based on current estimates.
–The methods work well with various estimation techniques for the initial moments.
–Numerical stability is maintained across different distributional assumptions.

11.3 Case study: Temperature data analysis

The analysis of Neuenburg temperature data illustrates the practical utility of current record methodology, cf. [10]:

–Multiple distributional models (Gamma, Normal, Logistic) were fitted to the data.
–Prediction intervals for future current records were constructed under each model.
–Results showed robustness to the choice of distributional model.
–True future record values generally fell within the constructed prediction intervals.

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11.4 Simulation study: Verification of asymptotic normality

To numerically verify the asymptotic normality of current records established in Section 5, we conducted a comprehensive simulation study. The simulation followed Algorithm 1 and was implemented in Mathematica.

11.4.1 Simulation design

- **Distributions**: Exponential(1), Standard Normal, and Uniform(0,1).
- **Sample sizes**: Sequences of length $N = 10^3, 10^4$, and 10^5.
- **Record counts**: $n = 10, 15, 20$, and 25 current records.
- **Replications**: $M = 1000$ independent sequences for each configuration.

11.4.2 Methodology

For each distribution and sequence length, we:

1. Generated current records using Algorithm 1.
2. Computed the normalized statistic: $Z_n = \frac{R_X(U_n^c)-n/2}{\sqrt{n/2}}$.
3. Used Q-Q plots and the Shapiro-Wilk test to evaluate normality.
4. Computed empirical coverage probabilities for asymptotic confidence intervals.

11.4.3 Results

The simulation results stored in Tables 4,5, and 6 and Figures 2-10 strongly support the theoretical asymptotic normality:

- As long as the sequence length increases, records of lower counts start to vanish because of their nature of depending somehow on the sample size drawn from it.
- For Low sequence length, coverage probability is higher for low record count. And for High sequence length, the coverage probability is higher for a higher record count.
- Q-Q plots showed excellent agreement with the standard normal distribution.
- P-value of the Shapiro-Wilk test shows normality for all current records of any count and for any sequence length from any distribution function.
- Convergence was fastest for exponential distributions and slowest for uniform distributions.

These findings validate the practical utility of asymptotic approximations for inference with current records, particularly for $n \geq 15$.

Table 4: Results of the simulation study for Exponential(1)

N	n	Shapiro-Wilk p-value	95 Coverage Probability
10^3	10	1.63725×10^{-11}	0.759615
	15	2.2059×10^{-30}	0.954545
	20	5.33033×10^{-20}	0.940625
	25	0.000921227	0.56
10^4	10	0.0427996	0.36667
	15	3.22558×10^{-18}	0.831963
	20	4.19599×10^{-29}	0.967241
	25	2.35036×10^{-20}	0.955714
10^5	10	–	–
	15	0.0286577	0.584127
	20	9.601431×10^{-31}	0.868125
	25	1.75798×10^{-38}	0.973023

Table 5: Results of the simulation study for Standard Normal

N	n	Shapiro-Wilk p-value	95 Coverage Probability
10^3	10	1.18483×10^{-9}	0.7866667
	15	3.10539×10^{-32}	0.93657
	20	1.17115×10^{-19}	0.973333
	25	0.000707977	0.82
10^4	10	0.04039036	0.49000
	15	1.20189×10^{-30}	0.820779
	20	4.35607×10^{-42}	0.953125
	25	2.50942×10^{-15}	0.96000
10^5	10	–	–
	15	0.00036777	0.572727
	20	1.40245×10^{-21}	0.895395
	25	4.5217×10^{-34}	0.959512

Table 6: Results of the simulation study for Uniform(0,1)

N	n	Shapiro-Wilk p-value	95 Coverage Probability
10^3	10	3.06503×10^{-17}	0.705263
	15	3.43056×10^{-33}	0.960684
	20	1.30928×10^{-21}	0.964063
	25	0.040736	0.60000
10^4	10	8.11007×10^{-6}	0.183333
	15	8.08248×10^{-24}	0.86000
	20	2.81913×10^{-40}	0.957143
	25	2.36381×10^{-26}	0.958333
10^5	10	–	–
	15	1.39025×10^{-6}	0.5000
	20	1.098088×10^{-19}	0.906557
	25	8.08248×10^{-24}	0.97000

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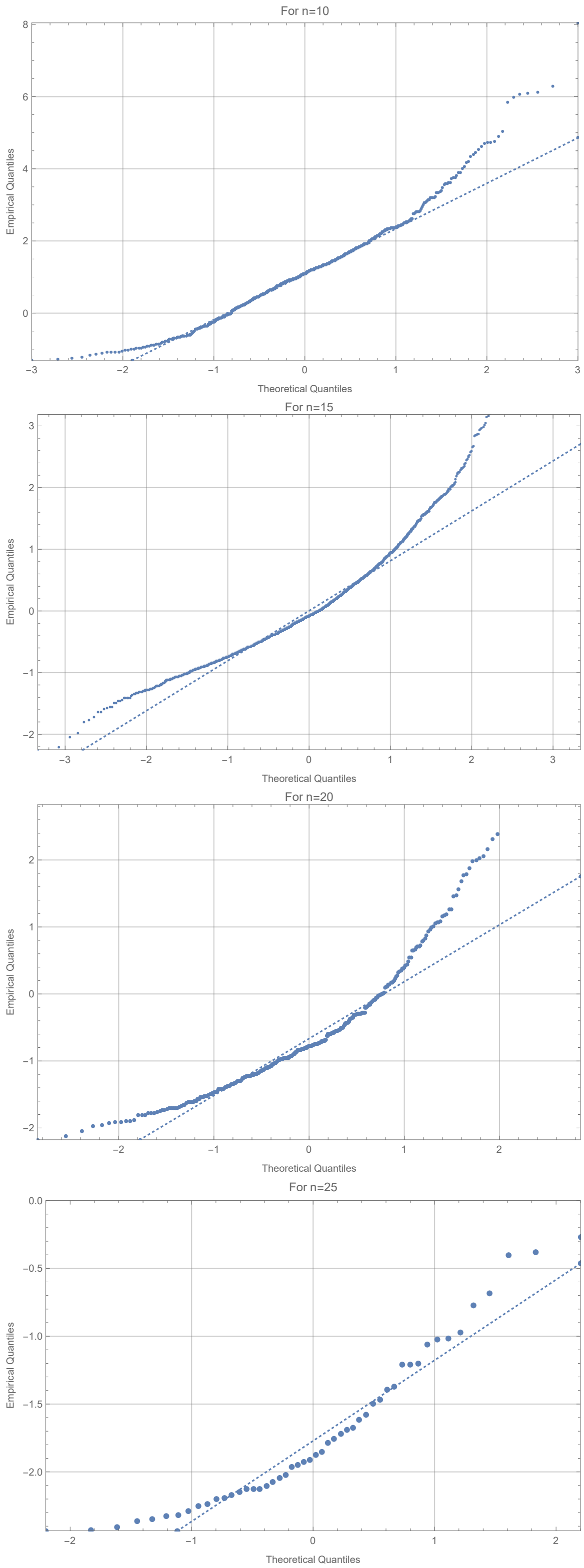

Fig. 2: QQ-plot of Z_n for different n from Exponential(1) distribution with $N = 10^3$

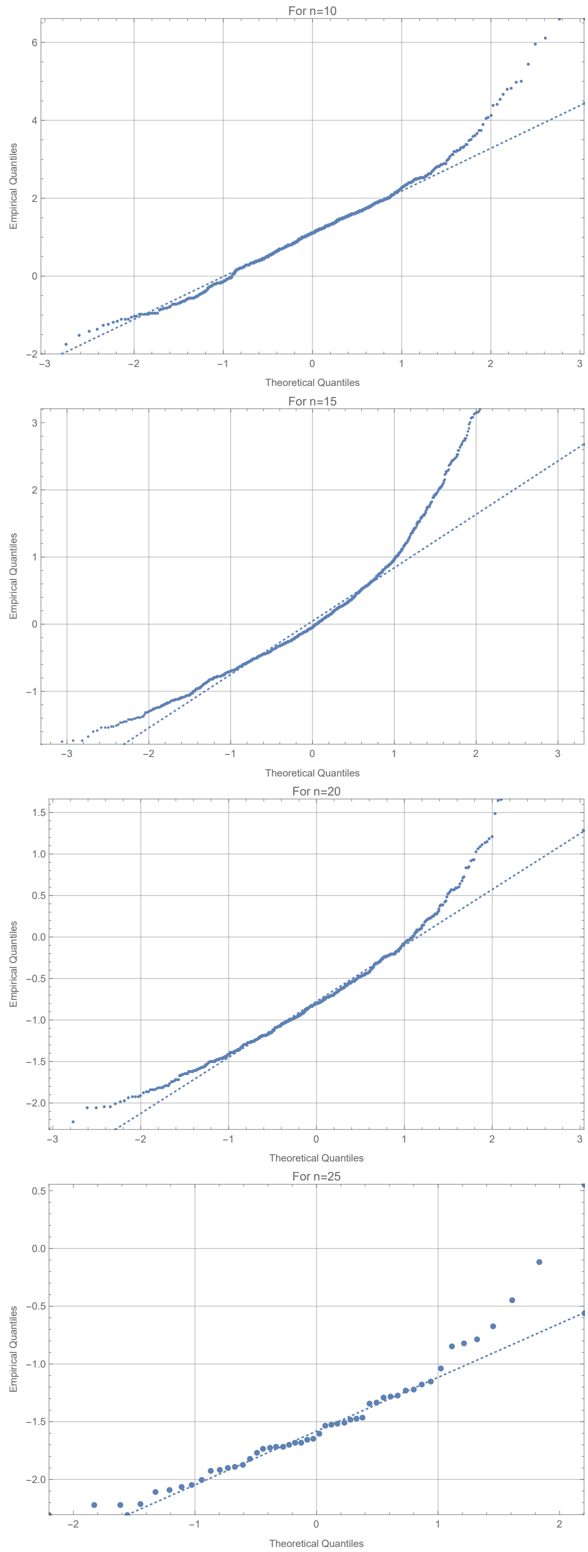

Fig. 3: QQ-plot of Z_n for different n from Standard Normal distribution with $N = 10^3$

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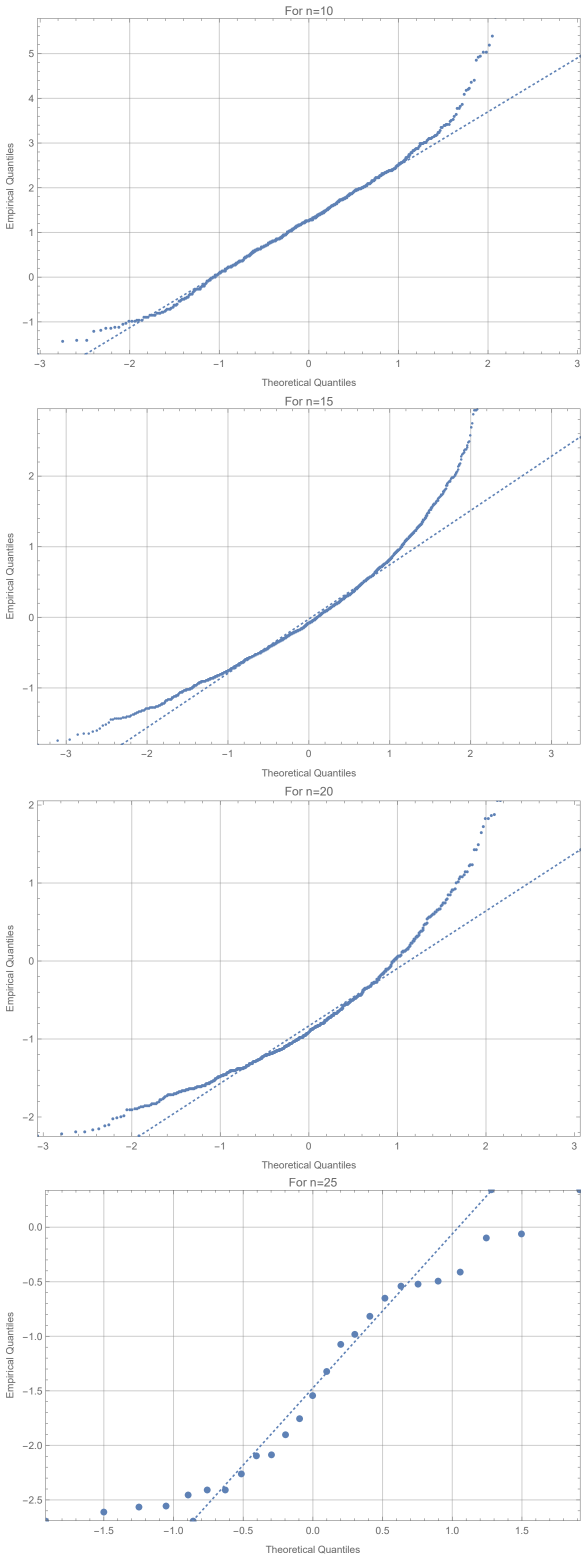

Fig. 4: QQ-plot of Z_n for different n from Uniform $(0,1)$ distribution with $N = 10^3$

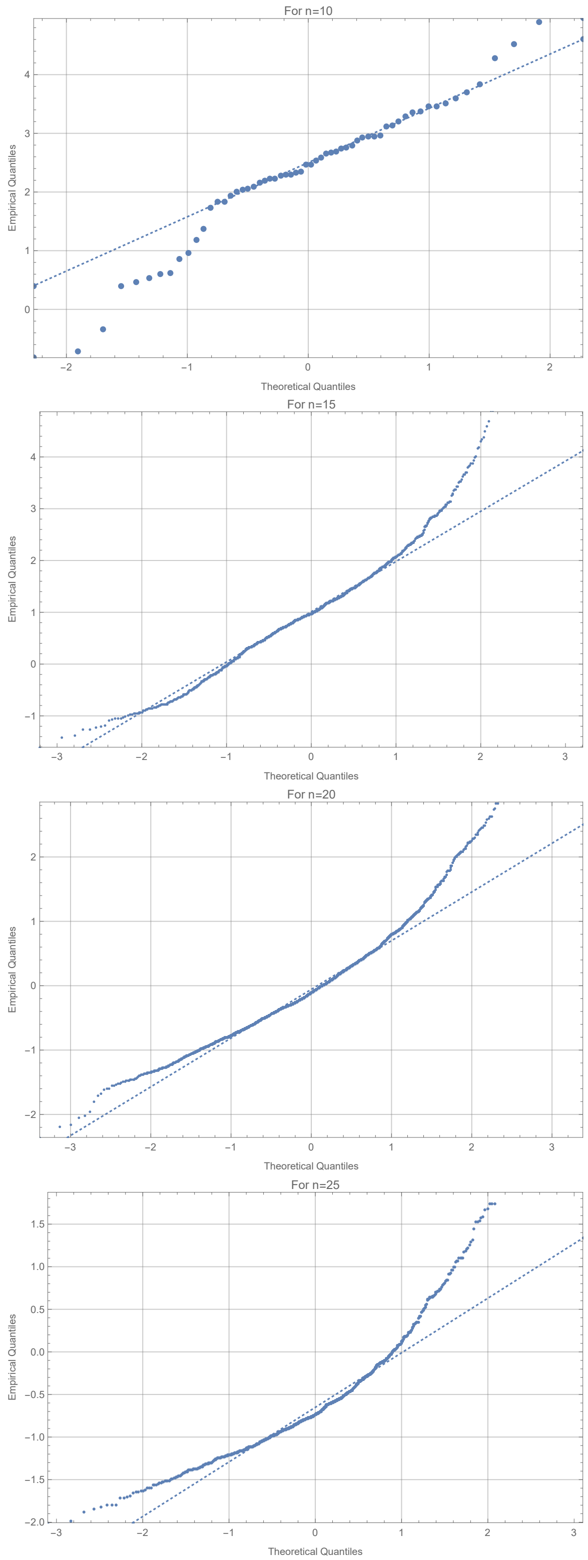

Fig. 5: QQ-plot of Z_n for different n from Exponential(1) distribution with $N = 10^4$

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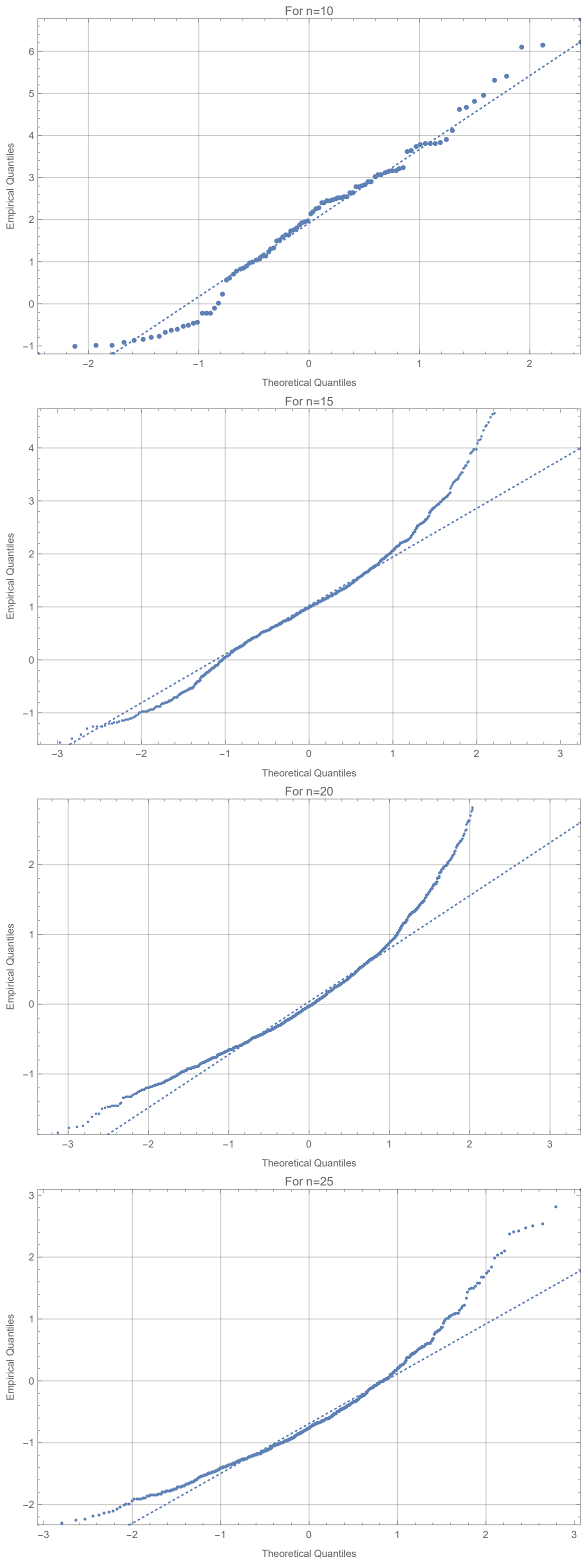

Fig. 6: QQ-plot of Z_n for different n from Standard Normal distribution with $N = 10^4$

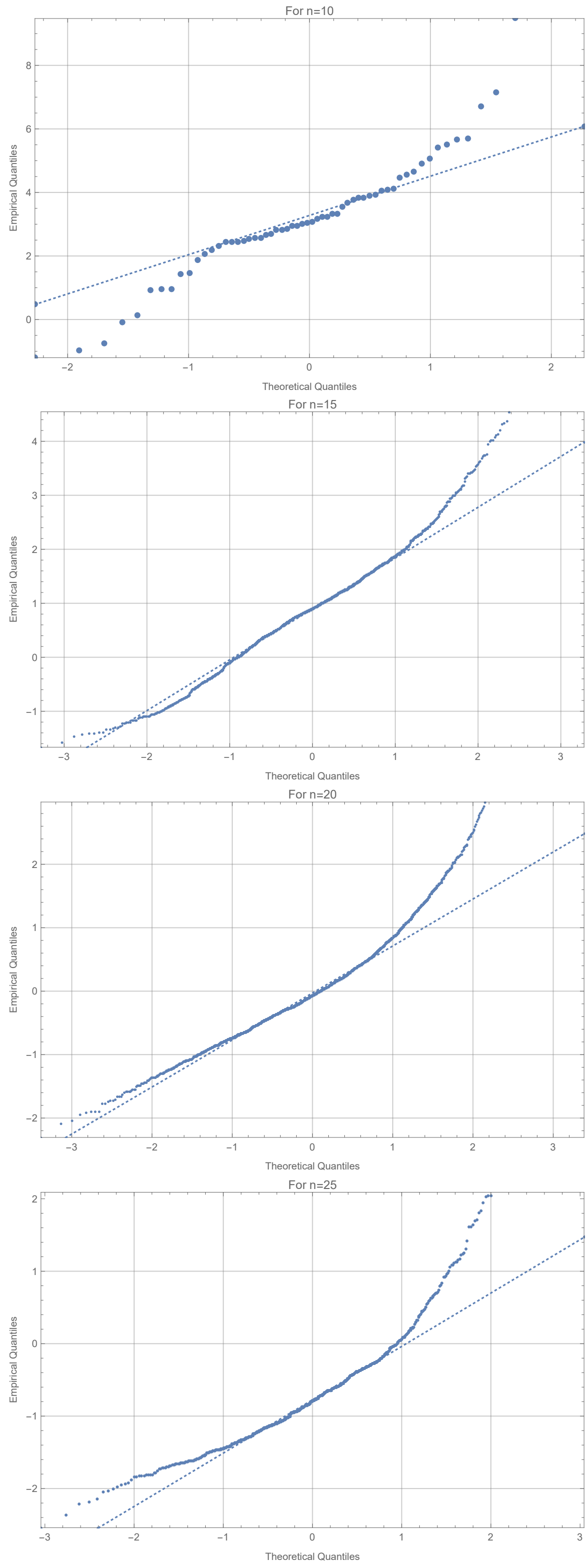

Fig. 7: QQ-plot of Z_n for different n from Uniform $(0,1)$ distribution with $N = 10^4$

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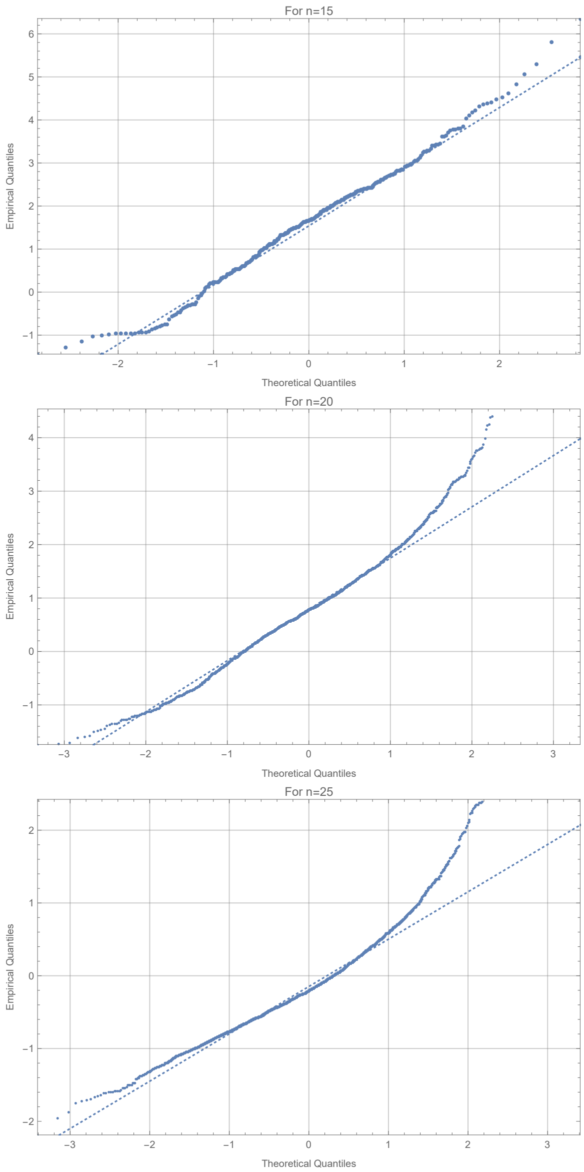


Fig. 8: QQ-plot of Z_n for different n from Exponential(1) distribution with $N = 10^5$

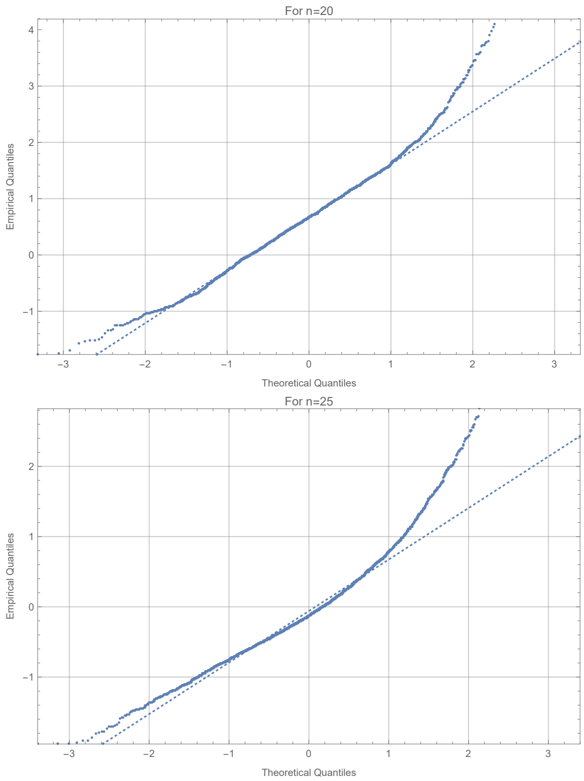
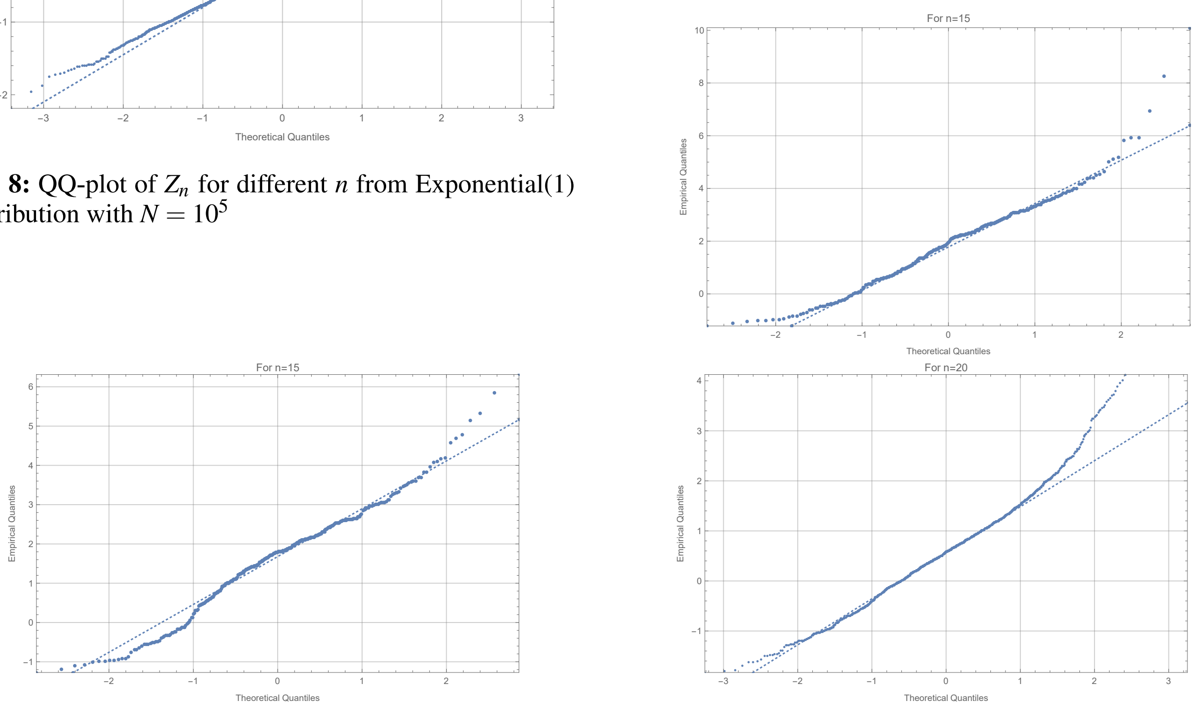


Fig. 9: QQ-plot of Z_n for different n from Standard Normal distribution with $N = 10^5$



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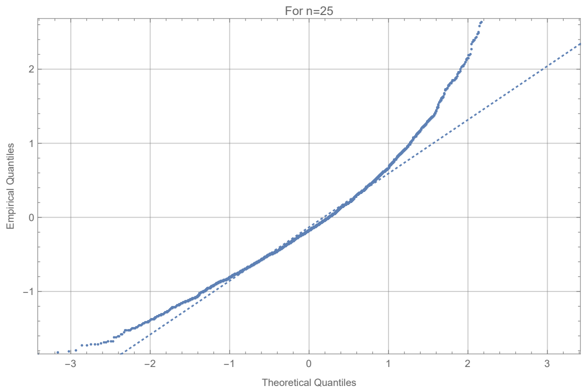

Fig. 10: QQ-plot of Z_n for different n from Uniform $(0,1)$ distribution with $N = 10^5$

The empirical findings motivate a broader discussion of the implications, limitations, and potential extensions of current record theory, which is provided in the next section.

12 Discussion and future research directions

12.1 Theoretical extensions

While significant progress has been made in current record theory, several theoretical challenges remain [19, ?]:

- Extension to dependent sequences and non-identical distributions,
- Development of multivariate current record theory,
- Bayesian approaches to current record analysis,
- Connections with stochastic process theory and point processes,
- Record values in non-stationary environments.

12.2 Methodological developments

Future methodological research could focus on, cf. [5,10]:

- Efficient computational algorithms for current record distributions.
- Robust estimation methods for record-based inference.
- Model selection criteria for record data.
- Sequential analysis and change-point detection using records.
- Record values in high-dimensional settings.

12.3 Applications in emerging fields

Current record methodology has potential applications in various emerging fields, cf. [2,4]:

- Industry and energy consumption data.
- Climate change research and extreme weather analysis.
- Financial risk management and extreme value at risk.
- Reliability engineering for complex systems.
- Sport analytics and performance prediction
- Medical statistics and disease outbreak monitoring.

12.4 Open problems

Several open problems in current record theory, k-sliding records, and related current records need further investigation, see [10,20]:

- Characterizing current records, $k-$sliding records, and their related structures for continuous and discrete distributions.
- Developing optimal stopping rules based on current records, $k-$sliding records, and their related structures.
- Exploring record value theory for continuous-time processes, including continuous-time analogs of $k-$sliding records and their related structures.
- Investigating connections with information theory and coding, especially for $k-$sliding records and related structures in data streams.
- Studying multivariate extremes and record trees, including extensions to multivariate $k-$sliding records and their related structures.
- Selecting optimal window sizes for $k-$sliding records in various applications and their effect on related distributions.
- Detecting change points using $k-$sliding records and their related structures in non-stationary environments.
- Developing distribution theory for $k-$sliding records and their related structures in dependent sequences.
- Modeling dependence for the related structures of current records using copula-based approaches that go beyond FGM models, such as the Sarmanov model and Cambanis family, see Barakat et al. [22] and Abd Elgawad et al. [1], respectively.
- Creating inference procedures based on related current records in multivariate extreme value analysis.

Finally, we conclude by summarizing the main contributions of this review and highlighting key directions for future research.

13 Conclusion

Current record values build on traditional record value theory by providing a framework for analyzing simultaneous upper and lower extremes in sequential data. This review has further advanced the field by introducing two major innovations: k-sliding records, which track local extremes within finite moving windows, and concomitants of current records, which enable

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multivariate analysis of extreme events via associated variables.

The foundation laid by Houchens [35] and expanded by Aldallal [10] covers distribution theory, asymptotic properties, moment relationships, and prediction methods for current records. Our work builds on this foundation, establishing the probabilistic properties, expectation and variance structures, and asymptotic normality of $k-$sliding records while developing distribution theory for concomitants of current records in bivariate settings.

Current records, $k-$sliding records, and their concomitants have unique theoretical features that facilitate both analytical approaches and practical implementations. These features include Markovian structures, exponential representations, distribution-free pivotal quantities, efficient recurrence relations, and novel dependence modeling frameworks. The prediction interval methods and concomitant distribution theories we developed offer valuable tools for predicting future extreme values and understanding multivariate extreme behavior.

Applications in meteorology, reliability engineering, quality control, finance, and sports statistics show the wide utility of current record analysis, k-sliding records, and related methodologies, cf. [2,23,10]. The $k-$sliding record framework is particularly effective for real-time monitoring and streaming data analysis, while concomitant current records create new opportunities for multivariate extreme value analysis in dependent systems.

As record theory evolves, promising research directions include extensions to dependent data, multivariate settings, Bayesian methods, and applications in emerging fields such as climate science and financial risk management (see Barakat et al. [19,20,21]). The strong foundation established in previous work, combined with the new frameworks of k-sliding records and concomitant current records, will help keep record value methodology a dynamic and important area of statistical research with practical significance.

The current record framework provides a flexible and informative approach for modeling extreme behavior by simultaneously tracking upper and lower records, thereby offering a more complete description than traditional one-sided record analysis. Its distribution-free properties and tractable recurrence relations make it suitable for inference in settings where only record information is available. However, the methodology also has limitations, including the inherent scarcity of record data, which may reduce estimation precision in small samples, and the reliance on independence and continuity assumptions in the underlying sequence. Future research may focus on extending current record theory to dependent or non-identically distributed sequences, developing copula-based and high-dimensional models for concomitants, and further exploring finite-memory extensions such as k-sliding records. Additional work on robust inference, computational methods, and real-time monitoring applications would further enhance the practical utility of the proposed framework.

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Conflicts of Interest

The writers affirm that they have no competing interests with respect to the publishing of this work.

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