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Relationship and Accuracy Analyses of Variable Precision Multi-Granulation Rough Sets based on Tolerance Relation

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Abstract: This paper discusses the properties of optimistic, pessimistic and basic approximations of rough sets in ordinary and variable precise multi-granulation models and the ones produced by using union and intersection operations on multi-property relations deeply, and analyzes the relationships between or among them. It explores the approximate accuracy formulas and finds several inequalities to describe their relationships of those approximate accuracy formulas. It proves that approximation accuracy of incomplete variable precision multi-granulation rough sets based on tolerance relation is higher than the non-variable ones.

Keywords: multi-granulation rough set model, variable precision multi-granulation rough set model, tolerance relation; incomplete information system

1 Introduction

In 1982, Pawlak put forward rough set models [1], which are widely used in many scientific and technological application fields, especially in complete information system. Kryszkiewicz proposed tolerance relation rough set model [2]; Stefanowski proposed similar relation rough set model [3]; Wang Guoyin proposed limited tolerance relation rough set model [4]; Greco, etc. proposed rough set model based on the advantage relationship and so on [5]. Presently, the variable precision rough set model has been suggested based on tolerance relation [6,7]. In this model, a threshold representing a bound on the conditional probability of a proportion of objects, which are classified into the same decision class, in a condition class, is given. It therefore admits some level of uncertainty in the classification process, leading to a deeper understanding and a better utilization of properties of the data being analyzed. The variable precision rough set model overcomes limitation in traditional rough set model.

In the literature [8,9], Qian et al. pointed out that we often need to describe concurrently a target concept through multi binary relations (e.g. equivalence relation, tolerance relation, reflexive relation and neighborhood relation) on the universe according to a user's requirements or targets of problem solving. Therefore, they proposed the concept of multi-granulation rough set model, which includes optimistic multi-granulation rough set and pessimistic multi-granulation rough set. Furthermore, Qian et al. proposed several basic views for establishing multi-granulation rough set model in incomplete information systems [10]. The purpose of this paper is to further generalize Ziarko's variable precision rough set and Qian's multi-granulation rough set in incomplete information system. From this point of view, we will propose the concept of the variable precision multi-granulation rough set model based on tolerance relation in incomplete information system.

The rest of the paper is organized as follows. In section 2, incomplete information system and multi-granulation rough set base on tolerance relation are briefly introduced. In section 3, Variable precision multi-granulation rough sets based on tolerance relation are explored. In section 4, the properties of variable precision multi-granulation rough set in incomplete information system base on tolerance relation are researched. In section 5, relationships of variable precision multi-granulation rough set including incomplete variable precision optimistic, pessimistic

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multi-granulation rough set and approximations produced by using union and intersection operations on multi-property relations are proposed and explored respectively. In section 6, the approximate accuracies of variable precision multi-granulation rough sets are analyzed. Section 7 gives the main conclusions according to the former section researches.

2 Incomplete Information System and Multi-Granulation Rough Set based on Tolerance Relation

In the tolerance relation model, one of the most important thought is in information table missing attribute values are given "*". "*" may be arbitrary value. Hence, $IS =< U, AT, D, V, f >$ is called an incomplete target information system if values of some attributes in AT are missing and those of all attributes in D are regular, where AT is called the conditional attributes and D is called the decision attribute set. For each attribute subset $A \subseteq AT$, a tolerance relation is defined as

$$T(A) = \{(x,y) \in U^2 :$$

$$\forall a \in A, a(x) = a(y) \vee a(x) = * \vee a(y) = *\}. \tag{1}$$

For $A \subseteq AT$ and $\forall X \subseteq U$, the lower approximation is

$$\underline{A}_T(X) = \{x \in U : T_A(x) \subseteq X\} \tag{2}$$

and the upper approximation is

$$\overline{A}_T(X) = \{x \in U : T_A(x) \cap X \neq \emptyset\}. \tag{3}$$

The ordered pair $[\underline{A}_T(X), \overline{A}_T(X)]$ is called a rough set of X with respect to A.

$$\underline{T}_A(x) = y \in U : (x,y) \in T(A). \tag{4}$$

is(are) called tolerant class(es). Qian et al. in their multi-granulation rough set model, proposed several basic views for establishing multi-granulation rough set model in incomplete information systems. Let $A_1, A_2, \cdots, A_m \subseteq AT$ be m attribute subsets. Then for $\forall X \subseteq U$,

$$\underline{\sum_{i=1}^{m} A_i}^{o}(X) = \{x \in U : T_{A_1}(x) \subseteq X \vee T_{A_2}(x) \subseteq X \vee \cdots \vee T_{A_m}(x) \subseteq X\} \tag{5}$$

and

$$\overline{\sum_{i=1}^{m} A_i}^{o}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{o}(\sim X). \tag{6}$$

are respectively the optimistic multi-granulation lower and upper approximations of X with respect to $A_1, A_2, \cdots, A_m \subseteq AT$. The optimistic multi-granulation boundary region of X is

$$BN^{o}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{o}(X) - \underline{\sum_{i=1}^{m} A_i}^{o}(X). \tag{7}$$

Following results have been obtained:

$$\underline{\sum_{i=1}^{m} A_i}^{o}(X) = \bigcup_{i=1}^{m} \underline{A_i}(X). \tag{8}$$

$$\overline{\sum_{i=1}^{m} A_i}^{o}(X) = \bigcap_{i=1}^{m} \overline{A_i}(X). \tag{9}$$

$$\underline{\sum_{i=1}^{m} A_i}^{p}(X) = \{x \in U : T_{A_i}(x) \subseteq X\}. \tag{10}$$

$$\overline{\sum_{i=1}^{m} A_i}^{p}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{p}(\sim X). \tag{11}$$

$$BN^{p}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{p}(X) - \underline{\sum_{i=1}^{m} A_i}^{p}(X). \tag{12}$$

$$\underline{\sum_{i=1}^{m} A_i}^{p}(X) = \bigcap_{i=1}^{m} \underline{A_i}(X). \tag{13}$$

$$\overline{\sum_{i=1}^{m} A_i}^{p}(X) = \bigcup_{i=1}^{m} \overline{A_i}^{p}(X). \tag{14}$$

3 Variable Precision Multi-Granulation Rough Sets based on Tolerance Relation

To save the space, the following assumption description is given: Let $IS =< U, AT \bigcup d >$ be an incomplete information system, $A_1, A_2, \cdots, A_m \subseteq AT$ be m attribute subsets, $0 \leq \beta < 0.5$, $X, Y \subseteq U$.

Definition 1. Variable precision optimistic multi-granulation lower and upper approximations respectively are :

$$\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) = \{x \in U : \exists i, e(T_{A_i}(x), X) \leq \beta\}; \tag{15}$$

$$\overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) = \sim \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\sim X) \tag{16}$$

Variable precision optimistic multi-granulation boundary is

$$BN^{o}_{\sum_{i=1}^{m} A_{i_\beta}}(X) = \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) - \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X). \tag{17}$$

Theorem 1. Under the condition given in the above,we have

$$\overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) = \{x \in U : \forall i, e(T_{A_i}(x), X) < 1-\beta\}. \quad (18)$$

Proof. By Definition 1, we have

$$x \in \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) \Leftrightarrow x \notin \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\sim X) \Leftrightarrow \forall i, e(T_{A_i}(x), X) < 1-\beta \quad (19)$$

Definition 2. Variable precision pessimistic multi-granulation lower and upper approximations respectively are :

$$\underline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X) = \{x \in U : \forall i, e(T_{A_i}(x), X) \leq \beta\}; \quad (20)$$

$$\overline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X) = \sim \underline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(\sim X) \quad (21)$$

. Variable precision optimistic multi-granulation boundary is

$$BN^{p}_{\sum_{i=1}^{m} A_{i_\beta}}(X) = \overline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X) - \underline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X). \quad (22)$$

Theorem 2. Under the condition described in the above,we have

$$\overline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X) = \{x \in U : \exists i, e(T_{A_i}(x), X) < 1-\beta\}. \quad (23)$$

Proof. By Definition 2, we have

4 Properties of Variable Precision Multi-Granulation Rough Set

Theorem 3. The following results are held:

$$(1) \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(U) = \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(U) = U; \quad (24)$$

$$(2) \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\varnothing) = \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\varnothing) = \varnothing. \quad (25)$$

$$(3) \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\sim X) = \sim \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X); \quad (26)$$

$$(4) \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\sim X) = \sim \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X). \quad (27)$$

$$(5) \beta_1 \geq \beta_2 \Rightarrow \underline{\sum_{i=1}^{m} A_{i_{\beta_1}}}^{o}(X) \supseteq \underline{\sum_{i=1}^{m} A_{i_{\beta_2}}}^{o}(X); \quad (28)$$

$$(6) \overline{\sum_{i=1}^{m} A_{i_{\beta_1}}}^{o}(X) \subseteq \overline{\sum_{i=1}^{m} A_{i_{\beta_2}}}^{o}(X);. \quad (29)$$

$$(7) \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X \cap Y) \subseteq \quad (30)$$

$$(8) \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) \cap \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(Y). \quad (31)$$

$$(9) \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X \cup Y) \supseteq \quad (32)$$

$$(10) \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) \cup \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(Y). \quad (33)$$

$$(11) \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X \cap Y) \subseteq \quad (34)$$

$$(12) \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) \cap \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(Y). \quad (35)$$

$$(13) \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X \cup Y) \supseteq \quad (36)$$

$$(14) \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) \cup \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(Y). \quad (37)$$

Proof. (1) and (2) $\forall x \in \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(U)$, by Definition 1, $\exists A_i \in A_1, A_2, \cdots, A_m$ such that $e(T_{A_i}(x), U) \leq \beta$. Thus $x \in U$. So $\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) \subseteq U$. For $\forall x \in U$, since $\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(U) = U$. For $\forall x \in \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\varnothing)$, $\exists A_i \in \{A_1, A_2, \cdots, A_m\}$ such that $e(T_{A_i}(x), \varnothing) \leq \beta$. Then $x \in \varnothing$, from which we can conclude that $\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\varnothing) \subseteq \varnothing$. So $\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\varnothing) = \varnothing$.

(3) and (4) By Definition 1 we know $\overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X) = \sim \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\sim X)$. Let $\sim X)$ replace X, we have $\sim \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\sim X) = \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)$, i.e. $\overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(\sim X) = \sim \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)$.

(5) and (6) For $\forall x \in \underline{\sum_{i=1}^{m} A_{i_{\beta_2}}}^{o}(X)$, $\exists A_i \in A_1, A_2, \cdots, A_m$ such that $e(T_{A_i}(x), X) \leq \beta_2$. Moreover, since $\beta_1 \geq \beta_2$, then $e(T_{A_i}(x), X) \leq \beta_1$, i.e. for $x \in \underline{\sum_{i=1}^{m} A_{i_{\beta_1}}}^{o}(X)$, we can conclude that $\underline{\sum_{i=1}^{m} A_{i_{\beta_1}}}^{o}(X) \supseteq \underline{\sum_{i=1}^{m} A_{i_{\beta_2}}}^{o}(X)$. (7) For $\forall x \in \underline{\sum_{i=1}^{m} A_{i_{\beta_2}}}^{o}(X \cap Y)$, by Definition 1, $\exists A_i \in \{A_1, A_2, \cdots, A_m\}$ such that $e(T_{A_i}(x), X \cap Y) \leq \beta$. By the definition of the inclusion error, we have $e(T_{A_i}(x), X) \leq \beta$ and $e(T_{A_i}(x), Y) \leq \beta$. So we have $x \in \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)$ and $x \in \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(Y)$, i.e.

$x \in \underline{\sum_{i=1}^m A_{i_\beta}}^o(X) \cap \underline{\sum_{i=1}^m A_{i_\beta}}^o(Y)$. Therefore, we obtain $\underline{\sum_{i=1}^m A_{i_{\beta_2}}}^o(X \cap Y) \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X) \cap \underline{\sum_{i=1}^m A_{i_\beta}}^o(Y)$. Similarly, it is not difficult to prove formula (8)-(14).

Theorem 4. Let $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq U$ .We have

$$(1)\underline{\sum_{i=1}^m A_{i_\beta}}^o(X_1) \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_2) \subseteq \cdots \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_n); \quad (38)$$

$$(2)\overline{\sum_{i=1}^m A_{i_\beta}}^o(X_1) \subseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_2) \subseteq \cdots \subseteq \o\overline{\sum_{i=1}^m A_{i_\beta}}^o(X_n). \quad (39)$$

Proof. Suppose $1 \le i \le j \le n$, then $X_i \subseteq X_j$. (1) Clearly, $X_i \cap X_j = X_i$. We have $\underline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) = \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_i \cap X_j) \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) \cap \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_j)$. Thus $\underline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) = \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) \cap \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_j)$. So we have

$$\underline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_j).$$

Therefore, it follows that

$$\underline{\sum_{i=1}^m A_{i_\beta}}^o(X_1) \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_2) \subseteq \cdots \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X_n).$$

(2) Clearly, $X_i \cup X_j = X_j$. we have $\overline{\sum_{i=1}^m A_{i_\beta}}^o(X_j) = \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_i \cup X_j) \supseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) \cup \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_j)$. Thus $\overline{\sum_{i=1}^m A_{i_\beta}}^o(X_j) \supseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) \cup \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_j)$. So we have

$$\overline{\sum_{i=1}^m A_{i_\beta}}^o(X_i) \subseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_j).$$

Therefore,

$$\overline{\sum_{i=1}^m A_{i_\beta}}^o(X_1) \subseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_2) \subseteq \cdots \subseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X_n).$$

Theorem 5. Let $A_1 \cup A_2 \cup \cdots \cup A_m \neq \varnothing$. Then

$$(1)\underline{\cup_{i=1}^m A_{i_\beta}}^o(X) \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X) \quad (40)$$

$$(2)\overline{\cup_{i=1}^m A_{i_\beta}}^o(X) \supseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X) \quad (41)$$

Proof. First prove (2). For $\forall x \in \overline{\sum_{i=1}^m A_{i_\beta}}^o(X)$, from Theorem 1, we have $\forall i, e(T_{A_i}(x), X) < 1-\beta$, thus $\max_{i=1}^m e(T_{A_i}(x), X) < 1-\beta$. Since for $\forall x \in U$, $\max_{i=1}^m e(T_{A_i}(x), X) \le e(T_{\cup_{i=1}^m A_i}(x), X)$, we have $e(T_{\cup_{i=1}^m A_i}(x), X) < 1-\beta$, i.e.$x \in \overline{\sum_{i=1}^m A_{i_\beta}}^o(X)$. Therefore, $\overline{\cup_{i=1}^m A_{i_\beta}}^o(X) \supseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X)$. By the definition of the variable precision multi-granulation model based on tolerance relation, we have

$$\underline{\cup_{i=1}^m A_{i_\beta}}^o(X) = \sim \overline{\cup_{i=1}^m A_{i_\beta}}^o(\sim X).$$

From the formula (2) in this theorem, we have

$$\sim \overline{\cup_{i=1}^m A_{i_\beta}}^o(\sim X) \subseteq \sim \overline{\sum_{i=1}^m A_{i_\beta}}^o(\sim (X) = \underline{\sum_{i=1}^m A_{i_\beta}}^o(X),$$

i.e.

$$\underline{\cup_{i=1}^m A_{i_\beta}}^o(X) \subseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X).$$

So (1) is held.

Theorem 6. Let $A_1 \cap A_2 \cap \cdots \cap A_m \neq \varnothing$. Then

$$(1)\underline{\cap_{i=1}^m A_{i_\beta}}^o(X) \supseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X); \quad (42)$$

$$(2)\overline{\cap_{i=1}^m A_{i_\beta}}^o(X) \subseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X). \quad (43)$$

Proof. (1) For $\forall x \in \underline{\sum_{i=1}^m A_{i_\beta}}^o(X)$, it follows that $\exists i, e(T_{A_i}(x), X) \le \beta$ from Definition 1. Thus $\min_{i=1}^m e(T_{A_i}(x), X) \le \beta$. Since

$$\min_{i=1}^m e(T_{A_i}(x), X) \ge e(T_{\cap_{i=1}^m A_i}(x), X),$$

$$e(T_{\cap_{i=1}^m A_i}(x), X) \le \beta$$

for $\forall x \in U$, i.e.

$$x \in \underline{\cap_{i=1}^m A_{i_\beta}}^o(X).$$

Thus $\underline{\cap_{i=1}^m A_{i_\beta}}^o(X) \supseteq \underline{\sum_{i=1}^m A_{i_\beta}}^o(X)$. (2) $\overline{\cap_{i=1}^m A_{i_\beta}}^o(X) = \sim \underline{\cap_{i=1}^m A_{i_\beta}}^o(\sim X)$. From the formula (1),

$$\sim \underline{\cap_{i=1}^m A_{i_\beta}}^o(\sim (X) \subseteq \sim \underline{\sum_{i=1}^m A_{i_\beta}}^o(\sim X) = \overline{\sum_{i=1}^m A_{i_\beta}}^o(X),$$

i.e.

$$\overline{\cap_{i=1}^m A_{i_\beta}}^o(X) \subseteq \overline{\sum_{i=1}^m A_{i_\beta}}^o(X).$$

Theorem 7. It is held that

$$(1)\underline{\sum_{i=1}^m A_{i_\beta}}^p(U) = \overline{\sum_{i=1}^m A_{i_\beta}}^p(U) = U; \quad (44)$$

$$(2)\underline{\sum_{i=1}^m A_{i_\beta}}^p(\varnothing) = \overline{\sum_{i=1}^m A_{i_\beta}}^p(\varnothing) = \varnothing. \quad (45)$$

$$(3)\underline{\sum_{i=1}^m A_{i_\beta}}^p(\sim X) = \sim \overline{\sum_{i=1}^m A_{i_\beta}}^p(X); \quad (46)$$

$$(4)\overline{\sum_{i=1}^m A_{i_\beta}}^p(\sim X) = \sim \underline{\sum_{i=1}^m A_{i_\beta}}^p(X). \quad (47)$$

$$(5)\beta_1 \geq \beta_2 \Rightarrow \underline{\sum_{i=1}^{m} A_{i_{\beta_1}}}^{p}(X) \supseteq \underline{\sum_{i=1}^{m} A_{i_{\beta_2}}}^{p}(X); \tag{48}$$

$$(6)\overline{\sum_{i=1}^{m} A_{i_{\beta_1}}}^{p}(X) \subseteq \overline{\sum_{i=1}^{m} A_{i_{\beta_2}}}^{p}(X);. \tag{49}$$

$$(7)\underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X \cap Y) \subseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \cap \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(Y). \tag{50}$$

$$(8)\underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \cup \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(Y). \tag{51}$$

$$(9)\overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X \cap Y) \subseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \cap \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(Y). \tag{52}$$

$$(10)\overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X \cup Y) \supseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \cup \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(Y). \tag{53}$$

Proof. The proof of it is similar to Theorem 3.

Theorem 8. Let $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq U$. Then

$$(1)\underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X_1) \subseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X_2) \subseteq \cdots \subseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X_n); \tag{54}$$

$$(2)\overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X_1) \subseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X_2) \subseteq \cdots \subseteq \o\overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X_n). \tag{55}$$

Proof. The proof of it is similar to Theorem 4.

Theorem 9. Let $A_1 \cup A_2 \cup \cdots \cup A_m \neq \varnothing$. Then

$$(1)\underline{\cup_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \supseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X); \tag{56}$$

$$(2)\overline{\cup_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \subseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \tag{57}$$

Theorem 10. Let $A_1 \cap A_2 \cap \cdots \cap A_m \neq \varnothing$. Then

$$(1)\underline{\cap_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \subseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X); \tag{58}$$

$$(2)\overline{\cap_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \supseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X). \tag{59}$$

5 The Relationships of Variable Precision Multi-Granulation Rough Set

Theorem 11. The following two results are held:

$$(1)\underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \subseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{o}(X); \tag{60}$$

$$(2)\overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X) \supseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{o}(X). \tag{61}$$

Proof. (1) For $\forall x \in \underline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X)$, $\exists A_i \in \{A_1, A_2, \cdots, A_m\}$ such that $e(T_{A_i}(x), X) \leq \beta$. Then we obviously have $x \in \underline{\sum_{i=1}^{m} Ai_{\beta}}^{o}(X)$. Thus $\underline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X) \subseteq \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)$.

(2) Similar to the proof of (1), it is not difficult to prove $\overline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X) \supseteq \overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)$.

Theorem 11 reveals the relationships between variable precision multi-granulation optimistic approximation and variable precision multi-granulation pessimistic approximation base on tolerance relation. We can conclude that the variable precision multi-granulation pessimistic lower approximation is included in the variable precision multi-granulation optimistic lower approximation; the variable precision multi-granulation optimistic upper approximation is included in the variable precision multi-granulation pessimistic upper approximation.

Theorem 12. It is held that

$$(1)\underline{\sum_{i=1}^{m} A_{i}}^{o}(X) \subseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{o}(X); \tag{62}$$

$$(2)\overline{\sum_{i=1}^{m} A_{i}}^{o}(X) \supseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{o}(X); \tag{63}$$

$$(3)\underline{\sum_{i=1}^{m} A_{i}}^{p}(X) \subseteq \underline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X); \tag{64}$$

$$(4)\overline{\sum_{i=1}^{m} A_{i}}^{p}(X) \supseteq \overline{\sum_{i=1}^{m} A_{i_{\beta}}}^{p}(X). \tag{65}$$

Proof. For $\forall x \in \underline{\sum_{i=1}^{m} A_i}^{o}(X)$, $\exists A_i \in \{A_1, A_2, \cdots, A_m\}$ such that $T_{A_i}(x) \subseteq X$. So $e(T_{A_i}(x), X) = 0 \leq \beta$, and $x \in \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)$. Thus $\underline{\sum_{i=1}^{m} A_i}^{o}(X) \subseteq \underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)$. Similarly, it is not difficult to prove other formulas. Theorem 12 shows the relationships between variable precision multi-granulation rough set base on tolerance relation and the classical multi-granulation rough set. We can conclude that the variable precision multi-granulation lower approximation is included in the classical multi-granulation lower approximation rough set; the variable precision multi-granulation upper approximation is included in the classical multi-granulation upper approximation rough set.

6 The Approximate Accuracy of Variable Precision Multi-Granulation Rough Set

The uncertainty of the rough set is due to the existence of a borderline region. The bigger the borderline region of a set is, the lower the accuracy of the set is. To more precisely express this idea, we introduce measure to incomplete variable precision multi-granulation rough set as follows.

Definition 3. Optimistic multi-granulation and pessimistic multi-granulation, variable precision optimistic multi-granulation and variable precision optimistic multi-granulation approximation accuracy of X are denoted by $\alpha_o, \alpha_p, \alpha_o^\beta$ and α_p^β respectively, and defined as

$$\alpha_o(\sum_{i=1}^{m} A_i, X) = |\underline{\sum_{i=1}^{m} A_i}^{o}(X)|/|\overline{\sum_{i=1}^{m} A_i}^{o}(X)| \tag{66}$$

$$\alpha_p(\sum_{i=1}^{m} A_i, X) = |\underline{\sum_{i=1}^{m} A_i}^{p}(X)|/|\overline{\sum_{i=1}^{m} A_i}^{p}(X)| \tag{67}$$

$$\alpha_o^\beta(\sum_{i=1}^{m} A_i, X) = |\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)|/|\overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)| \tag{68}$$

$$\alpha_p^\beta(\sum_{i=1}^{m} A_i, X) = |\underline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X)|/|\overline{\sum_{i=1}^{m} A_{i_\beta}}^{p}(X)| \tag{69}$$

Definition 4. Variable precision optimistic multi-granulation and variable precision optimistic multi-granulation approximation accuracy of according to $\cup_{i=1}^{m} A_i$ and $\cap_{i=1}^{m} A_i$ are denoted by μ_o^β, μ_p^β, η_o^β and η_p^β respectively, and defined as

$$\mu_o^\beta(\cup_{i=1}^{m} A_i, X) = |\underline{\cup_{i=1}^{m} A_i}^{o}(X)|/|\overline{\cup_{i=1}^{m} A_i}^{o}(X)|; \tag{70}$$

$$\mu_p^\beta(\cup_{i=1}^{m} A_i, X) = |\underline{\cup_{i=1}^{m} A_i}^{p}(X)|/|\overline{\cup_{i=1}^{m} A_i}^{p}(X)|; \tag{71}$$

$$\eta_o^\beta(\cap_{i=1}^{m} A_i, X) = |\underline{\cap_{i=1}^{m} A_{i_\beta}}^{o}(X)|/|\overline{\cap_{i=1}^{m} A_{i_\beta}}^{o}(X)|; \tag{72}$$

$$\eta_p^\beta(\cap_{i=1}^{m} A_i, X) = |\underline{\cap_{i=1}^{m} A_{i_\beta}}^{p}(X)|/|\overline{\cap_{i=1}^{m} A_{i_\beta}}^{p}(X)|. \tag{73}$$

Theorem 13. It is held that

$$(1)\alpha_o(\sum_{i=1}^{m} A_i, X) \le \alpha_o^\beta(\sum_{i=1}^{m} A_i, X); \tag{74}$$

$$(2)\alpha_p(\sum_{i=1}^{m} A_i, X) \le \alpha_p^\beta(\sum_{i=1}^{m} A_i, X). \tag{75}$$

Proof. (1) By Theorem 12, we have

$$|\underline{\sum_{i=1}^{m} A_i}^{o}(X)| \le |\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)|$$

and

$$|\overline{\sum_{i=1}^{m} A_i}^{o}(X)| \ge |\overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)|.$$

Therefore,

$$\begin{aligned}\alpha_o(\sum_{i=1}^{m} A_i, X) &= |\underline{\sum_{i=1}^{m} A_i}^{o}(X)|/|\overline{\sum_{i=1}^{m} A_i}^{o}(X)| \\ &\le |\underline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)|/|\overline{\sum_{i=1}^{m} A_{i_\beta}}^{o}(X)|.\end{aligned}$$

Similarly, it is not difficult to prove (2) .

Theorem 13 shows the relationships between incomplete variable precision multi-granulation rough set base on tolerance relation and the classical multi-granulation rough set. We can conclude that incomplete variable precision Multi-granulation rough set based on tolerance relation have a higher approximation.

7 Conclusion

The paper introduced variable precision multi-granulation model based on tolerance relation in incomplete information system and formed incomplete variable precision multi-granulation model. Such model is the combination of the incomplete variable precision rough set and multi-granulation rough set. Because we use variable precision, we improve the approximate accuracies, and therefore get the result that incomplete variable precision multi-granulation model rough set has further bigger lower approximation and further smaller upper approximation, through discussing relationships among them. The next job for us is to mine learning rules according to our model.

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