

A New Flexible Log-Logistic Distribution Based on a Nonhomogeneous Poisson Process: Properties, Classical and PINN-Based Estimation, and Applications

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Abstract: This manuscript presents a novel and flexible lifetime distribution, called the Generalized Kavya–Manoharan Log-Logistic (GKMLL) distribution, which is built within the generalized Kavya–Manoharan (GKM-G) family using a nonhomogeneous Poisson process framework. The proposed model can describe a broad spectrum of hazard-rate behaviors, such as increasing, decreasing, bathtub-shaped, modified bathtub, upside-down bathtub and J-shaped, and is thus well suited to complex lifetime data. Moreover, the probability density function of the GKMLL distribution is highly flexible, with symmetric, skewed, and heavy-tailed shapes. Several key statistical properties of the distribution are derived, including the hazard rate, the quantile function, and entropy measures. Various classical estimation approaches, namely maximum likelihood, least squares, weighted least squares, Cramér–von Mises, Anderson–Darling, right-tail Anderson–Darling, maximum product of spacings, and percentile estimators, together with a modern physics-informed neural network (PINN) estimator, are used to estimate the model parameters. An extensive Monte Carlo simulation is conducted to assess the performance of the classical estimators, and it is shown that the maximum product of spacings method gives the best estimates in a range of situations. The practical usefulness of the GKMLL distribution is illustrated with two real-life datasets (gauge lengths and aircraft windshields). The results show that the proposed model achieves good performance on several goodness-of-fit (GOF) measures and graphical analyses compared with popular competing distributions. Moreover, the GOF results obtained with the PINN estimator are consistent with those obtained with maximum likelihood, which supports the efficiency of machine-learning-based inference. Overall, the GKMLL distribution is an effective and flexible tool for modeling complex lifetime data in reliability analysis and related tasks.

Keywords: GKM-G family; Log-Logistic distribution; Poisson process; order statistics; parameter estimation; failure time data; PINN estimation.

1 Introduction

The log-logistic distribution is widely used in the analysis of lifetime data (LD). Its probability density function (PDF) is right-skewed, which makes it useful in reliability engineering and survival analysis (SA). Nevertheless, its hazard rate function (HRF) is either decreasing or unimodal, so it cannot represent bathtub-shaped, modified bathtub-shaped, or increasing hazard rates. In many applied fields, such as medicine, engineering, and industrial reliability, failure mechanisms tend to produce such non-monotonic

hazard structures, and the classical log-logistic model therefore may not have the flexibility needed to represent such data properly. Although the literature has proposed many extensions of lifetime distributions to overcome these shortcomings, many of the current solutions need a large number of parameters to model complex hazard-rate behavior. In addition, a number of these models still struggle to capture J-shaped and modified bathtub hazard functions effectively.

In recent years, numerous extensions of the Weibull (W) distribution (WD) have been proposed to enhance its

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flexibility and broaden its applicability to a wider range of lifetime data. Notable examples include the beta W [1], Kumaraswamy–W [2], transmuted W [3], exponentiated generalized W [4], new extended W [5], modified beta W [6], Kumaraswamy complementary W geometric [7], W–W [8], alpha power W [9], odd log-logistic exponentiated W [10], Lindley–W [11], exponentiated W [12], alpha logarithmic transformed W [13], alpha power exponentiated W [14], odd Lomax–W [15], Maxwell–W [16], exponentiated additive W [17], new generalized modified W [18], new flexible W [19], odd Burr exponentiated W [20], alpha power Kumaraswamy–W [21], new exponentiated inverse W [22], entropy-transformed W [23], extended W [24], and odd beta prime W [25] distributions. These generalized W models provide greater flexibility in modeling lifetime data and have demonstrated improved fitting performance across a broad range of practical applications.

To address the limitations of the log-logistic model, we propose a new extension of it, namely the GKMLL distribution, developed within the framework of a nonhomogeneous Poisson process (NHPP). The proposed model is designed to accommodate a broad range of non-monotonic hazard-rate behaviors while maintaining a parsimonious three-parameter structure. Consequently, the GKMLL distribution provides a flexible framework for modeling lifetime data and is particularly suitable for applications in SA and reliability theory. The GKMLL distribution belongs to the GKM-G family introduced by Ref. [26]. A distinguishing feature of the proposed model is its interpretation as a proportional reversed hazard (PRH) model. PRH models are particularly relevant in SA and reliability theory, especially in the analysis of left-censored lifetime data and the modeling of parallel systems [27]. Further discussions of PRH models and their theoretical properties can be found in [28, 29, 30].

The proposed GKMLL distribution has a number of significant strengths.

- First, it can represent a great variety of hazard-rate (HR) shapes, such as decreasing, increasing, J-shaped, reversed J-shaped, bathtub, modified bathtub, and upside-down bathtub shapes. Although the GKMLL distribution has a parsimonious three-parameter form, it is useful in modeling both standard and modified bathtub-shaped failure rates, and it is more flexible than most existing models with the same number of parameters, many of which need four or more parameters to achieve a comparable range of shapes.
- Second, the GKMLL distribution is specifically adapted to non-monotonic hazard behaviors, so it can be applied in a wide variety of disciplines, including medicine, engineering, SA, and industrial reliability.
- Third, the empirical comparison performed on two real-life datasets shows that the proposed model fits better than a number of competing lifetime distributions, which indicates its practical relevance.
- Lastly, the paper examines the performance of different frequentist estimators of the GKMLL distribution

across sample sizes and parameter configurations and provides practical advice on choosing the estimator, which may be of interest to practitioners.

Although the classical log-logistic distribution [31, 32] is widely applicable in LD modeling, it has shortcomings in describing the complicated, non-monotonic hazard-rate behavior seen in real-life cases. Specifically, the standard log-logistic model cannot represent key hazard shapes, including bathtub, modified bathtub, and J-shaped, which occur in many applications, e.g., reliability engineering and SA. Though many extensions of lifetime distributions have been proposed, particularly of the Weibull type, many of them are complicated and have a large number of parameters. This often causes problems in parameter estimation and decreases interpretability in practice. In addition, some existing models offer little flexibility in combining different distributional features, such as heavy tails and skewness of different magnitudes. Furthermore, the majority of the existing research relies on classical estimation methods, and few studies have investigated modern machine-learning-based estimation for complex nonlinear models. In particular, physics-informed neural networks (PINNs) [33] have received little attention in the context of statistical distribution modeling. Hence, a lifetime distribution that is both flexible and parsimonious, capturing complicated hazard-rate behaviors with a simple structure, and efficient estimation methods that combine classical and modern tools are needed. This motivates the proposed GKMLL distribution and the integration of classical and PINN-based estimation techniques in this work. The key contributions of this research are summarized as follows:

1. A novel flexible lifetime distribution, the GKMLL distribution, is introduced based on the generalized Kavya–Manoharan (GKM-G) family within a nonhomogeneous Poisson process framework. The model extends the classical log-logistic distribution with a parsimonious three-parameter structure.
2. The GKMLL distribution shows substantial flexibility in modeling a wide variety of PDF and hazard-rate shapes, including increasing, decreasing, bathtub, modified bathtub, upside-down bathtub, J-shaped, and reversed J-shaped forms.
3. Several statistical properties of the proposed distribution are derived, such as explicit expressions for the PDF, CDF, HRF, quantile function, a linear representation, order statistics, probability-weighted moments, and entropy measures.
4. A comprehensive estimation framework is provided that incorporates several classical methods, namely maximum likelihood estimation (MLE), least squares (LS), weighted least squares (WLS), Cramér–von Mises (CVM), Anderson–Darling (AD), right-tail Anderson–Darling (RTAD), maximum product of spacings (MPS), and percentile (PC) estimators, together with a modern PINN-based estimator.

5. The PINN scheme combines statistical inference with machine learning to estimate the parameters of a complex nonlinear model.
6. A large-scale Monte Carlo simulation study evaluates the performance of the classical estimators across various sample sizes and parameter configurations. The results indicate that the MPS method generally provides the most accurate parameter estimates, as evidenced by its overall ranking in terms of bias, mean squared error (MSE), and mean relative error (MRE).
7. The GKMLL distribution is applied to two real-life datasets (gauge lengths and aircraft windshields). According to the results, the proposed model outperforms several competing distributions in terms of information criteria and graphical analyses.
8. The agreement between the GOF results of the classical (MLE) and machine-learning-based (PINN) estimation indicates the stability of the proposed model and its usefulness in SA, reliability engineering, and related fields.

The workflow of the study is summarized in Figure 1.

The rest of the manuscript is organized as follows. In Section 2, the GKMLL distribution is presented. Section 3 presents its main statistical properties. Nine approaches to estimating the model parameters (eight classical estimators and one PINN-based estimator) are provided in Section 4. Section 5 gives the results of the Monte Carlo simulation experiments. In Section 6, the applicability of the GKMLL distribution is demonstrated on two real datasets. Lastly, Section 7 concludes the paper and provides directions for future research.

2 The GKMLL distribution (proposed model)

In this section, we present the GKMLL distribution as a member of the GKM-G family introduced by Ref. [26]. The GKM-G family can be viewed as a PRH family constructed using the exponentiated-H (exp-H) transformation [34], one of the most widely employed generalization techniques in lifetime distribution theory. Within this framework, the cumulative distribution function (CDF) and PDF of the GKM-G family are defined as

$$G(y; \beta, \boldsymbol{\varphi}) = \gamma^\beta [1 - e^{-F(y; \boldsymbol{\varphi})}]^\beta, \quad y \in \mathbb{R}^+, \beta > 0, \quad (1)$$

and

$$g(y; \beta, \boldsymbol{\varphi}) = \gamma^\beta \beta f(y; \boldsymbol{\varphi}) e^{-F(y; \boldsymbol{\varphi})} \times [1 - e^{-F(y; \boldsymbol{\varphi})}]^{\beta-1}, \quad y \in \mathbb{R}^+, \quad (2)$$

where $\gamma = e/(e-1)$, β is a shape parameter, $F(\cdot; \boldsymbol{\varphi})$ and $f(\cdot; \boldsymbol{\varphi})$ are the baseline CDF and PDF, and $\boldsymbol{\varphi}$ is the vector of baseline parameters. The corresponding HRF of the

GKM-G family is

$$h(y; \beta, \boldsymbol{\varphi}) = \frac{\beta f(y; \boldsymbol{\varphi}) e^{1-F(y; \boldsymbol{\varphi})} \kappa(y)^{\beta-1}}{(e-1)^\beta - \kappa(y)^\beta}, \quad y \in \mathbb{R}^+, \quad (3)$$

where $\kappa(y) = e - e^{1-F(y; \boldsymbol{\varphi})}$.

The baseline of the GKMLL model is the log-logistic distribution with shape parameter $a > 0$ and scale parameter $b > 0$. Writing $z(y) = (y/b)^a$, its PDF and CDF are

$$f_{LL}(y; a, b) = \frac{a}{b} \left(\frac{y}{b}\right)^{a-1} [1 + z(y)]^{-2},$$

$$F_{LL}(y; a, b) = \frac{z(y)}{1 + z(y)}, \quad y > 0.$$

The proposed GKMLL distribution is constructed within the framework of an NHPP, which serves as its fundamental basis. Define

$$H(x) = F_{LL}(x; a, b) = \frac{z(x)}{1 + z(x)}, \quad (4)$$

$$\eta(x) = \frac{d}{dx} H(x) = \frac{a}{b} \left(\frac{x}{b}\right)^{a-1} [1 + z(x)]^{-2},$$

with $a, b > 0$, and let $w(x) = 1 - e^{-H(x)}$. Let $\Lambda(x)$ denote the cumulative intensity function of the NHPP. The CDF of the time to the first event is

$$G(x) = 1 - \exp(-\Lambda(x)), \quad x > 0. \quad (5)$$

For the GKMLL model, we define the cumulative intensity function as

$$\Lambda(x) = -\ln[1 - \gamma^\beta w(x)^\beta], \quad \beta > 0. \quad (6)$$

Differentiating Λ gives the intensity function

$$\lambda(x) = \frac{d}{dx} \Lambda(x) = \frac{\gamma^\beta \beta \eta(x) e^{-H(x)} w(x)^{\beta-1}}{1 - \gamma^\beta w(x)^\beta}, \quad (7)$$

and the PDF of the time to the first event is

$$g(x) = \lambda(x) \exp(-\Lambda(x)), \quad x > 0. \quad (8)$$

This NHPP-based formulation provides a natural interpretation of the GKMLL distribution in reliability theory, where $\Lambda(x)$ represents the accumulated failure intensity over time and $\lambda(x)$ coincides with the hazard rate of the model. Moreover, this construction is consistent with the PRH framework, thereby linking the proposed model to a well-established class of lifetime distributions. Substituting (6) into (5), or equivalently the log-logistic CDF into Eq. (1), the CDF of the GKMLL distribution is

$$G(y; a, b, \beta) = \gamma^\beta \left[1 - \exp\left(-\frac{z(y)}{1 + z(y)}\right) \right]^\beta, \quad y > 0, a, b, \beta > 0. \quad (9)$$

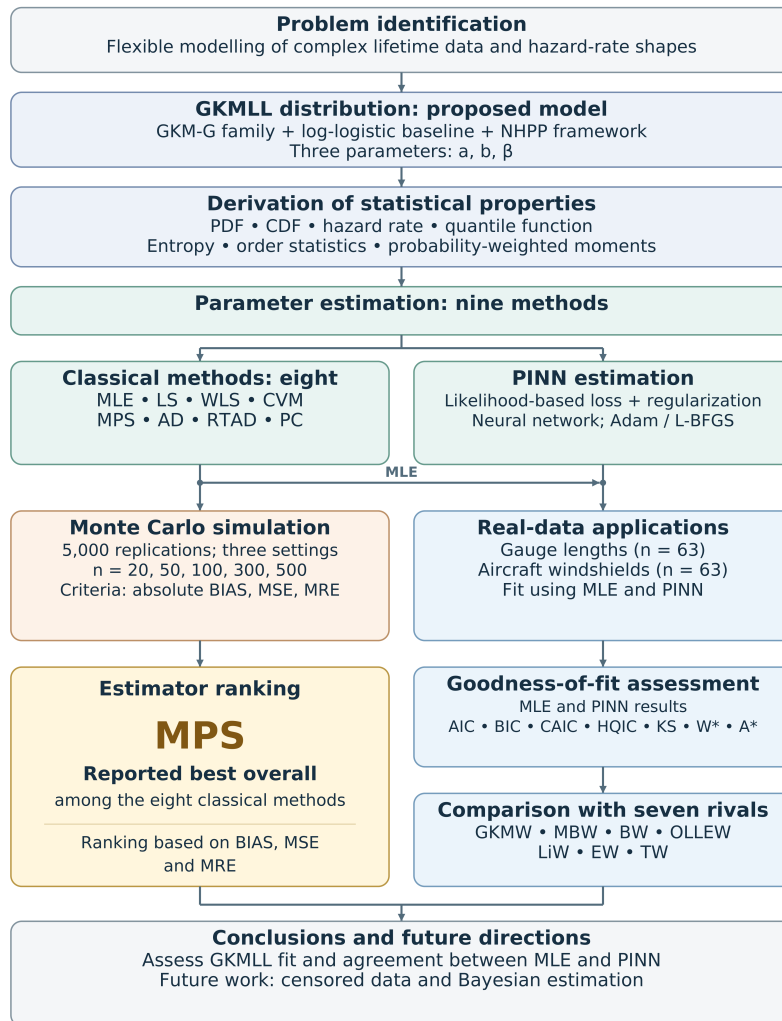


Figure 1. Workflow of the proposed study for the GKMLL distribution.

The corresponding PDF is

$$g(y; a, b, \beta) = \gamma^\beta \beta \eta(y) e^{-H(y)} w(y)^{\beta-1}, \quad y > 0, \quad (10)$$

with η , H , and w as defined above.

In this parametrization, $a > 0$ and $\beta > 0$ are shape parameters and $b > 0$ is a scale parameter. The scale parameter b regulates the dispersion of the GKMLL distribution without affecting its skewness or tail behavior, whereas the shape parameters a and β control the skewness, kurtosis, and tail behavior, and are therefore of central importance in determining the asymmetry and the general shape of the distribution. A random variable with PDF (10) is said to follow the GKMLL distribution and is denoted by $Y \sim \text{GKMLL}(a, b, \beta)$. The effect of the parameters is depicted in Figure 2 and Figure 3, which are obtained by setting $b = 1$ and varying a and β .

Throughout, we restrict the parameter vector $\theta = (a, b, \beta)$ to the natural parameter space $\Theta = (0, \infty)^3$. This

restriction ensures that the PDF is well defined for $y > 0$ and avoids boundary cases that may lead to weak identifiability or numerical instability. Identifiability is assumed in the usual parametric sense: if $G(y; \theta_1) = G(y; \theta_2)$ for all $y > 0$, then $\theta_1 = \theta_2$.

The HRF and the reversed hazard rate function (RHRF) of the GKMLL model are

$$h(y; a, b, \beta) = \frac{\beta \eta(y) v(y) [e - v(y)]^{\beta-1}}{(e-1)^\beta - [e - v(y)]^\beta}, \quad y > 0, \quad (11)$$

and

$$r(y; a, b, \beta) = \frac{g(y; a, b, \beta)}{G(y; a, b, \beta)} = \frac{\beta \eta(y) e^{-H(y)}}{1 - e^{-H(y)}}, \quad y > 0, \quad (12)$$

respectively, where $v(y) = e^{1/(1+z(y))} = e^{1-H(y)}$.

The quantile function (QF) of the GKMLL model is

$$Q(u) = b \left[\frac{-L(u)}{1+L(u)} \right]^{1/a}, \quad 0 < u < 1, \quad (13)$$

where $L(u) = \ln\{1 - (u/\gamma^\beta)^{1/\beta}\}$ and $\gamma = e/(e-1)$. The QF is convenient for computing quantile-based measures, including Galton's skewness and Moors' kurtosis. Figure 2 shows the PDF of the GKMLL distribution for $b = 1$ and selected values of a and β , and Figure 3 shows Galton's skewness and Moors' kurtosis for $b = 1$ and different values of a and β . The results show that the shape parameters a and β have an important effect on the skewness and kurtosis of the distribution.

3 Properties

This section provides an analysis of the fundamental statistical properties of the GKMLL distribution.

3.1 Linear representation

In this section, we obtain a mixture representation of the GKMLL density in terms of exponentiated log-logistic (exp-LL) densities. Let

$$\begin{aligned} h_m(y) &= \frac{d}{dy} H(y)^m = m \eta(y) H(y)^{m-1} \\ &= m \frac{a}{b} \left(\frac{y}{b} \right)^{am-1} [1 + z(y)]^{-(m+1)}, \quad m \geq 1, \end{aligned} \quad (14)$$

denote the exp-LL density with power parameter m . For any real $q > 0$, the generalized binomial series gives $G(y)^q = \gamma^{\beta q} \sum_{j=0}^{\infty} (-1)^j \binom{\beta q}{j} e^{-jH(y)}$, and expanding e^{-jH} in a power series yields $G(y)^q = \sum_{m=0}^{\infty} e_m(q) H(y)^m$ with

$$e_m(q) = \gamma^{\beta q} \sum_{j=0}^{\infty} \frac{(-1)^{j+m} j^m}{m!} \binom{\beta q}{j}. \quad (15)$$

Differentiating with $q = 1$ gives the linear representation

$$g(y) = \sum_{m=1}^{\infty} e_m(1) h_m(y). \quad (16)$$

Accordingly, the PDF of the GKMLL distribution is a linear mixture of exp-LL densities.

3.2 Order statistics

Order statistics (OSs) play a key role in quality control and reliability analysis, as they provide insight into early failures of an item and allow its future behavior to be predicted. For $i = 1, \dots, n$, let $Y_{(i)}$ denote the i th OS of a random sample of size n from the GKMLL distribution. Its PDF is

$g_{i:n}(y) = g(y)G(y)^{i-1}[1 - G(y)]^{n-i}/B(i, n-i+1)$. Expanding $[1 - G]^{n-i}$ binomially and using $gG^p = \frac{1}{p+1} \frac{d}{dy} G^{p+1}$ together with (15), we obtain

$$\begin{aligned} g_{i:n}(y) &= \sum_{m=1}^{\infty} w_m h_m(y), \quad y > 0, \\ w_m &= \frac{1}{B(i, n-i+1)} \\ &\quad \times \sum_{k=0}^{n-i} \frac{(-1)^k}{i+k} \binom{n-i}{k} e_m(i+k), \end{aligned} \quad (17)$$

where $B(\cdot, \cdot)$ denotes the beta function and $\gamma = e/(e-1)$ is the generator constant of the GKM-G family.

3.3 Probability-weighted moments

Probability-weighted moments (PWMs) were suggested by Greenwood et al. [35]. PWMs are widely used for estimating the parameters and quantiles of distributions that have an explicit quantile-based representation, and PWM-based estimators usually have low variance and moderate bias, which makes them competitive with MLEs. The (s, t) th PWM of Y , say $\omega_{s,t}$, is

$$\omega_{s,t} = E\{Y^s G(Y)^t\} = \int_0^\infty y^s g(y) G(y)^t dy,$$

where s and t are nonnegative integers. Using $gG^t = \frac{1}{t+1} \frac{d}{dy} G^{t+1}$ and (15), and noting that the s th moment of an exp-LL random variable T_m with power m is $E(T_m^s) = mb^s B(m+s/a, 1-s/a)$ for $s < a$, we obtain

$$\begin{aligned} \omega_{s,t} &= \frac{1}{t+1} \sum_{m=1}^{\infty} e_m(t+1) E(T_m^s) \\ &= \frac{b^s}{t+1} \sum_{m=1}^{\infty} m e_m(t+1) \\ &\quad \times B\left(m + \frac{s}{a}, 1 - \frac{s}{a}\right), \quad s < a. \end{aligned}$$

In particular, $\omega_{s,0} = E(Y^s)$ gives the ordinary moments of the GKMLL distribution.

3.4 Rényi entropy

Entropy quantifies the uncertainty or randomness inherent in a system, with broad applications spanning the human sciences, molecular tumor imaging, physics, and statistical methods such as sparse kernel density estimation. The Rényi entropy (RE) of order θ of the GKMLL distribution is

$$I_\theta = \frac{1}{1-\theta} \ln \left[\int_0^\infty g(y)^\theta dy \right], \quad \theta > 0, \theta \neq 1. \quad (18)$$

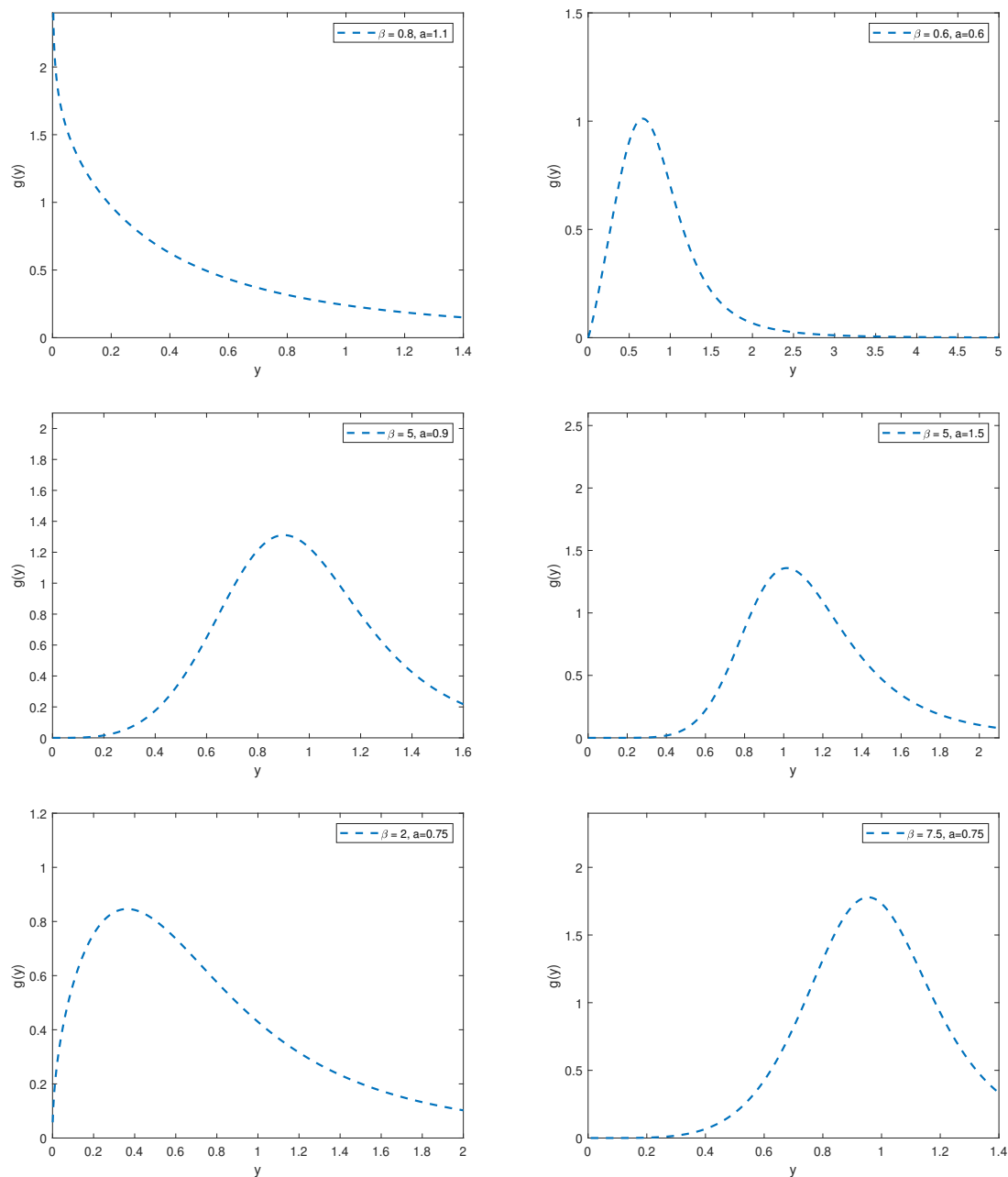


Figure 2. Plots of the GKMLL density function for $b = 1$ and varying values of the parameters a and β .

Using the binomial and power-series expansions of $w(y)^{\theta(\beta-1)}$ and $e^{-(\theta+k)H}$, and the substitution $u = H(y)$ (so that $\eta(y)dy = du$ and $\eta = \frac{a}{b}u^{(a-1)/a}(1-u)^{(a+1)/a}$), the RE can be expressed as

$$I_{\theta} = \frac{1}{1-\theta} \ln \left\{ (\gamma^{\beta} \beta)^{\theta} \left(\frac{a}{b} \right)^{\theta-1} \times \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} c_{k,m}(\theta) \mathcal{B}_m(\theta) \right\}, \quad (19)$$

where

$$c_{k,m}(\theta) = \frac{(-1)^{k+m} (\theta+k)^m}{m!} \binom{\theta(\beta-1)}{k},$$

$$\mathcal{B}_m(\theta) = B \left(m+1 + \frac{(\theta-1)(a-1)}{a}, 1 + \frac{(\theta-1)(a+1)}{a} \right),$$

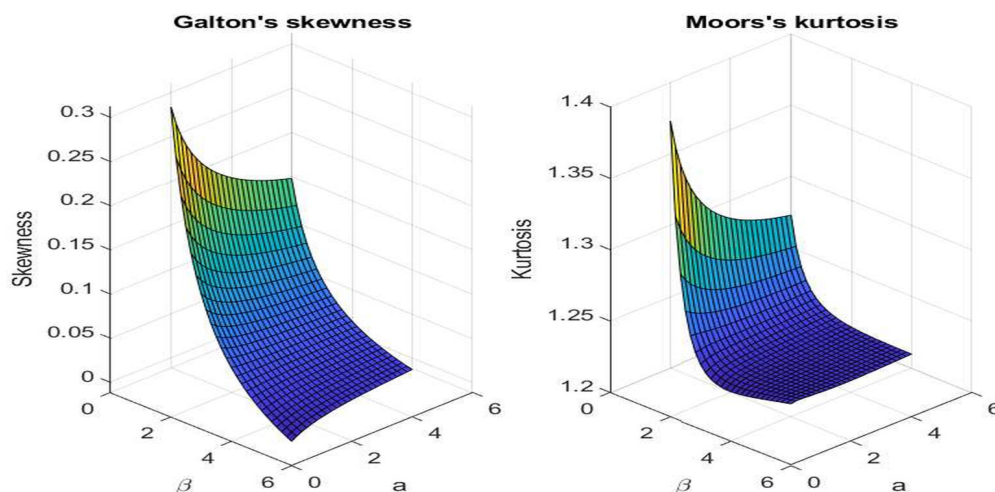


Figure 3. Galton's skewness and Moors' kurtosis coefficients for the GKMLL distribution.

provided that both arguments of the beta function are positive. The Shannon entropy is obtained as the limiting case of the RE when $\theta \rightarrow 1$.

4 Estimation methods

This section considers nine estimation procedures for the GKMLL parameters: eight classical estimators (MLE, LS, WLS, CVM, MPS, AD, RTAD, and PC) and one modern estimator based on a PINN.

4.1 Classical estimators

Let $x_1, \dots, x_n$ denote a random sample from the GKMLL distribution with parameters a , b , and β , and let $x_{(1)} < x_{(2)} < \dots < x_{(n)}$ denote the corresponding order statistics. Write $z_i = (x_i/b)^a$, $u_i = z_i/(1+z_i)$, and $\rho_i = e^{-u_i}/(1 - e^{-u_i})$. The log-likelihood function of the model is

$$\begin{aligned} \ell(a, b, \beta) = & n\beta \ln \gamma + n \ln \beta + n \ln a - n \ln b \\ & + (a-1) \sum_{i=1}^n \ln \left(\frac{x_i}{b} \right) \\ & - 2 \sum_{i=1}^n \ln(1+z_i) - \sum_{i=1}^n u_i \\ & + (\beta-1) \sum_{i=1}^n \ln[1 - e^{-u_i}]. \end{aligned} \quad (20)$$

The MLEs of a , b , and β are the values that maximize (20), or equivalently solve the following system of nonlinear

likelihood equations:

$$\begin{aligned} \frac{\partial \ell}{\partial a} = & \frac{n}{a} + \sum_{i=1}^n \ln \left(\frac{x_i}{b} \right) \\ & - 2 \sum_{i=1}^n \frac{z_i \ln(x_i/b)}{1+z_i} \\ & - \sum_{i=1}^n \frac{z_i \ln(x_i/b)}{(1+z_i)^2} \\ & + (\beta-1) \sum_{i=1}^n \rho_i \frac{z_i \ln(x_i/b)}{(1+z_i)^2} = 0, \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\partial \ell}{\partial b} = & -\frac{na}{b} + \frac{2a}{b} \sum_{i=1}^n \frac{z_i}{1+z_i} \\ & + \frac{a}{b} \sum_{i=1}^n \frac{z_i}{(1+z_i)^2} \\ & - (\beta-1) \frac{a}{b} \sum_{i=1}^n \rho_i \frac{z_i}{(1+z_i)^2} = 0, \end{aligned} \quad (22)$$

$$\frac{\partial \ell}{\partial \beta} = n \ln \gamma + \frac{n}{\beta} + \sum_{i=1}^n \ln[1 - e^{-u_i}] = 0. \quad (23)$$

Asymptotic inference for the MLE. The asymptotic results for the likelihood-based estimator are stated under standard regularity conditions for parametric MLE. Specifically, we assume:

1. $X_1, \dots, X_n$ are i.i.d. from the GKMLL model $g(x; \theta)$ with true parameter $\theta_0 \in \text{int}(\Theta)$;
2. the model is identifiable on $\Theta = (0, \infty)^3$;
3. the log-likelihood $\ell(\theta) = \sum_{i=1}^n \ln g(X_i; \theta)$ is twice continuously differentiable in a neighborhood of θ_0 , and

differentiation and integration can be interchanged under suitable regularity conditions, such as those ensured by dominated convergence; and

4. the Fisher information matrix

$$I(\boldsymbol{\theta}_0) = E_{\boldsymbol{\theta}_0} [-\nabla^2 \ln g(X; \boldsymbol{\theta}_0)]$$

exists, is finite, and is positive definite.

Under these conditions, the MLE $\hat{\boldsymbol{\theta}}$ is consistent and asymptotically normal,

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{D} \mathcal{N}(\mathbf{0}, I(\boldsymbol{\theta}_0)^{-1}).$$

In practice, standard errors (SEs) are obtained using the observed information matrix $J(\hat{\boldsymbol{\theta}}) = -\nabla^2 \ell(\hat{\boldsymbol{\theta}})$, so that $\widehat{\text{Var}}(\hat{\boldsymbol{\theta}}) = J(\hat{\boldsymbol{\theta}})^{-1}$, and the reported SEs are the square roots of the diagonal elements of $J(\hat{\boldsymbol{\theta}})^{-1}$.

For the remaining classical estimators, we write $G(x) = G(x; a, b, \beta)$ for the CDF in (9), $\bar{G}(x) = 1 - G(x)$, and $\Delta_s(x) = \partial G(x; a, b, \beta) / \partial \theta_s$ for $s = 1, 2, 3$, where $(\theta_1, \theta_2, \theta_3) = (\beta, a, b)$. With $u(x) = z(x)/(1 + z(x))$ and $\psi(x) = 1 - e^{-u(x)}$, these derivatives are

$$\begin{aligned} \Delta_1(x) &= G(x) [\ln \gamma + \ln \psi(x)], \\ \Delta_2(x) &= \gamma^\beta \beta \psi(x)^{\beta-1} e^{-u(x)} \\ &\quad \times \frac{z(x) \ln(x/b)}{(1+z(x))^2}, \\ \Delta_3(x) &= -\gamma^\beta \beta \psi(x)^{\beta-1} e^{-u(x)} \\ &\quad \times \frac{az(x)}{b(1+z(x))^2}. \end{aligned} \quad (24)$$

The LS and WLS techniques are popular approaches for estimating distribution parameters [36]. The LS and WLS estimators of the GKMLL parameters are obtained by minimizing

$$Q_{WLS}(a, b, \beta) = \sum_{i=1}^n w_i \left[G(x_{(i)}) - \frac{i}{n+1} \right]^2, \quad (25)$$

where $w_i = 1$ for the LS method and $w_i = (n+1)^2(n+2)/[i(n-i+1)]$ for the WLS method. Equivalently, the LS and WLS estimators solve the nonlinear system (for $s = 1, 2, 3$)

$$\sum_{i=1}^n w_i \left[G(x_{(i)}) - \frac{i}{n+1} \right] \Delta_s(x_{(i)}) = 0. \quad (26)$$

The Cramér-von Mises estimators (CVMEs) [37, 38] of the GKMLL parameters are obtained by minimizing the following objective function, which quantifies the squared discrepancies between the fitted CDF and the empirical distribution function:

$$C(a, b, \beta) = \frac{1}{12n} + \sum_{i=1}^n \left[G(x_{(i)}) - \frac{2i-1}{2n} \right]^2. \quad (27)$$

Equivalently, the CVMEs are obtained by numerically solving

$$\sum_{i=1}^n \left[G(x_{(i)}) - \frac{2i-1}{2n} \right] \Delta_s(x_{(i)}) = 0, \quad (28)$$

for $s = 1, 2, 3$.

MPS is a popular method for estimating the parameters of a continuous univariate distribution, used as an alternative to the ML method [39, 40]. The uniform spacings of the GKMLL distribution are defined as

$$D_i(a, b, \beta) = G(x_{(i)}) - G(x_{(i-1)}), \quad i = 1, \dots, n+1, \quad (29)$$

where $G(x_{(0)}) = 0$, $G(x_{(n+1)}) = 1$, and $\sum_{i=1}^{n+1} D_i(a, b, \beta) = 1$. The MPS estimators (MPSEs) of the GKMLL parameters are obtained by maximizing

$$M(a, b, \beta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln D_i(a, b, \beta). \quad (30)$$

Equivalently, the MPSEs can be obtained by numerically solving

$$\begin{aligned} \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(a, b, \beta)} \left[\Delta_s(x_{(i)}) \right. \\ \left. - \Delta_s(x_{(i-1)}) \right] = 0, \end{aligned} \quad (31)$$

for $s = 1, 2, 3$, with $\Delta_s(x_{(0)}) = \Delta_s(x_{(n+1)}) = 0$.

The Anderson-Darling estimators (ADEs), another class of minimum distance estimators, are obtained by minimizing

$$\begin{aligned} A(a, b, \beta) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\ln G(x_{(i)}) \right. \\ \left. + \ln \bar{G}(x_{(n+1-i)}) \right]. \end{aligned} \quad (32)$$

Equivalently, the ADEs are the numerical solution of

$$\begin{aligned} \sum_{i=1}^n (2i-1) \left[\frac{\Delta_s(x_{(i)})}{G(x_{(i)})} \right. \\ \left. - \frac{\Delta_s(x_{(n+1-i)})}{\bar{G}(x_{(n+1-i)})} \right] = 0, \end{aligned} \quad (33)$$

for $s = 1, 2, 3$.

The right-tail Anderson-Darling estimators (RTADEs) of the GKMLL parameters a , b , and β are found by minimizing

$$\begin{aligned} R(a, b, \beta) = \frac{n}{2} - 2 \sum_{i=1}^n G(x_{(i)}) \\ - \frac{1}{n} \sum_{i=1}^n (2i-1) \ln \bar{G}(x_{(n+1-i)}). \end{aligned} \quad (34)$$

The PC method [41] estimates the unknown parameters by matching the sample percentiles with the corresponding

theoretical percentiles. Since $p_i = i/(n+1)$ is an unbiased estimator of $G(x_{(i)})$, the percentile estimators (PCEs) minimize

$$P(a, b, \beta) = \sum_{i=1}^n (x_{(i)} - Q(p_i; a, b, \beta))^2, \quad (35)$$

where Q is the quantile function in (13).

4.2 PINN-based estimator

In addition to the classical methods, a PINN is used to estimate the parameters of the GKMLL model. The central idea is to embed the mathematical (likelihood) structure of the statistical model in the training of a neural network, so that the parameters are learned through a physics-informed loss function that combines a data-fidelity term with a regularization term.

To ensure reproducibility, the architecture, loss formulation, and training scheme are described below. The PINN is a fully connected feed-forward neural network. The network input is the observed lifetime sample x_i , and the output layer represents the unknown parameter vector $\theta = (\beta, a, b)^T$. The network structure is:

- Input layer: dimension 1 (observed lifetime value);
- Hidden layers: 3 fully connected layers with 20 neurons each;
- Activation function: hyperbolic tangent (tanh);
- Output layer: 3 neurons corresponding to the parameters β, a, b .

This architecture has $(1 \cdot 20 + 20) + 2(20 \cdot 20 + 20) + (20 \cdot 3 + 3) = 943$ trainable weights and biases. The hyperbolic tangent activation is chosen for its smooth differentiability, which improves gradient stability when optimizing likelihood-based objective functions. The data loss is the average negative log-likelihood of the GKMLL density g in (10):

$$L_{data}(\theta) = -\frac{1}{n} \sum_{i=1}^n \ln g(x_i; \theta). \quad (36)$$

To improve numerical stability and ensure valid parameter estimates, a regularization term is incorporated:

$$L_{reg}(\theta) = \|\theta\|^2. \quad (37)$$

The total loss function is

$$L_{total}(\theta) = L_{data}(\theta) + \lambda L_{reg}(\theta), \quad (38)$$

where $\lambda > 0$ is a tuning parameter that regulates the trade-off between goodness of fit and the magnitude of the parameters. The PINN estimation algorithm is as follows.

1. **Initialization.** Initialize the weights and biases of the network randomly (e.g., from a uniform distribution), choose the tanh activation function, the regularization parameter λ , the learning rate α , the tolerance ε , and the maximum number of epochs.

2. **Forward pass.** Evaluate the network on the observed sample to obtain the parameter vector $\theta = (\beta, a, b)^T$, which must lie in Θ .
3. **Loss evaluation.** Compute L_{data} , L_{reg} , and L_{total} from (36)–(38).
4. **Optimization.** Compute the gradient of L_{total} with respect to the network parameters by backpropagation and update them using a gradient-based optimizer (Adam or L-BFGS).
5. **Convergence check.** Stop if the change in L_{total} between successive epochs is at most ε or the maximum number of epochs is reached; otherwise, adjust α or λ if necessary and return to Step 2.
6. **Output and validation.** Return the final parameter vector $\theta^* = (\beta^*, a^*, b^*)^T$ and use it to compute the GOF statistics (AIC, CAIC, BIC, HQIC, W^* , A^* , KS), which are compared with those obtained from the MLE.

5 Simulation analysis

In this section, a Monte Carlo (MC) simulation study is conducted to evaluate the accuracy of the classical estimators of the unknown parameters of the GKMLL distribution. Let B denote the number of replications ($B = 5000$), and let $\hat{\vartheta}^{(r)}$ be the estimate of a generic parameter $\vartheta \in \{a, b, \beta\}$ in the r th replication. The estimators are assessed using the mean absolute bias, the MSE, and the MRE:

$$\text{BIAS} = \frac{1}{B} \sum_{r=1}^B |\hat{\vartheta}^{(r)} - \vartheta|,$$

$$\text{MSE} = \frac{1}{B} \sum_{r=1}^B (\hat{\vartheta}^{(r)} - \vartheta)^2,$$

$$\text{MRE} = \frac{1}{B} \sum_{r=1}^B \frac{|\hat{\vartheta}^{(r)} - \vartheta|}{\vartheta}.$$

In addition to reporting the MC point estimates of these criteria, we quantify the MC uncertainty associated with the reported summaries. For any performance metric T (e.g., BIAS, MSE, or MRE) computed per replication, the MC standard error (MCSE) is

$$\text{MCSE}(\bar{T}) = \frac{s_T}{\sqrt{B}},$$

where s_T is the sample standard deviation of the replication-level values of T . Where informative, an approximate 95% confidence interval for the MC estimate is $\bar{T} \pm 1.96 \text{MCSE}(\bar{T})$. This allows the reader to distinguish substantive differences between estimators from differences that are within simulation noise.

For each sample size $n \in \{20, 50, 100, 300, 500\}$ and each parameter setting, $B = 5000$ random samples were generated from the GKMLL distribution. The parameter settings (a, β, b) considered were $(0.5, 0.25, 1.5)$, $(0.5, 0.25, 3.5)$, and $(0.5, 2, 1.5)$. The GKMLL parameters

(a, b, β) were estimated with eight estimation methods, namely the MLE, LSE, WLSE, CVME, MPSE, PCE, ADE, and RTADE, for each combination of parameter values and sample size, and the BIAS, MSE, and MRE were then computed for each estimator. All simulations and numerical calculations were performed with MATLAB R2020a.

The MC results are reported in Table 1, Table 2, and Table 3. The tables also list the rank of each estimator in each scenario (in curly braces) and the cumulative rank, denoted by $\sum \text{RANKS}$, of each estimator for each sample size. The tables reveal a consistent pattern across the estimation methods: for every parameter combination, the BIAS, MSE, and MRE decrease as the sample size increases. The partial and overall rankings of the competing estimators are summarized in Table 4, where a lower total ranking score corresponds to better performance. Based on these rankings, the eight estimation methods are ordered from best to worst as follows: MPS, ML, AD, WLS, RTAD, LS, CVM, and PC. Notably, the MPS method attains the lowest overall score of 15, confirming its superior performance among the competing approaches. This indicates that the MPS estimator generally provides more accurate parameter estimates, as reflected by its lower values of MSE, BIAS, and MRE across different sample sizes and parameter configurations.

Finally, to complement statistical accuracy with practical feasibility, the computational cost of the classical estimators and of the PINN-based estimator can be compared by recording the wall-clock runtime per replication under a fixed software and hardware environment, together with relative runtime ratios (e.g., PINN time divided by MLE time). This comparison is relevant because, unlike the closed-form or standard numerical optimization routines used in classical estimation, PINN training introduces additional computational overhead due to iterative gradient-based optimization.

6 Real-life data

Two real datasets are analyzed in this section to assess the applicability and practical relevance of the proposed GKMLL model. The first dataset consists of the first 63 gauge lengths, measured at a constant length of 10 mm, reported by Kundu and Ragab [42]. This dataset has been widely used in the literature for evaluating and comparing lifetime distributions. The second dataset contains the service times of 63 aircraft windshields and has also been considered in previous reliability studies as a benchmark dataset for lifetime modeling [41]. These data have also been discussed in [43, 44]. The GKMLL model is compared with the following competing lifetime distributions: the GKMW, MBW, BW, OLLEW, LiW, EW, and TW models (Table 5). The parameters are estimated by the classical MLE method and by the PINN approach.

Several commonly used GOF and information-based criteria are used to compare the fitted models: the Akaike information criterion (AIC), consistent Akaike information

criterion (CAIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), the Cramér–von Mises (W^*) statistic, the Anderson–Darling (A^*) statistic, the negative log-likelihood ($-\mathcal{L}$), and the Kolmogorov–Smirnov (KS) statistic with its p -value. In general, smaller values of these criteria indicate a better-fitting model. In addition, the performance of the GKMLL model is assessed graphically using the total time on test (TTT) plot, the PDF, CDF, survival function (SF), and probability–probability (P–P) plots for the considered datasets.

6.1 Results Based on MLE

Table 6 and Table 7 list the MLEs of the model parameters with their SEs in parentheses. Based on these estimates, the GOF statistics are computed and presented in Table 8 and Table 9. These tables show that the GKMLL distribution provides a good fit compared with the competing models on both datasets. Specifically, the GKMLL model attains the lowest values of AIC, CAIC, BIC, and HQIC, together with a high p -value for the KS test. Since lower information criteria indicate better model adequacy and higher p -values indicate no significant deviation between the empirical and theoretical distributions, these results support the suitability of the proposed model. Graphical diagnostics are discussed in Section 6.3.

6.2 Results Based on PINN Estimation

To further evaluate the robustness of the proposed model, the PINN approach described in Section 4 is also employed to estimate its unknown parameters. This approach combines statistical inference with machine learning by embedding the likelihood structure of the GKMLL model in a neural network framework. Table 10 and Table 11 show the GOF statistics obtained using the PINN estimates. The GKMLL distribution again attains the best overall ranking among the competing models under this estimation framework, and its GOF statistics are of a similar magnitude to those obtained with the MLE (Table 8 and Table 9). This agreement between the classical and machine-learning-based estimation supports the stability of the GKMLL model and the PINN approach as a possible alternative for parameter estimation in complex nonlinear models.

6.3 Graphical Analysis

Graphical tools are also used to confirm the adequacy of the GKMLL model under both the classical and the machine-learning-based estimation. The fitted PDF, CDF, SF, and P–P plots for the two datasets under the MLE method are presented in Figure 4 and Figure 5. The fitted PDF is consistent with the empirical histogram and captures the main

Table 1. Estimators performance for $(a, \beta, b) = (0.5, 0.25, 1.5)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RTADEs
20	BIAS	$\hat{a}$	0.37901{6}	0.35065{5}	0.30056{3}	0.39438{7}	0.16252{1}	1.84669{8}	0.28186{2}	0.31833{4}
		$\hat{b}$	0.51922{2}	1.83841{7}	1.68571{6}	1.40180{4}	1.06022{1}	0.31400{8}	1.19773{3}	1.42237{5}
		$\hat{\beta}$	0.21074{4}	0.22221{5}	0.20455{3}	0.22844{6}	0.14765{1}	0.38516{8}	0.20390{2}	0.24268{7}
	MSE	$\hat{a}$	0.29568{7}	0.22396{4}	0.17717{3}	0.28449{6}	0.08733{1}	1.97997{8}	0.17022{2}	0.25477{5}
		$\hat{b}$	1.46850{2}	1.93700{7}	1.55400{5}	1.91500{4}	1.13470{1}	1.14870{8}	1.42150{3}	1.65300{6}
		$\hat{\beta}$	0.09569{4}	0.10249{5}	0.08948{3}	0.10703{6}	0.07743{1}	0.63228{8}	0.08929{2}	0.25688{7}
	MRE	$\hat{a}$	0.75802{6}	0.70130{5}	0.60112{3}	0.78876{7}	0.32505{1}	3.69337{8}	0.56371{2}	0.63666{4}
		$\hat{b}$	1.34615{2}	1.55894{7}	1.12381{6}	1.93453{4}	1.37348{1}	1.21000{8}	1.79849{3}	1.94825{5}
		$\hat{\beta}$	0.84297{4}	0.88885{5}	0.81819{3}	0.91376{6}	0.59060{1}	1.54064{8}	0.81559{2}	0.97073{7}
	Σ RANKS		37.0{4}	50.0{6}	35.0{3}	50.0{6}	9.0{1}	72.0{8}	21.0{2}	50.0{6}
50	BIAS	$\hat{a}$	0.17657{4}	0.22169{6}	0.17917{5}	0.22915{7}	0.03049{1}	1.22530{8}	0.16043{3}	0.15139{2}
		$\hat{b}$	0.14757{2}	1.59073{7}	1.18647{3}	1.35802{6}	0.22211{1}	1.36000{8}	1.53423{4}	1.29419{5}
		$\hat{\beta}$	0.13762{4}	0.15629{6}	0.14926{5}	0.16076{7}	0.04248{1}	0.30815{8}	0.13621{3}	0.11946{2}
	MSE	$\hat{a}$	0.07319{4}	0.11420{6}	0.07594{5}	0.12204{7}	0.01401{1}	3.78122{8}	0.05941{3}	0.05330{2}
		$\hat{b}$	1.38494{3}	0.90989{7}	1.53632{2}	1.89721{5}	0.45684{1}	1.58020{8}	1.14907{4}	1.18084{6}
		$\hat{\beta}$	0.03849{2}	0.05160{6}	0.04236{5}	0.05444{7}	0.01633{1}	0.20813{8}	0.03925{3}	0.04135{4}
	MRE	$\hat{a}$	0.35315{4}	0.44339{6}	0.35834{5}	0.45830{7}	0.06098{1}	2.45060{8}	0.32086{3}	0.30278{2}
		$\hat{b}$	0.76505{2}	1.06049{7}	0.79098{3}	0.90535{6}	0.14808{1}	1.20000{8}	0.82282{4}	0.86279{5}
		$\hat{\beta}$	0.55050{4}	0.62514{6}	0.59702{5}	0.64304{7}	0.16993{1}	1.23258{8}	0.54484{3}	0.47784{2}
	Σ RANKS		29.0{2}	57.0{6}	38.0{5}	59.0{7}	9.0{1}	72.0{8}	30.0{3.5}	30.0{3.5}
100	BIAS	$\hat{a}$	0.11041{3}	0.13853{6}	0.11248{4}	0.14089{7}	0.00118{1}	0.74013{8}	0.10946{2}	0.11443{5}
		$\hat{b}$	0.08767{3}	1.38480{6}	0.95669{2}	1.28108{5}	0.01110{1}	1.44000{8}	1.12364{4}	1.12152{7}
		$\hat{\beta}$	0.10387{6}	0.09757{4}	0.12094{7}	0.09884{5}	0.00224{1}	0.34731{8}	0.09209{3}	0.08649{2}
	MSE	$\hat{a}$	0.01978{4}	0.04305{6}	0.01898{2}	0.04521{7}	0.00002{1}	1.06214{8}	0.01928{3}	0.02367{5}
		$\hat{b}$	1.23468{3}	5.74020{6}	1.45030{2}	1.65151{5}	0.00012{1}	1.28418{8}	1.01992{4}	1.14074{7}
		$\hat{\beta}$	0.01867{6}	0.01592{4}	0.02384{7}	0.01634{5}	0.00001{1}	0.41184{8}	0.01498{2}	0.01590{3}
	MRE	$\hat{a}$	0.22082{3}	0.27706{6}	0.22495{4}	0.28177{7}	0.00235{1}	1.48025{8}	0.21891{2}	0.22887{5}
		$\hat{b}$	0.72512{3}	0.92320{6}	0.63779{2}	0.85405{5}	0.00740{1}	1.10000{8}	0.76910{4}	0.94768{7}
		$\hat{\beta}$	0.41550{6}	0.39026{4}	0.48375{7}	0.39535{5}	0.00896{1}	1.38923{8}	0.36838{3}	0.34598{2}
	Σ RANKS		37.0{3.5}	48.0{6}	37.0{3.5}	51.0{7}	9.0{1}	72.0{8}	27.0{2}	43.0{5}
300	BIAS	$\hat{a}$	0.05515{2}	0.07195{5}	0.09031{7}	0.07225{6}	0.00000{1}	0.45566{8}	0.05859{4}	0.05649{3}
		$\hat{b}$	0.02373{6}	0.62101{2}	0.94134{7}	0.62200{4}	0.00000{1}	1.79000{8}	0.62188{3}	0.62848{5}
		$\hat{\beta}$	0.06782{6}	0.04847{3}	0.12422{7}	0.04867{4}	0.00000{1}	0.45854{8}	0.05089{5}	0.04272{2}
	MSE	$\hat{a}$	0.00482{3}	0.00841{5}	0.01013{7}	0.00858{6}	0.00000{1}	0.34601{8}	0.00525{4}	0.00481{2}
		$\hat{b}$	0.65042{4}	0.67030{6}	1.09795{7}	0.64807{3}	0.00000{1}	1.37108{8}	0.52909{2}	0.66461{5}
		$\hat{\beta}$	0.00734{6}	0.00374{3}	0.01921{7}	0.00377{4}	0.00000{1}	0.95623{8}	0.00417{5}	0.00282{2}
	MRE	$\hat{a}$	0.11031{2}	0.14389{5}	0.18061{7}	0.14451{6}	0.00000{1}	0.91131{8}	0.11718{4}	0.11298{3}
		$\hat{b}$	0.48249{6}	0.41401{2}	0.54090{7}	0.41466{4}	0.00000{1}	1.10000{8}	0.41459{3}	0.41898{5}
		$\hat{\beta}$	0.27126{6}	0.19386{3}	0.49688{7}	0.19470{4}	0.00000{1}	1.83416{8}	0.20357{5}	0.17087{2}
	Σ RANKS		41.0{5.5}	34.0{3}	63.0{7}	41.0{5.5}	9.0{1}	72.0{8}	35.0{4}	29.0{2}
500	BIAS	$\hat{a}$	0.04755{4}	0.05569{6}	0.10009{7}	0.05554{5}	0.00000{1}	0.37142{8}	0.04744{3}	0.03948{2}
		$\hat{b}$	0.01019{6}	0.47008{4}	0.09325{7}	0.46999{3}	0.00000{1}	1.64000{8}	0.47492{5}	0.46039{2}
		$\hat{\beta}$	0.06344{6}	0.04018{3}	0.14318{7}	0.04033{4}	0.00000{1}	0.32104{8}	0.04290{5}	0.02990{2}
	MSE	$\hat{a}$	0.00323{3}	0.00456{5.5}	0.01140{7}	0.00456{5.5}	0.00000{1}	0.24568{8}	0.00334{4}	0.00245{2}
		$\hat{b}$	0.48539{6}	0.32522{4}	1.06005{7}	0.32161{3}	0.00000{1}	2.11718{8}	0.31091{2}	0.34047{5}
		$\hat{\beta}$	0.00529{6}	0.00266{3}	0.02264{7}	0.00268{4}	0.00000{1}	0.34298{8}	0.00294{5}	0.00148{2}
	MRE	$\hat{a}$	0.09509{4}	0.11137{6}	0.20018{7}	0.11108{5}	0.00000{1}	0.74284{8}	0.09489{3}	0.07896{2}
		$\hat{b}$	0.42013{6}	0.31338{4}	0.48217{7}	0.31332{3}	0.00000{1}	1.23000{8}	0.31661{5}	0.30693{2}
		$\hat{\beta}$	0.25376{6}	0.16074{3}	0.57272{7}	0.16131{4}	0.00000{1}	1.28416{8}	0.17160{5}	0.11962{2}
	Σ RANKS		47.0{6}	39.0{5}	63.0{7}	36.0{3}	9.0{1}	72.0{8}	37.0{4}	21.0{2}

Table 2. Estimators performance for $(a, \beta, b) = (0.5, 0.25, 3.5)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RTADEs
20	BIAS	$\hat{a}$	0.18634{1}	0.24386{5}	0.21049{4}	0.30862{7}	0.19381{2}	1.42363{8}	0.20422{3}	0.28735{6}
		$\hat{b}$	2.99530{1}	3.46543{7}	2.41335{6}	3.76475{3}	3.80492{5}	3.54144{8}	3.47786{2}	2.43160{4}
		$\hat{\beta}$	0.13625{1}	0.17863{5}	0.16470{4}	0.18608{7}	0.15471{3}	0.18928{8}	0.15280{2}	0.18538{6}
	MSE	$\hat{a}$	0.06047{1}	0.11820{5}	0.08312{4}	0.20105{7}	0.06360{2}	6.30304{8}	0.07243{3}	0.20065{6}
		$\hat{b}$	3.02317{1}	2.18095{7}	2.27368{6}	3.65067{3}	1.65965{5}	2.53691{8}	2.07042{2}	1.39016{4}
		$\hat{\beta}$	0.04447{1}	0.06507{5}	0.05700{4}	0.06791{6}	0.05124{3}	0.09016{8}	0.04888{2}	0.06967{7}
	MRE	$\hat{a}$	0.37268{1}	0.48772{5}	0.42099{4}	0.61723{7}	0.38762{2}	2.84725{8}	0.40845{3}	0.57470{6}
		$\hat{b}$	0.85580{1}	2.13298{7}	2.11810{6}	1.64707{3}	1.94426{5}	3.86898{8}	1.56510{2}	1.83760{4}
		$\hat{\beta}$	0.54501{1}	0.71453{5}	0.65881{4}	0.74434{7}	0.61886{3}	0.75713{8}	0.61120{2}	0.74151{6}
	Σ RANKS		30.0{3}	51.0{7}	42.0{4}	50.0{6}	9.0{1}	72.0{8}	21.0{2}	49.0{5}
50	BIAS	$\hat{a}$	0.09661{1}	0.18993{6}	0.15774{4}	0.20827{7}	0.10094{2}	0.61055{8}	0.12999{3}	0.16776{5}
		$\hat{b}$	1.60017{1}	2.58435{7}	1.47239{5}	2.20138{4}	2.35456{2}	2.57935{8}	2.70035{3}	2.93686{6}
		$\hat{\beta}$	0.07088{1}	0.12414{6}	0.10381{4}	0.12638{7}	0.07912{2}	0.22367{8}	0.09648{3}	0.11258{5}
	MSE	$\hat{a}$	0.01759{2}	0.07489{6}	0.04685{4}	0.09237{7}	0.01691{1}	1.82021{8}	0.03030{3}	0.06463{5}
		$\hat{b}$	1.17117{1}	2.46301{6}	1.18529{5}	1.50361{4}	2.08433{3}	1.50940{8}	1.35925{2}	1.88161{7}
		$\hat{\beta}$	0.01262{1}	0.03019{7}	0.02175{4}	0.02985{6}	0.01404{2}	0.12706{8}	0.01883{3}	0.02323{5}
	MRE	$\hat{a}$	0.19321{1}	0.37986{6}	0.31547{4}	0.41654{7}	0.20188{2}	1.22109{8}	0.25998{3}	0.33552{5}
		$\hat{b}$	0.45719{1}	1.45267{7}	1.27782{5}	1.20040{4}	0.95845{2}	1.16553{8}	1.05724{3}	1.41053{6}
		$\hat{\beta}$	0.28351{1}	0.49657{6}	0.41526{4}	0.50551{7}	0.31647{2}	0.89469{8}	0.38591{3}	0.45032{5}
	Σ RANKS		18.0{2}	57.0{7}	39.0{4}	53.0{6}	10.0{1}	72.0{8}	26.0{3}	49.0{5}
100	BIAS	$\hat{a}$	0.06155{1}	0.12226{6}	0.10236{5}	0.12884{7}	0.07065{2}	0.30990{8}	0.09219{3}	0.09255{4}
		$\hat{b}$	0.85681{1}	2.49498{7}	1.39421{4}	1.74180{5}	1.11878{2}	1.23502{8}	2.24134{3}	2.77873{6}
		$\hat{\beta}$	0.04453{1}	0.08696{6}	0.07203{4}	0.08886{7}	0.05577{2}	0.23687{8}	0.06771{3}	0.07221{5}
	MSE	$\hat{a}$	0.00719{1}	0.02631{6}	0.01842{5}	0.03271{7}	0.00847{2}	0.26970{8}	0.01402{3}	0.01682{4}
		$\hat{b}$	1.14896{1}	2.17726{7}	1.16859{4}	1.25794{6}	1.16839{2}	1.16760{8}	1.22633{3}	1.23796{5}
		$\hat{\beta}$	0.00547{1}	0.01440{6}	0.01022{4}	0.01520{7}	0.00649{2}	0.41691{8}	0.00937{3}	0.01047{5}
	MRE	$\hat{a}$	0.12309{1}	0.24453{6}	0.20471{5}	0.25768{7}	0.14131{2}	0.61980{8}	0.18439{3}	0.18511{4}
		$\hat{b}$	0.24480{1}	0.85571{7}	0.74120{4}	0.78337{5}	0.31965{2}	1.49572{8}	0.64038{3}	0.79392{6}
		$\hat{\beta}$	0.17812{1}	0.34786{6}	0.28814{4}	0.35544{7}	0.22307{2}	0.94747{8}	0.27084{3}	0.28884{5}
	Σ RANKS		18.0{2}	57.0{6}	39.0{4}	58.0{7}	9.0{1}	72.0{8}	27.0{3}	44.0{5}
300	BIAS	$\hat{a}$	0.02833{1}	0.07099{6}	0.05573{5}	0.07148{7}	0.03461{2}	0.27698{8}	0.05320{3}	0.05441{4}
		$\hat{b}$	0.15706{2}	1.35199{6}	1.24903{4}	1.31694{5}	0.09360{1}	1.12004{8}	1.13468{3}	1.36092{7}
		$\hat{\beta}$	0.01894{1}	0.04653{6}	0.03892{4}	0.04683{7}	0.02256{2}	0.47645{8}	0.03654{3}	0.03964{5}
	MSE	$\hat{a}$	0.00163{1}	0.00819{6}	0.00481{5}	0.00830{7}	0.00249{2}	0.12405{8}	0.00437{3}	0.00459{4}
		$\hat{b}$	0.24304{2}	2.11162{7}	2.07711{4}	1.00150{5}	0.16757{1}	1.12388{8}	1.12504{3}	1.08471{6}
		$\hat{\beta}$	0.00085{1}	0.00376{6}	0.00245{4}	0.00380{7}	0.00104{2}	1.53054{8}	0.00229{3}	0.00254{5}
	MRE	$\hat{a}$	0.05667{1}	0.14197{6}	0.11145{5}	0.14295{7}	0.06922{2}	0.55396{8}	0.10641{3}	0.10883{4}
		$\hat{b}$	0.04487{2}	0.38628{6}	0.35687{4}	0.37627{5}	0.02674{1}	1.44858{8}	0.32419{3}	0.38883{7}
		$\hat{\beta}$	0.07577{1}	0.18614{6}	0.15567{4}	0.18733{7}	0.09022{2}	1.90579{8}	0.14614{3}	0.15856{5}
	Σ RANKS		15.0{2}	55.0{6}	39.0{4}	57.0{7}	12.0{1}	72.0{8}	27.0{3}	47.0{5}
500	BIAS	$\hat{a}$	0.01710{1}	0.05507{6}	0.04396{5}	0.05592{7}	0.01783{2}	0.32849{8}	0.04268{4}	0.04113{3}
		$\hat{b}$	0.08346{2}	1.17457{7}	1.02869{4}	1.14016{6}	0.00388{1}	1.07914{8}	0.97831{3}	1.07656{5}
		$\hat{\beta}$	0.01163{1}	0.03397{6}	0.02851{4}	0.03445{7}	0.01250{2}	0.92248{8}	0.02775{3}	0.02908{5}
	MSE	$\hat{a}$	0.00072{1}	0.00586{6}	0.00327{5}	0.00627{7}	0.00089{2}	0.25726{8}	0.00307{4}	0.00291{3}
		$\hat{b}$	0.09635{2}	2.10335{7}	2.02830{4}	1.00121{6}	0.00011{1}	0.41091{8}	1.02263{3}	1.04878{5}
		$\hat{\beta}$	0.00032{2}	0.00199{6}	0.00130{4}	0.00204{7}	0.00029{1}	4.80387{8}	0.00124{3}	0.00149{5}
	MRE	$\hat{a}$	0.03420{1}	0.11014{6}	0.08792{5}	0.11185{7}	0.03566{2}	0.65697{8}	0.08536{4}	0.08226{3}
		$\hat{b}$	0.02385{2}	0.33559{7}	0.29391{4}	0.32576{6}	0.00111{1}	3.79404{8}	0.27952{3}	0.30759{5}
		$\hat{\beta}$	0.04654{1}	0.13588{6}	0.11405{4}	0.13781{7}	0.05000{2}	3.68992{8}	0.11099{3}	0.11631{5}
	Σ RANKS		14.0{2}	57.0{6}	39.0{4.5}	60.0{7}	13.0{1}	72.0{8}	30.0{3}	39.0{4.5}

Table 3. Estimators performance for $(a, \beta, b) = (0.5, 2, 1.5)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RTADEs
20	BIAS	$\hat{a}$	0.13918{4}	0.14155{6}	0.13276{3}	0.18351{7}	0.11220{1}	0.27771{8}	0.13248{2}	0.14081{5}
		$\hat{b}$	4.96951{1}	4.37429{7}	4.23009{6}	4.10735{5}	4.87463{4}	4.22134{8}	4.58366{2}	4.72415{3}
		$\hat{\beta}$	3.48568{8}	2.86166{5}	2.74511{4}	3.25088{6}	2.46773{1}	2.66802{2}	2.67160{3}	3.45783{7}
	MSE	$\hat{a}$	0.04590{4}	0.06597{6}	0.05976{5}	0.12109{7}	0.02381{1}	0.13617{8}	0.03929{2}	0.04000{3}
		$\hat{b}$	3.85206{1}	3.16354{7}	4.29843{6}	3.75028{5}	3.35066{4}	4.22215{8}	3.93107{2}	3.64594{3}
		$\hat{\beta}$	2.62482{7}	1.79849{4}	2.41459{3}	2.26959{6}	1.54264{1}	2.39255{5}	1.00464{2}	2.84752{8}
	MRE	$\hat{a}$	0.27837{4}	0.28311{6}	0.26551{3}	0.36702{7}	0.22440{1}	0.55542{8}	0.26496{2}	0.28162{5}
		$\hat{b}$	3.31301{1}	4.24953{7}	4.15340{6}	4.07157{5}	3.91642{4}	4.48089{8}	3.72244{2}	3.81610{3}
		$\hat{\beta}$	1.74284{8}	1.43083{5}	1.37256{4}	1.62544{6}	1.23387{1}	1.33401{2}	1.33580{3}	1.72891{7}
	Σ RANKS		38.0{3}	53.0{6}	40.0{4}	54.0{7}	18.0{1}	57.0{8}	20.0{2}	44.0{5}
50	BIAS	$\hat{a}$	0.08126{2}	0.09612{6}	0.08625{3}	0.10182{7}	0.07884{1}	0.28129{8}	0.08657{4}	0.08812{5}
		$\hat{b}$	2.66440{1}	4.15158{7}	3.56325{4}	4.04019{6}	3.14628{2}	3.19631{8}	3.31319{3}	2.99898{5}
		$\hat{\beta}$	2.19667{4}	2.37430{6}	2.19366{3}	2.50598{8}	1.70382{1}	2.39010{7}	1.99222{2}	2.30607{5}
	MSE	$\hat{a}$	0.01296{2}	0.01864{6}	0.01474{4}	0.02166{7}	0.01086{1}	0.13265{8}	0.01414{3}	0.01526{5}
		$\hat{b}$	2.58500{1}	3.09739{7}	3.41861{4}	3.00972{5}	2.39738{2}	2.35316{8}	2.96087{3}	3.58878{6}
		$\hat{\beta}$	2.31325{4}	2.77628{5}	1.85508{3}	2.20557{7}	1.21803{1}	2.27316{8}	1.00416{2}	1.81845{6}
	MRE	$\hat{a}$	0.16251{2}	0.19225{6}	0.17249{3}	0.20364{7}	0.15768{1}	0.56257{8}	0.17314{4}	0.17624{5}
		$\hat{b}$	1.77627{1}	2.76772{7}	2.37550{4}	2.69346{6}	2.09752{2}	4.39754{8}	2.20879{3}	2.66598{5}
		$\hat{\beta}$	1.09833{4}	1.18715{6}	1.09683{3}	1.25299{8}	0.85191{1}	1.19505{7}	0.99611{2}	1.15303{5}
	Σ RANKS		21.0{2}	56.0{6}	31.0{4}	61.0{7}	12.0{1}	70.0{8}	26.0{3}	47.0{5}
100	BIAS	$\hat{a}$	0.05781{2}	0.07008{6}	0.06064{5}	0.07126{7}	0.05686{1}	0.25719{8}	0.05989{3}	0.06061{4}
		$\hat{b}$	1.79899{1}	2.96264{6}	2.27296{4}	2.92613{5}	2.10672{2}	2.71139{8}	2.20539{3}	2.03545{7}
		$\hat{\beta}$	1.25869{3}	1.80420{5}	1.34827{4}	1.86338{6}	1.00360{1}	3.02269{8}	1.24533{2}	1.91682{7}
	MSE	$\hat{a}$	0.00536{2}	0.00835{6}	0.00600{4}	0.00894{7}	0.00499{1}	0.08587{8}	0.00571{3}	0.00607{5}
		$\hat{b}$	2.46634{1}	2.62390{6}	2.09304{4}	3.00759{5}	2.29900{2}	2.23215{8}	2.86573{3}	3.35849{7}
		$\hat{\beta}$	3.97479{3}	2.06253{5}	1.40981{4}	2.10834{6}	1.18014{1}	2.11788{8}	1.00353{2}	1.72016{7}
	MRE	$\hat{a}$	0.11563{2}	0.14017{6}	0.12128{5}	0.14252{7}	0.11373{1}	0.51438{8}	0.11978{3}	0.12122{4}
		$\hat{b}$	1.19933{1}	1.97510{6}	1.51530{4}	1.95076{5}	1.40448{2}	4.47426{8}	1.47026{3}	2.02363{7}
		$\hat{\beta}$	0.62934{3}	0.90210{5}	0.67413{4}	0.93169{6}	0.50180{1}	1.51135{8}	0.62266{2}	0.95841{7}
	Σ RANKS		18.0{2}	51.0{5}	38.0{4}	54.0{6}	12.0{1}	72.0{8}	24.0{3}	55.0{7}
300	BIAS	$\hat{a}$	0.03019{2}	0.03870{7}	0.03301{4}	0.03866{6}	0.02968{1}	0.21756{8}	0.03232{3}	0.03639{5}
		$\hat{b}$	0.87207{1}	1.39273{6}	1.04368{4}	1.38155{5}	0.93319{2}	2.67252{8}	1.02243{3}	1.44795{7}
		$\hat{\beta}$	0.55285{2}	0.99537{6}	0.66741{4}	0.99429{5}	0.47197{1}	3.88851{8}	0.61916{3}	1.29285{7}
	MSE	$\hat{a}$	0.00139{2}	0.00239{6}	0.00168{4}	0.00242{7}	0.00134{1}	0.06646{8}	0.00163{3}	0.00203{5}
		$\hat{b}$	1.40902{1}	2.56932{6}	1.96788{4}	2.61568{5}	1.72170{2}	2.15496{8}	1.91690{3}	3.26834{7}
		$\hat{\beta}$	0.76545{2}	2.46714{5}	1.11939{4}	2.49928{6}	0.51889{1}	1.03745{8}	0.95197{3}	2.26154{7}
	MRE	$\hat{a}$	0.06037{2}	0.07740{7}	0.06601{4}	0.07733{6}	0.05937{1}	0.43512{8}	0.06464{3}	0.07278{5}
		$\hat{b}$	0.58138{1}	0.92849{6}	0.69579{4}	0.92103{5}	0.62213{2}	3.98168{8}	0.68162{3}	0.96530{7}
		$\hat{\beta}$	0.27642{2}	0.49769{6}	0.33371{4}	0.49714{5}	0.23598{1}	1.94425{8}	0.30958{3}	0.64643{7}
	Σ RANKS		15.0{2}	55.0{6}	36.0{4}	50.0{5}	12.0{1}	72.0{8}	27.0{3}	57.0{7}
500	BIAS	$\hat{a}$	0.02398{2}	0.03315{6}	0.02723{4}	0.03320{7}	0.02373{1}	0.21386{8}	0.02691{3}	0.03031{5}
		$\hat{b}$	0.70313{1}	1.13801{6}	0.83763{4}	1.13447{5}	0.71422{2}	6.14251{8}	0.83620{3}	1.33393{7}
		$\hat{\beta}$	0.44044{2}	0.72785{5}	0.51491{4}	0.73212{6}	0.38590{1}	4.01507{8}	0.49583{3}	0.92010{7}
	MSE	$\hat{a}$	0.00089{2}	0.00170{6}	0.00112{4}	0.00171{7}	0.00087{1}	0.06264{8}	0.00109{3}	0.00144{5}
		$\hat{b}$	0.80916{1}	3.07464{6}	1.25863{3}	3.05252{5}	0.91985{2}	1.08794{8}	1.34046{4}	2.18790{7}
		$\hat{\beta}$	0.43355{2}	1.25426{5}	0.57748{4}	1.27490{6}	0.31905{1}	1.02543{8}	0.52485{3}	2.14580{7}
	MRE	$\hat{a}$	0.04796{2}	0.06630{6}	0.05445{4}	0.06640{7}	0.04746{1}	0.42771{8}	0.05382{3}	0.06062{5}
		$\hat{b}$	0.46875{1}	0.75868{6}	0.55842{4}	0.75631{5}	0.47615{2}	4.09501{8}	0.55746{3}	0.88929{7}
		$\hat{\beta}$	0.22022{2}	0.36393{5}	0.25746{4}	0.36606{6}	0.19295{1}	2.00753{8}	0.24792{3}	0.46005{7}
	Σ RANKS		15.0{2}	51.0{5}	35.0{4}	54.0{6}	12.0{1}	72.0{8}	28.0{3}	57.0{7}

Table 4. Partial and overall rankings of the estimation methods under different parameter configurations $\theta = (a, \beta, b)$.

θ^T	n	MLE	LSE	WLSE	CVME	MPSE	PCE	ADE	RTADE
(0.5, 0.25, 1.5)	20	4	6	3	6	1	8	2	6
	50	2	6	5	7	1	8	3.5	3.5
	100	3.5	6	3.5	7	1	8	2	5
	300	5.5	3	7	5.5	1	8	4	2
	500	6	5	7	3	1	8	4	2
(0.5, 0.25, 3.5)	20	3	7	4	6	1	8	2	5
	50	2	7	4	6	1	8	3	5
	100	2	6	4	7	1	8	3	5
	300	2	6	4	7	1	8	3	5
	500	2	6	4.5	7	1	8	3	4.5
(0.5, 2, 1.5)	20	3	6	4	7	1	8	2	5
	50	2	6	4	7	1	8	3	5
	100	2	5	4	6	1	8	3	7
	300	2	6	4	5	1	8	3	7
	500	2	5	4	6	1	8	3	7
Σ Ranks		43	86	66	92.5	15	120	43.5	74
Overall Rank		2	6	4	7	1	8	3	5

Table 5. Competing Weibull-based distributions considered in the comparative analysis.

Distribution	Abbreviation
Modified Beta Weibull [6]	MBW
Beta Weibull [1]	BW
Odd log-logistic exponentiated Weibull [10]	OLLEW
Lindley Weibull [11]	LiW
Exponentiated Weibull [12]	EW
Transmuted Weibull [3]	TW
Generalized Kavya–Manoharan Weibull [45]	GKMW

distributional features, including skewness and tail behavior. The CDF and SF plots show close agreement between the empirical and theoretical curves with no apparent systematic deviation. Moreover, the P–P plots show that the observed probabilities are close to the 45-degree reference line, which supports the appropriateness of the GKMLL model.

Figure 6 and Figure 7 compare the fitted PDFs of the GKMLL distribution and its competitors under the MLE. The GKMLL model follows the empirical data closely, especially in the tail region where some of the competing distributions fit poorly. These results are supported by the P–P plots in Figure 8 and Figure 9.

Complementing the classical estimation results, Figure 12 and Figure 13 show the fitted PDFs of the GKMLL distribution and the competing models under the PINN estimation scheme for the gauge lengths and aircraft windshields datasets, respectively. The GKMLL model continues to perform well in the PINN framework: its fitted density follows the empirical distribution closely over the entire

support, whereas some competing models display visible differences, especially in the tails and the peaks.

The similarity between the graphical results of the MLE (Figure 6 and Figure 7) and of the PINN (Figure 12 and Figure 13) also indicates the stability of the proposed model and suggests that the PINN is a viable alternative estimation method.

Lastly, the TTT plots (Figure 10 and Figure 11) indicate different hazard-rate trends in the two datasets. Concave and convex TTT patterns suggest increasing and decreasing hazard rates, respectively. The GKMLL distribution can accommodate these varied behaviors, which shows its flexibility in fitting a wide variety of lifetime data.

6.4 Discussion

The numerical findings and graphical evaluations indicate that the proposed GKMLL distribution offers a good fit relative to the competing models considered. This performance is supported not only by the GOF measures and diagnostic plots but also by the greater flexibility of the distribution in modeling complex features of lifetime data, such as pronounced skewness, heavy-tailed behavior, and diverse hazard-rate shapes, including both monotonic and non-monotonic patterns. Such characteristics are frequently observed in lifetime-data analysis and may not be adequately captured by more restrictive parametric distributions.

A direct comparison with established flexible families is informative. The generalized gamma distribution is frequently used as a benchmark due to its broad nesting structure (including the gamma and Weibull distributions) and its ability to approximate lognormal-like behavior under

Table 6. ML estimates and corresponding SEs for the fitted distributions based on the gauge lengths dataset.

Distribution	ML estimates and SEs				
GKMW	$\hat{\delta} = 44.8650$ (8.9400)	$\hat{\beta} = 1.5680$ (0.9485)	$\hat{\lambda} = 0.6585$ (1.2268)		
MBW	$\hat{\delta} = 1.0011$ (0.3887)	$\hat{\beta} = 1.0014$ (0.2517)	$\hat{a} = 1.9972$ (13.4352)	$\hat{b} = 0.7134$ (6.4848)	$\hat{c} = 1.0029$ (2.5747)
BW	$\hat{\delta} = 13.0596$ (5.7269)	$\hat{\beta} = 0.0005$ (2.0478)	$\hat{a} = 121.0112$ (617.6803)	$\hat{b} = 0.1625$ (8.0652)	
OLLEW	$\hat{\delta} = 71.1250$ (66.6782)	$\hat{\beta} = 0.0126$ (6.6654)	$\hat{\gamma} = 0.9529$ (0.0384)	$\hat{\theta} = 50.5148$ (27.8608)	
LiW	$\hat{\delta} = 0.0038$ (0.4037)	$\hat{\beta} = 3.5511$ (0.3450)	$\hat{\theta} = 0.5958$ (82.4204)		
EW	$\hat{\delta} = 1.1494$ (1.1200)	$\hat{\beta} = 1.4541$ (0.7583)	$\hat{\theta} = 37.1349$ (79.4533)		
TW	$\hat{\delta} = 2.5164$ (0.1414)	$\hat{\beta} = 5.5706$ (0.4021)	$\hat{\lambda} = 0.6563$ (0.2732)		
GKMML	$\hat{a} = 6.2681$ (10.7400)	$\hat{b} = 2.0908$ (0.9374)	$\hat{\beta} = 10.2741$ (1.1158)		

Table 7. ML estimates and corresponding SEs for the fitted distributions based on the aircraft windshields dataset.

Distribution	ML estimates and SEs				
GKMW	$\hat{\delta} = 0.4580$ (0.1885)	$\hat{\beta} = 1.3988$ (0.5034)	$\hat{\lambda} = 0.0294$ (0.0373)		
MBW	$\hat{\delta} = 1.0006$ (43.2128)	$\hat{\beta} = 0.9999$ (0.5111)	$\hat{a} = 3.0430$ (1.0717)	$\hat{b} = 0.8257$ (254.2038)	$\hat{c} = 1.0011$ (0.1783)
BW	$\hat{\delta} = 0.9641$ (0.0924)	$\hat{\beta} = 0.1273$ (0.3440)	$\hat{a} = 0.6068$ (0.3258)	$\hat{b} = 1.0026$ (11.6530)	
OLLEW	$\hat{\delta} = 61.8207$ (0.3021)	$\hat{\beta} = 3.1485$ (0.2457)	$\hat{\gamma} = 0.0658$ (0.0073)	$\hat{\theta} = 2.1308$ (0.1602)	
LiW	$\hat{\delta} = 0.2787$ (0.6218)	$\hat{\beta} = 0.6192$ (0.1221)	$\hat{\theta} = 0.9704$ (1.3452)		
EW	$\hat{\delta} = 11.3829$ (0.0867)	$\hat{\beta} = 1.1010$ (0.2763)	$\hat{\theta} = 0.5982$ (0.2040)		
TW	$\hat{\delta} = 5.5355$ (5.0758)	$\hat{\beta} = 0.7431$ (0.1628)	$\hat{\lambda} = 0.3160$ (0.8546)		
GKMML	$\hat{a} = 2.6393$ (0.0774)	$\hat{b} = 0.2319$ (0.4145)	$\hat{\beta} = 17.7114$ (0.0282)		

Table 8. GOF statistics for the fitted distributions based on the gauge lengths dataset (MLE).

Distribution	k	$-\mathcal{L}$	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	P -value	Rank
GKMW	3	56.3108	118.6200	119.0300	125.0500	121.1500	0.0644	0.3369	0.0783	0.768900	2
MBW	5	55.5042	121.0100	122.0600	131.7200	125.2200	0.0472	0.2530	0.0598	0.840400	4
BW	4	55.7130	119.4300	120.1200	128.0000	122.8000	0.0521	0.2938	0.0815	0.766000	3
OLLEW	4	57.8465	123.6900	124.3800	132.2700	127.0600	0.0776	0.6041	0.0927	0.775600	6
LiW	3	67.6676	141.3400	141.7400	147.7600	143.8600	0.1681	1.4706	0.1204	0.175200	8
EW	3	56.3108	118.6200	119.0300	125.0500	121.1500	0.0644	0.3369	0.0783	0.768900	5
TW	3	60.5613	127.1200	127.5300	133.5500	129.6500	0.0931	0.7236	0.0835	0.740600	7
GKMML	3	56.2658	118.5300	118.9400	124.9600	121.0600	0.0702	0.4316	0.0885	0.832300	1

Table 9. GOF statistics for the fitted distributions based on the aircraft windshields dataset (MLE).

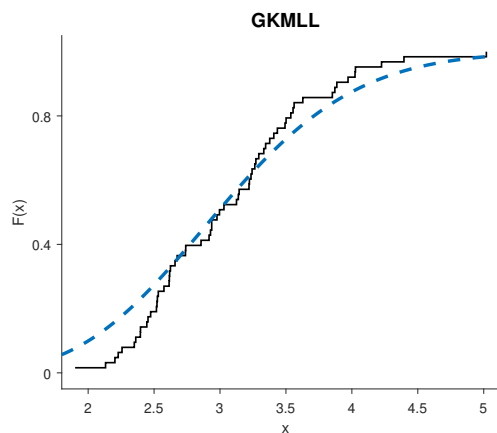
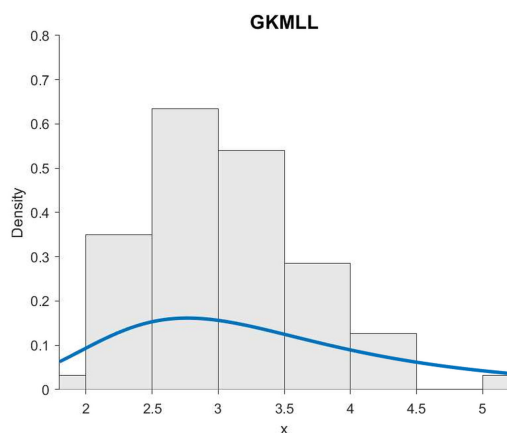
Distribution	k	$-\mathcal{L}$	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	p -value
GKMW	3	150.3032	306.4024	306.9242	312.1385	308.5868	0.0686	0.3959	0.0945	0.786279
MBW	5	193.1988	396.3977	397.7613	405.9578	400.0382	0.1854	0.3985	0.4993	0.648332
BW	4	150.0520	308.1041	308.9930	315.7522	311.0166	0.0676	0.3880	0.0847	0.740231
OLLEW	4	150.4207	308.0536	308.8425	315.6017	310.8660	0.0646	0.2700	0.9889	0.600002
LiW	3	150.2871	306.6853	307.2070	312.4214	308.8715	0.0608	0.4228	0.1007	0.566786
EW	3	150.1493	306.4076	307.0303	312.1437	308.6020	0.4078	0.3346	0.0877	0.630364
TW	3	150.5044	307.0089	307.5307	312.7450	309.1933	0.0746	0.4164	0.6969	0.447758
GKMML	3	149.9856	305.9713	306.4930	311.7073	308.1556	0.0575	0.2848	0.0736	0.897389

Table 10. GOF statistics for the fitted distributions based on the gauge lengths dataset obtained using the PINN method.

Distribution	k	$-\mathcal{L}$	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	p -value	Rank
GKMW	3	56.2658	118.5519	118.9587	124.9813	121.0706	0.0702	0.4316	0.0885	0.832305	2
MBW	5	55.8104	121.6208	122.6735	132.3365	125.8354	0.0724	0.4261	0.0910	0.826209	5
BW	4	56.1781	120.4684	121.2570	128.1421	122.8401	0.0712	0.3179	0.0787	0.831004	4
OLLEW	4	57.8465	123.6931	124.3828	132.2656	127.0647	0.0776	0.6041	0.0927	0.775627	6
LiW	3	59.5331	125.0662	125.4730	131.4956	127.5949	0.5817	0.9932	0.0987	0.828922	7
EW	3	56.3107	118.6215	119.0283	125.0509	121.1403	0.0718	0.3199	0.0712	0.788310	3
TW	3	60.4512	126.1225	126.5283	132.5510	128.6412	0.2200	0.7734	0.0936	0.761280	8
GKMML	3	54.4165	114.8331	115.2398	121.2625	117.3618	0.0811	0.5405	0.0994	0.932414	1

Table 11. GOF statistics for the fitted distributions based on the aircraft windshields dataset obtained using the PINN method.

Distribution	k	$-\mathcal{L}$	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	p -value	Rank
GKMW	3	149.0032	305.4024	305.9242	311.1385	307.5868	0.0676	0.3859	0.0845	0.776279	2
MBW	5	192.1988	395.3977	396.7613	404.9578	399.0382	0.1754	0.3885	0.4893	0.658332	8
BW	4	149.0520	307.1041	307.9930	314.7522	310.0166	0.0686	0.3980	0.0947	0.750231	3
OLLEW	4	149.4207	306.0536	306.8425	313.6017	308.8660	0.0846	0.2900	0.9989	0.620002	6
LiW	3	148.2871	302.6853	303.2070	308.4214	304.8715	0.0908	0.4628	0.2207	0.597786	5
EW	3	149.1393	304.4076	305.0303	310.1437	306.6020	0.4278	0.3546	0.0977	0.650364	4
TW	3	150.5044	307.0089	307.5307	312.7450	309.1933	0.0746	0.4164	0.6969	0.447758	7
GKMML	3	148.1556	304.9713	305.4930	310.7073	307.1556	0.0411	0.2174	0.0628	0.855200	1



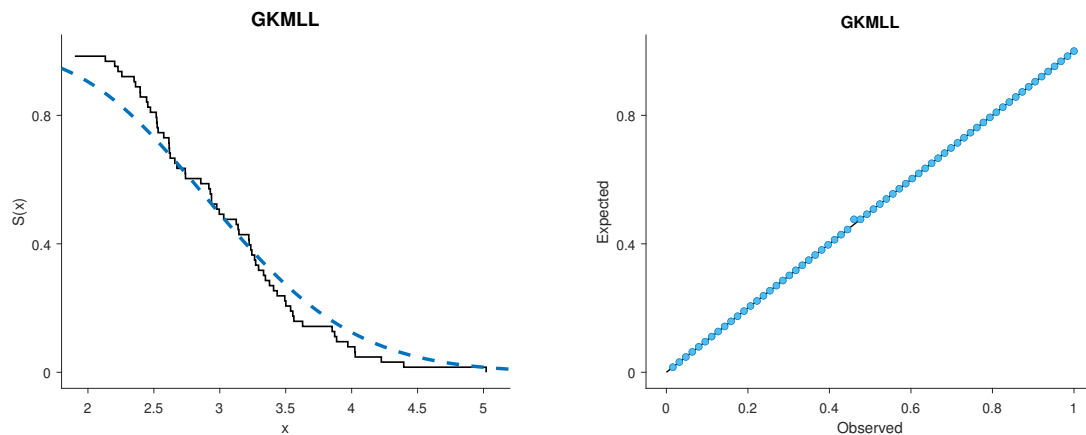


Figure 4. Form-fitting PDF, CDF, SF, and P-P plots of the proposed GKMLL distribution based on the gauge lengths dataset.

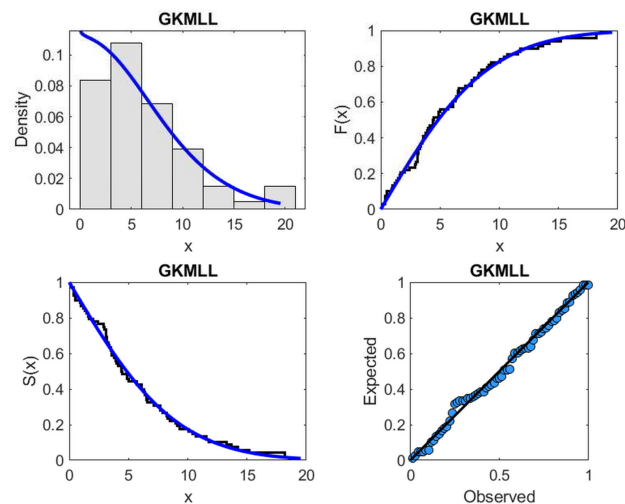


Figure 5. Form-fitting PDF, CDF, SF, and P-P plots of the proposed GKMLL distribution based on the aircraft windshields dataset.

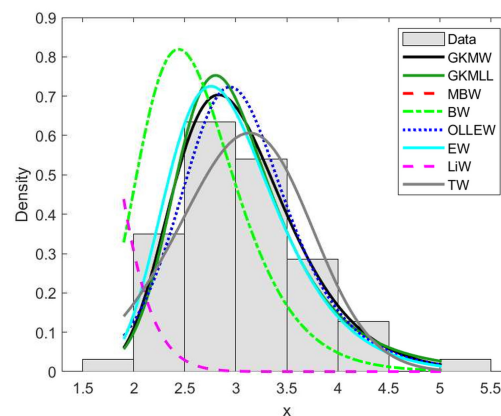


Figure 6. Comparison of the form-fitting PDF of the GKMLL and competing distributions based on the gauge lengths dataset.

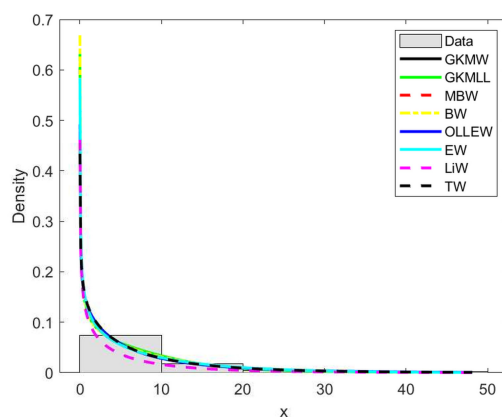


Figure 7. Comparison of the form-fitting PDF of the GKMLL and competing distributions based on the aircraft windshields dataset.

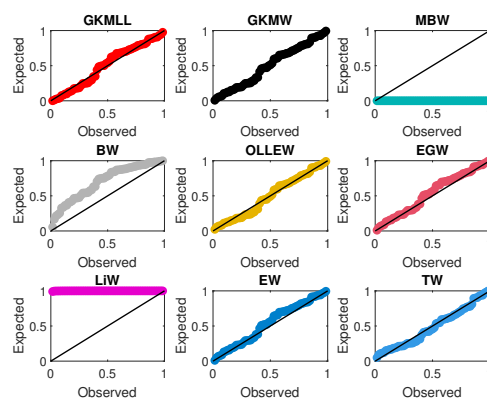


Figure 8. Probability–probability (P–P) plots comparing the GKMLL distribution with the competing distributions for the gauge lengths dataset.

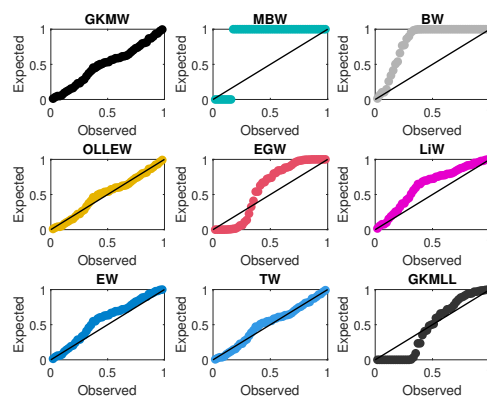


Figure 9. Probability–probability (P–P) plots comparing the GKMLL distribution with the competing distributions for the aircraft windshields dataset.

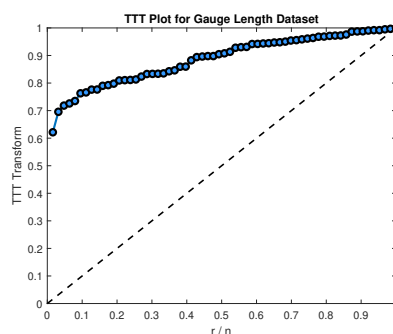


Figure 10. Total time on test (TTT) plot for the gauge lengths dataset.

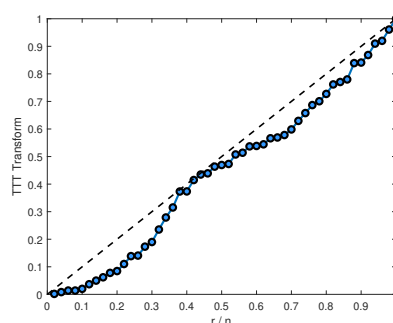


Figure 11. Total time on test (TTT) plot for the aircraft windshields dataset.

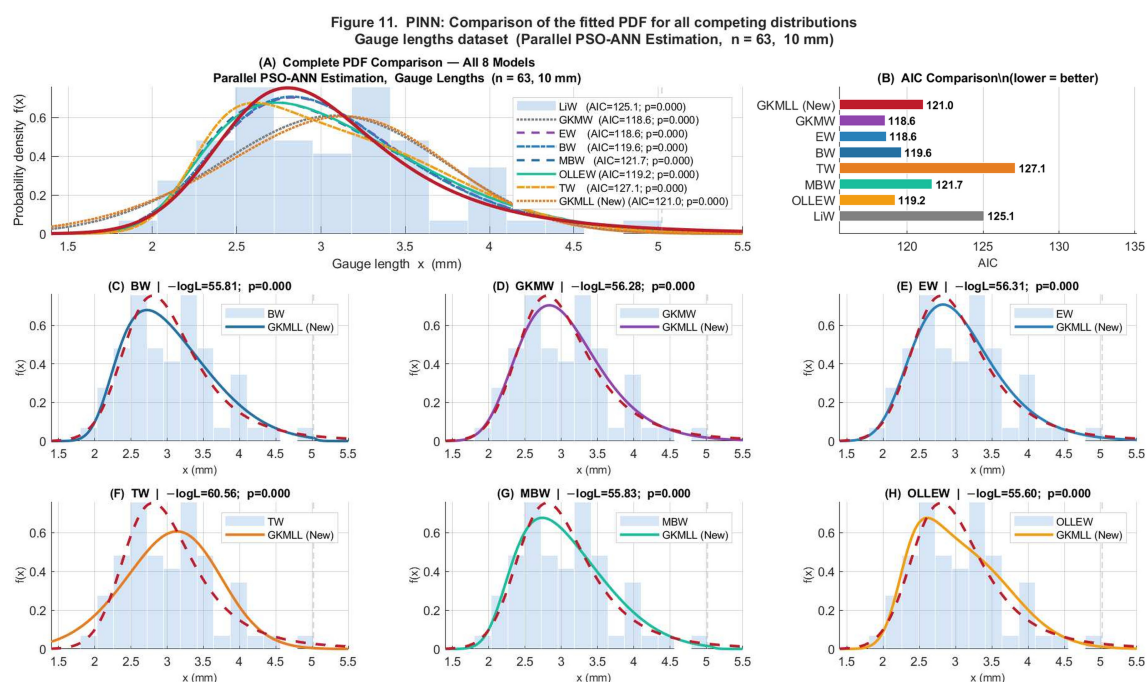


Figure 12. PINN: Comparison of the fitted PDF for all competing distributions for the gauge lengths dataset.

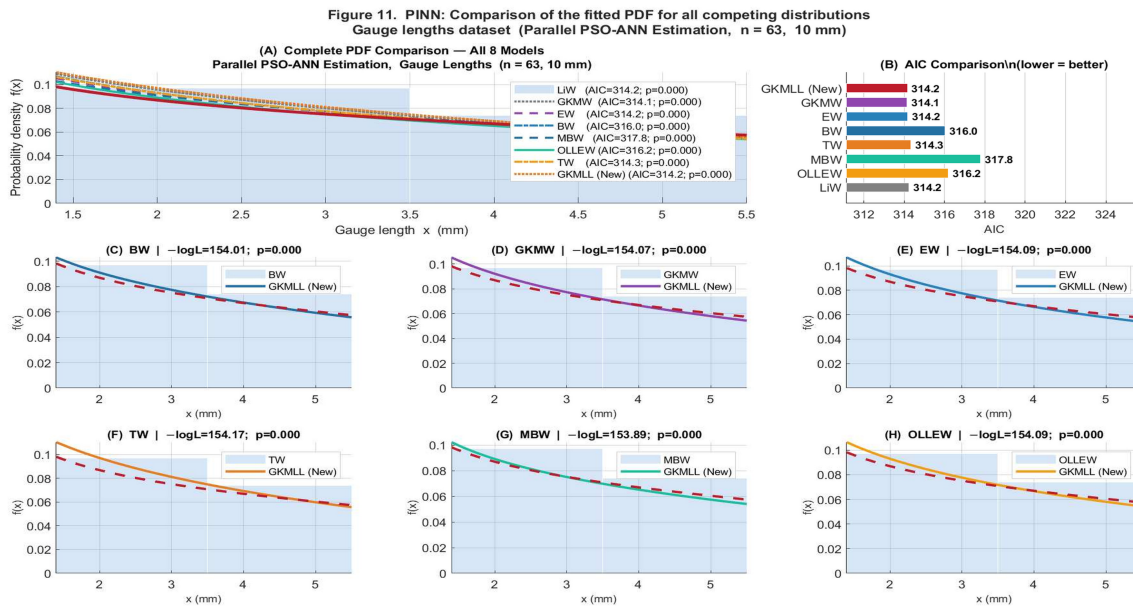


Figure 13. PINN: Comparison of the fitted PDF for all competing distributions for the aircraft windshields dataset.

certain parametrizations. Nonetheless, it may offer limited adaptability for highly irregular hazard profiles and, in some cases, insufficient tail flexibility. In contrast, GKMLL extends the log-logistic framework while preserving interpretability, and it expands the attainable hazard-rate and tail behaviors, features that appear to underpin its improved empirical performance on the analyzed datasets.

Generalized Weibull variants introduce additional shape control beyond the Weibull baseline and are widely applied when monotonicity is violated. However, many remain closely tied to Weibull-type tail behavior and may be less effective under heavier tails or abrupt hazard changes. The GKMLL distribution is comparatively well suited to such settings, providing increased hazard-shape diversity alongside a tractable parametric structure.

From an inferential perspective, the agreement between the classical (MLE-based) and PINN-based GOF results in the applications supports the stability of the proposed model and highlights PINNs as a viable alternative when classical optimization becomes challenging. Nevertheless, PINN estimation may require additional computation and tuning, and its use is most justified when the modeling benefits outweigh these costs.

Several limitations merit acknowledgment. The added flexibility increases the risk of identifiability issues or multiple local optima, particularly in small samples or under censoring. Moreover, the present analysis focuses on complete data; extensions to censored settings and regression structures with covariates remain important directions for future work. Finally, broader validation across additional domains and under misspecification would clarify the settings in which GKMLL offers the largest gains.

In terms of applications, GKMLL is promising for reliability and industrial lifetime modeling, biomedical SA, and risk-oriented contexts where heavy tails and complex hazard shapes are plausible. It may also be useful in actuarial loss modeling, environmental extremes, and fatigue data. Overall, GKMLL constitutes a flexible and practically relevant addition to lifetime modeling, especially when generalized gamma or generalized Weibull-type alternatives are insufficient.

7 Conclusion and future work

This study proposed a novel and flexible lifetime distribution, referred to as the GKMLL distribution, by extending the classical log-logistic distribution within the GKM-G framework. The proposed model can accommodate a broad range of distributional shapes and hazard-rate behaviors, including both monotonic and non-monotonic patterns frequently encountered in real-world lifetime data. A comprehensive analysis of the statistical properties of the model was carried out, including the derivation of its PDF, HRF, quantile function, linear representation, order statistics, probability-weighted moments, and entropy measures. The findings highlight the flexibility of the model in representing key characteristics of lifetime data, including varying degrees of skewness, kurtosis, and heavy-tailed behavior.

Several classical estimation methods, together with the recently developed PINN approach, were employed to estimate the unknown parameters of the GKMLL distribution. The results of an extensive MC simulation study showed

that the maximum product of spacings (MPS) method generally provides the most accurate parameter estimates, with favorable performance in terms of bias, MSE, and relative error across sample sizes and parameter configurations.

The practical usefulness of the GKMLL distribution was further established through its application to two real-life lifetime datasets. The numerical GOF measures and graphical diagnostics showed that the proposed model compares favorably with the competing distributions. In particular, the GKMLL distribution produced the smallest information-criterion values and high p -values for the corresponding GOF tests, demonstrating its ability to capture the underlying structure and key features of the observed data. Furthermore, the agreement between the GOF results obtained under the maximum likelihood (MLE) and the physics-informed neural network (PINN) approaches provides additional support for the stability and reliability of the proposed modeling framework.

Overall, the GKMLL distribution is a useful addition to the class of flexible lifetime distributions, offering substantial modeling flexibility while retaining a relatively simple parametrization. Its ability to accommodate diverse distributional and hazard-rate behaviors makes it suitable for applications in survival analysis, reliability engineering, and risk assessment.

Future research could explore several promising extensions of the GKMLL distribution, including its application to censored and truncated data, the development of Bayesian estimation methods, and the construction of discrete and multivariate versions of the proposed model. In addition, investigating advanced machine-learning-based estimation approaches, particularly in the context of large-scale and high-dimensional datasets, could further improve the flexibility, computational efficiency, and practical applicability of the model across diverse statistical applications.

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