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Solvability and Controllability of a Retarded-Type Nonlocal Non-Autonomous Fractional Differential Equation

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Abstract: This paper considers a non-autonomous retarded-type fractional differential equation involving Caputo derivative along with a nonlocal condition in a general Banach space. We present a novel approach to determine the existence-uniqueness and controllability of mild solution to the considered problem using the fixed-point technique, classical semigroup theory, and tools of fractional calculus. It is imperative to mention that the main results are established without assuming the continuity of linear operator $-A(t)$ and compactness condition on semigroup. At the end, the developed theoretical results have been applied to a nonlocal fractional order retarded elliptic evolution equation.

Keywords: Functional differential equations, local and global existence, complete controllability, analytic semigroup, fractional calculus.

1 Introduction

Fractional derivatives have enticed many scientists and researchers due to its applicability in various areas like robotics and control, anomalous-relaxation, diffusion-processes, porous medium, phase synchronization, etc [1,2,3,4]. Indeed, some of the proposed fractional models have been validated experimentally and shown their efficacy in modeling physical-systems more accurately than classical ones, for instance, see [5,6,7,8] and the references therein. The key reason behind such advancement is that the fractional derivatives are nonlocal in nature which makes fractional differential equations (FDEs) an important one to describe memory effects in complex systems [9,10].

The aforementioned reasons have increased a significant interest of the authors to consider the abstract formulation of various types of FDEs and study their qualitative and quantitative characteristics. Ding et al. [11] considered a fractional-delay system and defined its mild solution using Laplace transform and Mittag-Leffler functions. The authors also rendered sufficient criteria on nonlinear function term to show that the nonlinear-system is controllable if the controllability of the linear-system is assumed. In [12], the authors established the existence of mild solution for neutral-type FDEs with state-dependent delay using the non-compact measure and Mönch fixed-point theorem. Jothimani et al. [13] investigated the controllability results for fractional integrodifferential equations with nondense nature in Banach spaces. Employing Schauder's fixed-point theorem, Chen et al. [14] discussed the existence and approximate controllability of mild solutions for a class of nonlocal FDEs in Banach spaces. It is noted that the existence-uniqueness and controllability results for the autonomous FDEs have been studied vastly (refer to the references listed in the above-mentioned literature), however, the study for the non-autonomous FDEs is still at initial level. This is mainly due to difficulty in defining a proper representation of mild solution for non-autonomous FDEs. It is worth mentioning a pioneer work by El-Borai [15] in which the author established a fundamental solution of a non-autonomous fractional evolution equation using the theory of nonlinear analysis, classical semigroup theory and the

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probability density function. Thereafter, Fie Xiao [16] studied a nonlocal non-autonomous FDE and investigated existence results using Sadovskii's fixed-point theorem. In [17], the authors rendered sufficient criteria for the controllability of a mild solution to a nonlocal fractional evolution system along with impulsive condition in a Banach space. In [18,19,20,21], the authors studied neutral-type and Sobolev-type fractional evolution systems after introducing proper definition of mild solutions based on the theory of resolvent operators and probability density functions. Recently, the existence of solution of non-autonomous FDEs with integral impulse condition is investigated by Kumar et al. [22] using the non-compact measure, fixed-point techniques, and $k-$set contraction.

The present article is focused on establishing the existence-uniqueness and controllability results to the following nonlocal non-autonomous FDE in a Banach space X.

$$\begin{aligned} {}^{C}D_r^{\alpha}[x(r)] + P(r)x(r) &= h(r,x(r),x_r), \quad r \in I = [0,b], \\ g(x_{[-\eta,0]}) &= \kappa. \end{aligned} \tag{1}$$

Here, ${}^{C}D_r^{\alpha}$ denotes the $\alpha-$th order Caputo's derivative with $\alpha \in (0,1]$. The maps $h : \widetilde{X} \to X$ and $g : C_0 \to C_0$ are nonlinear with $\widetilde{X} = I \times X \times C_0$, where $C_0 = C([-\eta,0],X)$ is a space of continuous functions $w : [-\eta,0] \to X$ and η is a positive constant. For $r \geq 0$, the linear operator $-P(r) : \mathscr{D}(P)(\subseteq X) \to X$ (not necessarily bounded) is closed and $\overline{\mathscr{D}(P)} = X$ such that $\mathscr{D}(P)$ is independent of r. Moreover, $-P(r)$ satisfies the following conditions.

(P1)For every $r \in I$ and $a \in \mathbb{C}$ with $\Re(a) \geq 0$, the operator $[aE + P(r)]^{-1}$ exists and

$$\left\| [aE + P(r)]^{-1} \right\| \leq \frac{K}{|\, a \,| + 1},$$

(P2)For every $\theta_1, \ \theta_2, \ \theta_3 \in I$, we have

$$\left\| [P(\theta_1) - P(\theta_3)] P^{-1}(\theta_2) \right\| \leq K |\, \theta_1 - \theta_3 \,|^{\beta}, \ \ 0 < \beta < 1,$$

where $K > 0$ and β are independent of $\theta_1, \ \theta_2, \ \theta_3$, and E denotes the identity operator on X. The assumption $(P1)$ guarantees that $-P(r)$ is an infinitesimal generator of an analytic semigroup $\{S_r(s) = e^{-sP(r)}\}$. More details can be seen in [23,24].

The organization of this article is as follows. The Section 2 defines preliminary facts and introduces a proper definition of a mild solution to FDE (1). In Section 3, we shall establish the existence-uniqueness and regularity of mild solution. Further, we consider the associated control problem to FDE (1) and derive sufficient conditions for the complete controllability of mild solution over the interval $J = [-\eta, b]$ in the Section 4. At the end, we shall apply the developed theoretical results to an application in the Section 5.

2 Preliminaries

For $\alpha \in (0,1)$, the $\alpha-$th order Caputo's derivative for a function $w : I \to \mathbb{R}$ is defined as [25]:

$${}^{C}D_r^{\alpha} w(r) = \frac{1}{\Gamma(1-\alpha)} \int_0^r (r-\mu)^{-\alpha} w'(\mu) d\mu, \quad \text{whenever } w' \in L^1(I,\mathbb{R}).$$

In order to give a proper definition of a mild solution of the problem (1), first we introduce the following nonlinear operators [15]:

$$\begin{aligned} \psi(r,\theta) &= \alpha \int_0^r v r^{\alpha-1} p_{\alpha}(v) exp\left(-r^{\alpha} v P(\theta)\right) dv, \\ \varphi(r,\theta) &= \sum_{j=1}^{\infty} \varphi_j(r,\theta), \\ U(r) &= -P(r)P^{-1}(0) - \int_0^r \varphi(r,\mu) P(\mu) P^{-1}(0) d\mu. \end{aligned}$$

where p_{α} is a probability density function defined on $[0,\infty)$. Further, φ_k satisfies the following recurrence relation:

$$\varphi_1(r,\theta) = [P(r) - P(\theta)]\psi(r-\theta,\theta) \ \text{ and } \varphi_{j+1}(r,\theta) = \int_{\theta}^{r} \varphi_j(r,v)\varphi_1(v,\theta) dv, \text{ for } j = 1,2,...$$

Let $C_r = C([-\eta, r], X)$, $r \in I$, denote a space of continuous functions $f : [-\eta, r] \to X$. It is easy to see that C_r becomes Banach space with respect to the following norm

$$\|f\|_r := \sup_{-\eta \le s \le r} \|f(s)\|, \text{ for } f \in C_r.$$

Further, for each $r \in I$, $x_r : [-\eta, 0] \to X$ defined as $x_r(s) = x(r+s)$ will be continuous, i.e., $x_r \in C_0$. Now, the definition of mild solution to (1) can be defined as follows:

Let $\phi \in C_0$ be such that $g(\phi) = \kappa$. Then, a continuous function $x : J \to X$ becomes a mild solution of (1) (see [15,16, 21]), if

$$x(r) = \phi(0) + \int_0^r \psi(r-\theta,\theta)U(\theta)P(0)\phi(0)d\theta + \int_0^r \psi(r-\theta,\theta)h(\theta,x(\theta),x_\theta)d\theta$$
$$+ \int_0^r \int_0^\theta \psi(r-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)d\mu d\theta, \text{ for } r \in I,$$

and $x(r) = \phi(r)$ for $r \in [-\eta, 0]$. In addition, if $x(s) \in \mathscr{D}(P)$ for each $s \in I$, the Caputo derivative of x is continuous on the interval $(0,b)$ and the function x satisfies (1) on $(0,b)$, then such function x is said to be a classical solution of (1).

If the continuous function $x : [-\eta, b_0] \to X$ becomes a mild solution (or classical solution) of the problem (1), for some $b_0 \in (0,b)$, then x is referred as a local mild solution (or a local classical solution) of (1) on $[-\eta, b_0]$.

For establishing the existence-uniqueness and controllability results, the following lemma is required:

Lemma 1. *The following results hold:*

(i) The operator-valued functions $\psi(r-\theta,\theta)$ and $P(r)\psi(r-\theta,\theta)$ are continuous in the uniform operator topology in the variables r, θ, where $0 \le r \le \theta - \varepsilon$, $r \in I$, for any $\varepsilon > 0$. Moreover,

$$\|\psi(r-\theta,\theta)\| \le K(r-\theta)^{\alpha-1}.$$

(ii) The function $\varphi(r,\theta)$ is uniformly continuous in the uniform operator topology in r,θ provided $0 \le r \le \theta - \varepsilon$, $\varepsilon \le r \le b$ for any $\varepsilon > 0$. Moreover,

$$\|\varphi(r,\theta)\| \le K(r-\theta)^{\beta-1}.$$

(iii) For $r \in I$, $\int_0^r \psi(r-\theta,\theta)U(\theta)d\theta$ is uniformly continuous in the norm of $\mathbb{B}(X)$ and

$$\|U(\theta)\| \le K(1+\theta^\beta).$$

(iv) For $0 < \theta < t_1 \le t_2$ and $\alpha \in (0,1]$, there exists $\gamma \in (0,1]$ such that

$$\|[\psi(t_1-\theta,\theta) - \psi(t_2-\theta,\theta)]\| \le K\left[(t_1-\theta)^{\alpha-1}\{1+(t_2-t_1)^\gamma\} - (t_2-\theta)^{\alpha-1}\right].$$

Proof. It is easy to follow the first three results from [15]. To prove the inequality in (iv), one may follow the arguments in [26] (see pp. 437) and the following equality from [23]:

$$\left\|e^{-\theta_1 P(s)}v - e^{-\theta_2 P(s)}\right\| = \left\|\int_{\theta_2}^{\theta_1} \frac{d}{dt}\left(e^{-tP(s)}\right)dt\right\| \le K|\ \theta_1 - \theta_2\ |^\gamma, \text{ where } \mu \in (0,1].$$

Also, we employ the following lemmas to derive our main results.

Lemma 2. *(Bochner's Theorem) A measurable function $S : I \to X$ is Bochner integrable if $|\ S\ |$ is Lebesgue integrable.*

Lemma 3. *(see [27]) Let $B(\alpha,\beta)$ be a Beta function. Then, for every function $w \in L^1[0,b]$, we have*

$$\int_0^m \int_0^\mu (m-\mu)^{\alpha-1}(\mu-s)^{\beta-1}w(s)dsd\mu = B(\alpha,\beta)\int_0^m (m-s)^{\alpha+\beta-1}w(s)ds.$$

3 Existence of solutions

This section presents existence-uniqueness and regularity of a mild solution to the considered FDE (1). In order to do this, we assume the following hypotheses:

(H1) There is a continuous function $\phi : [-\eta, 0] \to X$ satisfying Lipschitz condition with $\phi(0) \in D(P)$ and is such that $g(\phi) = \kappa$.

(H2) Let $\mathscr{U} \subseteq \widetilde{X}$ be open. Further, for every point (r,x,y) in $\mathscr{U}$, we have a neighborhood $F \subseteq \mathscr{U}$ of (r,x,y) such that the function h becomes continuous w.r.t. to the first variable. Also, for each $r \in I$, $h(r,.,.) : X \times C_0 \to X$ satisfies the following with a positive constant L_h.

$$\|h(r,u,v) - h(r,w,p)\| \le L_h[\|u-w\| + \|v-p\|_0], \quad \forall\ (r,u,v),\ (r,w,p) \in F.$$

3.1 Local mild solution

Theorem 1. *Let the hypotheses (H1) and (H2) hold true for the functions h and g in the FDE (1). Then, corresponding to unique ϕ, the problem (1) has a unique local mild solution. Further, the local mild solution can be extended either on the whole interval J or on an interval $[-\eta, b_{max})$, $b_{max} \leq b < \infty$. In the later case, we have*

$$\lim_{r \uparrow b_{max}} \|x(r)\| = \infty.$$

Proof. We select $r_1 > 0$. Using the hypothesis (H1), we introduce a map $\tau : [-\eta, r_1] \to X$ given by:

$$\tau(r) = \begin{cases} \phi(r), & \text{if } r \in [-\eta, 0], \\ \phi(0), & \text{if } r \in [0, r_1]. \end{cases}$$

It is easy to verify the continuity of the segment $\tau_r : [-\eta, 0] \to X$ and the function $h(r, \tau(0), \tau_0)$ on the interval $[0, r_1]$. Let us denote

$$N_1 = \sup_{0 \leq r \leq r_1} \|h(r, \tau(0), \tau_0)\|. \tag{2}$$

We consider $\varepsilon > 0$ and $r_1 > 0$ such that the map h satisfies the hypothesis (H2) on the neighborhood F of $(0, \tau(0), \tau_0) \in \widetilde{X}$ defined as

$$F = \left\{ (r, u, v) \in \widetilde{X} : 0 \leq r \leq r_1, \|u - \tau(0)\| \leq \varepsilon, \ \textit{and} \ \|v - \tau_0\|_0 \leq \varepsilon \right\}. \tag{3}$$

Next, we select a positive constant b_0 such that

$$b_0 < \min\left\{ r_1, \ (\upsilon \varepsilon M_3^{-1})^{\frac{1}{\alpha}}, \ (\upsilon \varepsilon M_3^{-1})^{\frac{1}{\alpha+\beta}} \right\}, \tag{4}$$

where $\upsilon \in (0, 1)$ and

$$\begin{cases} M_1 = \frac{K^2}{\alpha} \|P(0)\phi(0)\|, \\ M_2 = \frac{K}{\alpha} [2\varepsilon L_h + N_1], \\ M_3 = 2M_1 + (1 + KB(\alpha, \beta))M_2. \end{cases} \tag{5}$$

Further, we specify a map $\mathscr{F}$ defined on the space C_{b_0} by $\mathscr{F}x = \tilde{x}$, where $\tilde{x}$ equals ϕ on the interval $[-\eta, 0]$ and

$$\begin{aligned} \tilde{x}(r) =& \phi(0) + \int_0^r \psi(r-\theta, \theta)U(\theta)P(0)\phi(0)d\theta + \int_0^r \psi(r-\theta, \theta)h(\theta, x(\theta), x_\theta)d\theta \\ &+ \int_0^r \int_0^\theta \psi(r-\theta, \theta)\varphi(\theta, \mu)h(\mu, x(\mu), x_\mu)d\mu d\theta, \ \text{ for } r \in [0, b_0]. \end{aligned} \tag{6}$$

Following the hypothesis (H2) along with the equations (2) and (3), we get

$$\|h(r, x(r), x_r)\| \leq \|h(r, x(r), x_r) - h(r, \tau(0), \tau_0)\| + N_1 \leq 2\varepsilon L_h + N_1. \tag{7}$$

The above inequality along with the hypothesis (H2) and Lemma 1 imply

$$\begin{cases} \|\psi(r-\theta, \theta)U(\theta)P(0)\phi(0)\| \leq K^2(r-\theta)^{\alpha-1}(1+\theta^\beta)\|P(0)\phi(0)\|, \\ \|\psi(r-\theta, \theta)h(\theta, x(\theta), x_\theta)\| \leq K(r-\theta)^{\alpha-1}[2\varepsilon L_h + N_1], \\ \left\|\psi(r-\theta, \theta)\varphi(\theta, \mu)h(\mu, x(\mu), x_\mu)\right\| \leq K^2(r-\theta)^{\alpha-1}(\theta-\mu)^{\beta-1}[2\varepsilon L_h + N_1]. \end{cases} \tag{8}$$

The derived inequalities in (8), the Lemmas 3 and 1, and the equation (5) give the following:

$$\begin{cases} \int_0^r \|\psi(r-\theta, \theta)U(\theta)P(0)\phi(0)\| d\theta \leq M_1 {b_0}^\alpha (1 + {b_0}^\beta), \\ \int_0^r \|\psi(r-\theta, \theta)h(\theta, x(\theta), x_\theta)\| d\theta \leq M_2 {b_0}^\alpha, \\ \int_0^r \int_0^\theta \left\|\psi(r-\theta, \theta)\varphi(\theta, \mu)h(\mu, x(\mu), x_\mu)\right\| d\mu d\theta \leq KB(\alpha, \beta)M_2 {b_0}^{\alpha+\beta}. \end{cases} \tag{9}$$

Hence, the above result along with the Lemma 2 imply the existence of all integral terms in the equation (6) in Bochner's sense.

Next, we prove that $\mathscr{F}x \in C_{b_0}$ for every $x \in C_{b_0}$. To do this, we consider $x \in C_{b_0}$ and $\theta_1, \theta_2 \in [0, b_0]$ with $\theta_1 \leq \theta_2$. Then, the inequality (7) and the Lemma 1 imply

$$\|\mathscr{F}x(\theta_2) - \mathscr{F}x(\theta_1)\| \leq \left[M_1(1+\theta_2^{\beta}) + M_2\left(1+\frac{Kb_0^{\beta}}{\beta}\right)\right][3(\theta_2-\theta_1)^{\alpha} + b_0^{\alpha}(\theta_2-\theta_1)^{\gamma}].$$

Now, it is easy to follow from the above inequality that $\mathscr{F}x \in C_{b_0}$.

We introduce a set $\Upsilon = \left\{x \in C_{b_0} : x(r) = \phi(r) \text{ for } r \in [-\eta, 0] \text{ and } \|x(r) - \phi(0)\| \leq \varepsilon \text{ for } r \in [0, b_0]\right\}$. One may deduce that the defined set Υ is a nonempty, closed and bounded subset of C_{b_0}. Further, it is easy to observe from the definition of the map $\mathscr{F}$ and the inequalities in the equations (4), (5) and (8) that $\mathscr{F}x \in \Upsilon$ for every $x \in \Upsilon$. Our aim is to show that the nonlinear map $\mathscr{F} : \Upsilon \to \Upsilon$ is contraction. For proving this, we let $x,\ y \in \Upsilon$. Then, for every point $r \in [0, b_0]$,

$$\|\mathscr{F}x(r) - \mathscr{F}y(r)\| \leq \int_0^r \|\psi(r-\theta,\theta)\mathscr{H}(\theta)\|\,d\theta + \int_0^r\int_0^{\theta} \|\psi(r-\theta,\theta)\varphi(\theta,\mu)\mathscr{H}(\mu)\|\,d\mu\,d\theta$$

with $\mathscr{H}(\mu) = h(\mu, x(\mu), x_\mu) - h(\mu, y(\mu), y_\mu)$. Then, from the Lemma 1 and the equations (2) - (5) along with the hypothesis (H2), we get

$$\|\mathscr{F}x(r) - \mathscr{F}y(r)\| \leq \upsilon\,\|x-y\|_{b_0}.$$

Therefore, the derived inequality along with the fact that $\|\mathscr{F}x(r) - \mathscr{F}y(r)\| = 0$, for $r \in [-\eta, 0]$ show that $\mathscr{F} : \Upsilon \to \Upsilon$ defines a contraction map. And hence, the nonlinear map $\mathscr{F}$ has a fixed-point, say x, in Υ due to Banach Contraction Principle and that x will be such that it equals ϕ on the interval $[-\eta, 0]$ and

$$\begin{aligned} x(r) =& \phi(0) + \int_0^r \psi(r-\theta,\theta)U(\theta)P(0)\phi(0)d\theta + \int_0^r \psi(r-\theta,\theta)h(\theta,x(\theta),x_\theta)d\theta \\ &+ \int_0^r\int_0^{\theta} \psi(r-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)d\mu d\theta, \quad \text{for } r \in [0,b_0]. \end{aligned} \tag{10}$$

The obtained x will be a mild solution of FDE (1) on $J_0 = [-\eta, b_0]$.

The derived mild solution x can be extended to a larger interval $[-\eta, b_0+\delta]$ with $\delta > 0$ as follows. We let $x(r+b_0) = w(r)$, where the $X-$valued function $w(r)$ denotes a mild solution of the following nonlocal FDE

$$\begin{cases} {}^{C}D_r^{\alpha}[w(r)] + P(r)w(r) = h(r+b_0, w(r), w_r), \\ \tilde{g}\left(w_{[-\eta-b_0,0]}\right) = \tilde{\kappa}. \end{cases} \tag{11}$$

The segment w_r is $X-$valued given by $w_r(s) = w(r+s)$ for $s \in [-\eta-b_0, 0]$. Also, $\tilde{g} : \tilde{C}_0 \to \tilde{C}_0$ is a nonlinear map with $\tilde{C}_0 = C([-\eta-b_0, 0], X)$. We introduce the following map

$$\tilde{\phi}(r) = \begin{cases} \phi(r+b_0), & t \in [-\eta-b_0, -b_0], \\ x(r+b_0), & r \in [-b_0, 0]. \end{cases}$$

It is easy to verify that the function $\tilde{\phi} : [-\eta-b_0, 0] \to X$ is Lipschitz continuous and satisfies the condition $\tilde{g}\left(\tilde{\phi}\right) = \tilde{\kappa}$. Also, the hypothesis (H2) holds true for the nonlinear map h due to the definition of w. Hence, one may follow the similar arguments to show the existence of a solution $w \in C([-\eta-b_0, b_0+\delta], X)$ to (11). Finally, the following function $\tilde{x}$ will become a mild solution to FDE (1) on the larger interval $[-\eta, b_0+\delta]$:

$$\tilde{x}(r) = \begin{cases} x(r), & r \in [-\eta, b_0], \\ w(r-b_0), & r \in [b_0, b_0+\delta]. \end{cases}$$

Proceed in the similar way to extend the solution x on larger intervals. One may obtain either the solution x on the whole interval J or on the interval $[-\eta, b_{\max})$, where $b_{\max} \leq b < \infty$. In the later case, we will show that

$$\lim_{r \uparrow b_{\max}} \|x(r)\| = \infty.$$

Assume that the above result does not hold true. Then, one can find a sequence $r_n \uparrow b_{\max}$ such that $\|x(r_n)\| \leq M$ for each $n \in \mathbb{N}$. Hence, the mild solution $x : [-\eta, r_n] \to X$ can be extended to $[-\eta, r_n+\delta]$ by following the above arguments, where $\delta > 0$ does not depend on r_n. This contradicts the definition of $b_{\max}$. Therefore, our claim.

Moreover, it is easy to observe from the definition of the mild solution to FDE (1) that it will be unique corresponding to unique ϕ.

3.2 Local classical solution

Theorem 2. *In addition to the assumptions in Theorem 1, we suppose that the function h in FDE (1) is Hölder continuous w.r.t. to r on the interval I. Then, corresponding to unique ϕ, the FDE (1) has a unique local classical solution.*

Proof. Let $\phi \in C_0$. It is easy to observe from the Theorem 1 that the problem (1) has a unique local mild solution, say x, on an interval $J_0 = [-\eta, b_0]$ for some $0 < b_0 < b$. We will show that this mild solution x is indeed a classical solution whenever the function h is Hölder continuous w.r.t. to r on the interval I.

Note that $x \in C_{b_0}$. Following hypothesis (H2), one can deduce that the function h is bounded on the interval $[0, b_0]$. Therefore, we set

$$N_2 = \sup_{0 \le r \le b_0} \|h(r, x(r), x_r)\|. \tag{12}$$

Further, for $0 < \theta_1 \le \theta_2 \le b_0$, the Lemma 1 along with the equality (12) imply that

$$\|x(\theta_2) - x(\theta_1)\| \le \frac{K}{\alpha}\left[K(1 + b_0^\beta)\|P(0)\phi(0)\| + N_2 + \frac{KN_2}{\beta} b_0^\beta\right][3(\theta_2 - \theta_1)^\alpha + b_0^\alpha(\theta_2 - \theta_1)^\gamma].$$

Suppose that $(\theta_2 - \theta_1)^q = \max\{(\theta_2 - \theta_1)^\alpha, (\theta_2 - \theta_1)^\gamma\}$. The above inequality guarantees that there is a positive constant N_3, independent of θ_1 and θ_2, such that

$$\|x(\theta_1) - x(\theta_2)\| \le N_3(\theta_2 - \theta_1)^q.$$

This establishes that the function x is Hölder continuous on the interval $[0, b_0]$. Moreover, the function h is also Hölder continuous on the interval $[0, b_0]$. Hence, one can find the constants $k > 0$ and $q_1 \in (0, 1]$ such that for every $0 < \theta_3 < \theta_1, \theta_2 \le b_0$, we have

$$\left\|\widetilde{h}(\theta_1) - \widetilde{h}(\theta_2)\right\| \le k|\ \theta_1 - \theta_2\ |^{q_1}$$

with $\widetilde{h}(\mu) = h(\mu, x(\mu), x_\mu)$. Following [15,26], there exists a unique classical solution $w \in C_{b_0}$ of the following FDE

$$\begin{cases} {}^C D_r^\alpha[w(r)] + P(r)w(r) = \widetilde{h}(r), & r > 0 \\ w(r) = \phi(r),\ r \in [-\eta, 0], \end{cases} \tag{13}$$

where the function w is such that

$$w(r) = \phi(0) + \int_0^r \psi(r - \theta, \theta)U(\theta)P(0)\phi(0)d\theta + \int_0^r \psi(r - \theta, \theta)\widetilde{h}(\theta)d\theta$$
$$+ \int_0^r \int_0^\theta \psi(r - \theta, \theta)\varphi(\theta, \mu)\widetilde{h}(\mu)d\mu d\theta, \text{ for } r \in [0, b_0]$$

and w equals ϕ on the interval $[-\eta, 0]$. It is easy to verify that x equals w on the interval $[-\eta, b_0]$ which implies the regularity of the mild solution x of (1) on the interval J_0. Moreover, the similar procedure can be followed, as in Theorem 1, to show the uniqueness of x corresponding to unique ϕ.

3.3 Global solution

Theorem 3. *Let the $X-$valued nonlinear map h is defined on $[0, \infty) \times X \times C_0$. Further, we suppose that the hypotheses (H1) - (H2) and the conditions $(P1)$ - $(P2)$ hold true for each $b < \infty$. Moreover, let either one of the following assumptions is true.*

(i) There is a nonnegative real-valued continuous function w_0 defined on the interval $[-\eta, \infty)$ such that $\|x(r)\| \le w_0(r)$ for every point r in the interval of existence of x.

(ii) There exists two locally integrable and nonnegative real-valued functions w_1 and w_2 defined on $[0, \infty)$ such that

$$\|h(r, u, v)\| \le w_1(r)[\|u\| + \|v\|_0] + w_2(r),\ \ \forall\ r \in [0, \infty), u \in X, v \in C_0,$$

where w_1 is an increasing function.

Then, corresponding to unique $\phi \in C_0$, the problem (1) has a unique global solution $x \in C([-\eta,\infty),X)$.

*Proof.*Theorem 1 guarantees the existence of a unique local mild solution to the problem (1), say x, on the interval $[-\eta,b]$ for $0<b<\infty$.

In the case when the condition (i) is true, then the local mild solution x can be extended to the whole interval by following the similar process as mentioned in the Theorem 1.

We assume that the condition (ii) holds true. Then, we introduce the following real-valued function Φ defined on $[0,\infty)$.

$$\Phi(r)=\|\phi\|_0+\frac{K^2}{\alpha}r^{\alpha}(1+r^{\beta})\|P(0)\phi(0)\|+K\int_0^r\left[(r-\theta)^{\alpha-1}+KB(\alpha,\beta)(r-\theta)^{\alpha+\beta-1}\right]w_2(\theta)d\theta.$$

It is easy to prove the continuity of the function Φ. Also, we have

$$\sup_{-\eta\le r\le 0}\|x(r)\|=\sup_{-\eta\le r\le 0}\|\phi(r)\|=\|\phi\|_0\le\Phi(r).$$

Further, from the Lemmas 3 and 1, for each $r\ge 0$, we get

$$\|x(r)\|\le\Phi(r)+2K\int_0^r\left[(r-\theta)^{\alpha-1}+KB(\alpha,\beta)(r-\theta)^{\alpha+\beta-1}\right]w_1(\theta)v(\theta)d\theta$$

with $v(r)=\left(\sup_{-\eta\le s\le r}\|x(s)\|\right)$. Note that both the maps Φ and v are increasing. Hence, for each $\theta\in[0,r]$, we have

$$\|x(\theta)\|\le\Phi(r)+2K\int_0^r\left[(r-\theta)^{\alpha-1}+KB(\alpha,\beta)(r-\theta)^{\alpha+\beta-1}\right]w_1(\theta)v(\theta)d\theta.$$

The derived inequality along with the assumption that the function w_1 is increasing imply the following estimate.

$$\sup_{0\le\theta\le r}\|x(\theta)\|\le\Phi(r)+2Kw_1(r)\int_0^r\left[(r-\theta)^{\alpha-1}+KB(\alpha,\beta)(r-\theta)^{\alpha+\beta-1}\right]v(\theta)d\theta.$$

Therefore, from above discussion, we get

$$v(r)\le 2\Phi(r)+2Kw_1(r)\int_0^r\left[(r-\theta)^{\alpha-1}+KB(\alpha,\beta)(r-\theta)^{\alpha+\beta-1}\right]v(\theta)d\theta.$$

Finally, as an application to the generalized Gronwall inequality with singularity [28], we get

$$v(r)\le 2\Phi(r)\left[E_{\alpha}\left(2Kw_1(r)\Gamma(\alpha)r^{\alpha}\right)+E_{\alpha+\beta}\left(2K^2B(\alpha,\beta)w_1(r)\Gamma(\alpha+\beta)r^{\alpha+\beta}\right)\right].$$

The above inequality implies that the condition (i) holds true. Therefore, the problem (1) has a unique global solution.

4 Fractional control problem

In this section, we aimed to study the following fractional dynamical system associated with (1) :

$$\begin{cases}{}^{C}D_r^{\alpha}[x(r)]+P(r)x(r)=h(r,x(r),x_r)+Du(r), & r\in I=[0,b],\\ g(x_{[-\eta,0]})=\kappa,\end{cases}\tag{14}$$

where the admissible control function $u\in L^2[I,Y]$ with Y as a Banach space and $D:L^2[I,Y]\to L^1[I,X]$ is a bounded linear operator. That is, there exists $\widehat{M}>0$ such that $\|D\|\le\widehat{M}$.

A proper definition of mild solution of (14) can be given by following the Section 2. Let $\phi:[-\eta,0]\to X$ be a continuous function such that $g(\phi)=\kappa$. Then, for each $u\in L^2[I,Y]$, a mild solution of the fractional dynamical system (14) is a continuous function $x^u:[-\eta,b]\to X$ such that $x^u=\phi$ on the interval $[-\eta,0]$ and

$$\begin{aligned}x(r)=&\phi(0)+\int_0^r\psi(r-\theta,\theta)U(\theta)P(0)\phi(0)d\theta+\int_0^r\psi(r-\theta,\theta)(h(\theta,x(\theta),x_\theta)+Du(\theta))d\theta\\&+\int_0^r\int_0^\theta\psi(r-\theta,\theta)\varphi(\theta,\mu)(h(\mu,x(\mu),x_\mu)+Du(\mu))d\mu d\theta\qquad\text{for } r\in I.\end{aligned}$$

The fractional dynamical system (14) is said to be completely controllable on I if, for every point $x_1 \in X$ and the function $\phi \in C_0$ such that $g(\phi) = \kappa$, there exists a control function $u \in L^2[I,Y]$ such that the mild solution, say x^u, of (14) reaches x_1 at $r = b$, i.e., $x^u(b) = x_1$.

We assume the following hypothesis on nonlinear function h to establish the controllability result:

(HH)The nonlinear map h is continuous w.r.t. to the first variable. Also, there exist a constant $q_1 \in (0,\alpha) \cap (0,\beta)$ and a function $L_h(.) \in L^{\frac{1}{q_1}}[I,\mathbb{R}^+]$ such that

$$\|h(r,v_1,w_1) - h(r,v_2,w_2)\| \le L_h(r)[\|v_1 - v_2\| + \|w_1 - w_2\|_0],$$

for each $(r,v_1,w_1),(r,v_2,w_2) \in I \times X \times C_0$.

On applying Hölder's inequality, we get

$$\int_0^b \left[(b-\theta)^{\alpha-1} + K\mathscr{B}(\alpha,\beta)(b-\theta)^{\alpha+\beta-1}\right] L_h(\theta)d\theta \le M_4, \tag{15}$$

where $M_4 = L_1\left(\left[\frac{b^{m+1}}{m+1}\right]^{1-q_1} + \left[\frac{b^{p+1}}{p+1}\right]^{1-q_1}\right)$, $L_1 = \|L_h\|_{L^{\frac{1}{q_1}}(I,\mathbb{R}_+)}$, $m = \frac{\alpha-1}{1-q_1}, p = \frac{\alpha+\beta-1}{1-q_1}$ and $m,p \in (-1,0)$. For brevity, let $\widehat{N} = Kb^\alpha\left[\frac{1}{\alpha} + \frac{K\mathscr{B}(\alpha,\beta)b^\beta}{\alpha+\beta}\right]$, $\widehat{K} = 2KM_4$, and $M_5 = \|x\|_b + \|\phi\|_0$.

Theorem 4.*Let the hypotheses (H1) and (HH), and the following assumptions hold true.*

(Q1)The linear operator $Q : L^2[I,Y] \to X$ defined as

$$Qu = \int_0^b \psi(b-\theta,\theta)Du(\theta)d\theta + \int_0^b\int_0^\theta \psi(b-\theta,\theta)\varphi(\theta,\mu)Du(\mu)d\mu d\theta, \text{ for } u \in L^2[I,Y],$$

gives rise to a bijective operator $\widetilde{Q} : L^2[I,Y]/KerQ \to X$ having bounded inverse. That is, there exists a positive constant M_3 such that $\left\|\widetilde{Q}^{-1}\right\| \le M_3$.

(Q2)The following set is assumed to be relatively compact in the Banach space X for each bounded subsets W of X, for arbitrary $l \in (0,r)$ and positive constant ϑ.

$$\Lambda_{\varepsilon,\vartheta}(r) := \left\{\int_0^{r-l}\int_\vartheta^\infty \mathscr{F}(v,r,\theta)h(\theta,x(\theta),x_\theta)dvd\theta + \int_0^{r-l}\int_0^\theta\int_\vartheta^\infty \mathscr{F}(v,r,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)dvd\mu d\theta,\ x \in W\right\},$$

where, $\mathscr{F}(v,r,\theta) = \alpha v(r-\theta)^{\alpha-1}p_\alpha(v)exp(-(r-\theta)^\alpha vP(\theta))$.

Then the fractional dynamical system (14) is completely controllable on I if $\widehat{K}\left[1 + \widehat{N}\widehat{M}M_3\right] < 1$.

*Proof.*Following hypothesis (H1), there is a continuous function $\phi : [-\eta,0] \to X$ satisfying $g(\phi) = \kappa$. Now, we introduce a function τ on the interval $[-\eta,b]$ as follows:

$$\tau(r) = \begin{cases} \phi(r), & \text{if } r \in [-\eta,0], \\ \phi(0), & \text{if } r \in I. \end{cases}$$

One can verify that the segment $\tau_r \in C_0$ and the map $h(r,\tau(0),\tau_0)$ is continuous for every $r \in I$. Let us denote

$$K_1 = \sup_{r\in I}\|h(r,\tau(0),\tau_0)\|. \tag{16}$$

We define the admissible control function $u_x : I \to Y$ for $x \in C_b$ as follows:

$$\begin{aligned} u_x(r) =& \widetilde{Q}^{-1}\left[x_1 - \phi(0) - \int_0^b \psi(b-\theta,\theta)U(\theta)P(0)\phi(0)d\theta - \int_0^b \psi(b-\theta,\theta)h(\theta,x(\theta),x_\theta)d\theta\right. \\ &\left. - \int_0^b\int_0^\theta \psi(b-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)d\mu d\theta\right], \quad \text{for } r \in I. \end{aligned} \tag{17}$$

We define the operator $\Im$ as $\Im x = \widetilde{x}$ for $x \in C_b$, where $\widetilde{x}$ equals ϕ on the interval $[-\eta, 0]$ and

$$\begin{aligned}\widetilde{x}(r) =& \phi(0) + \int_0^r \psi(r-\theta,\theta)U(\theta)P(0)\phi(0)d\theta + \int_0^r \psi(r-\theta,\theta)(h(\theta,x(\theta),x_\theta) + Du_x(\theta))d\theta \\ &+ \int_0^r \int_0^\theta \psi(r-\theta,\theta)\varphi(\theta,\mu)(h(\mu,x(\mu),x_\mu) + Du_x(\mu))d\mu d\theta, \text{ for } r \in I.\end{aligned} \tag{18}$$

Following the hypothesis (HH) and the equality in (16), for every point $\theta \in I$, we get

$$\|h(\theta,x(\theta),x_\theta)\| \leq \|h(\theta,x(\theta),x_\theta) - h(\theta,\tau(0),\tau_0)\| + K_1 \leq 2L_h(\theta)M_5 + K_1. \tag{19}$$

The above inequality and Lemma 1 along with the hypothesis (HH), for each $r, \theta \in I$, imply that

$$\begin{aligned}\|\psi(r-\theta,\theta)U(\theta)P(0)\phi(0)\| &\leq K^2(r-\theta)^{\alpha-1}(1+\theta^\beta)\|P(0)\phi(0)\|, \\ \|\psi(r-\theta,\theta)h(\theta,x(\theta),x_\theta)\| &\leq K[2L_h(\theta)M_5 + K_1](r-\theta)^{\alpha-1}, \\ \left\|\psi(r-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)\right\| &\leq K^2[2L_h(\mu)M_5 + K_1](r-\theta)^{\alpha-1}(\theta-\mu)^{\beta-1}.\end{aligned}$$

Using above derived inequalities, the equation (15) and Lemma 3, we get

$$\begin{aligned}\int_0^r \|\psi(r-\theta,\theta)U(\theta)P(0)\phi(0)\|d\theta &\leq \frac{K^2}{\alpha}b^\alpha(1+b^\beta)\|P(0)\phi(0)\|, \\ \int_0^r \|\psi(r-\theta,\theta)h(\theta,x(\theta),x_\theta)\|d\theta &\leq K\left[\frac{b^\alpha K_1}{\alpha} + 2M_5\left(\frac{L_1 b^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}}\right)\right], \\ \int_0^r \int_0^\theta \left\|\psi(r-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)\right\|d\mu d\theta &\leq K^2\mathscr{B}(\alpha,\beta)\left[\frac{b^{\alpha+\beta}K_1}{\alpha+\beta} + 2M_5\left(\frac{L_1 b^{(p+1)(1-q_1)}}{(p+1)^{(1-q_1)}}\right)\right].\end{aligned}$$

Now, for every point $r \in I$ and the function $x \in C_b$, we get the following estimate

$$\|u_x(r)\| \leq M_3[\|x_1\| + a_0 + \widehat{K}\|x\|_b] \tag{20}$$

with $a_0 = (1+\widehat{K})\|\phi\|_0 + \frac{K^2}{\alpha}b^\alpha(1+b^\beta)\|P(0)\phi(0)\| + \widehat{N}K_1$. Using the inequality (20), the assumption (Q1) and the Lemma 1, it is easy to deduce that

$$\begin{aligned}\int_0^r \|\psi(r-\theta,\theta)Du_x(\theta)\|d\theta &\leq \frac{Kb^\alpha}{\alpha}M_2M_3\left[\|x_1\| + a_0 + \widehat{K}\|x\|_b\right], \\ \int_0^r \int_0^\theta \|\psi(r-\theta,\theta)\varphi(\theta,\mu)Du_x(\mu)\|d\mu d\theta &\leq \frac{K^2 b^{\alpha+\beta}}{\alpha+\beta}M_2M_3\mathscr{B}(\alpha,\beta)\left[\|x_1\| + a_0 + \widehat{K}\|x\|_b\right].\end{aligned}$$

The above result along with the Lemma 2 imply that all of the integral terms in the equations (17) and (18) are Bochner integrable.

We will show that the operator $\Im$ assumes the values in C_b, i.e., $\Im(C_b) \subseteq C_b$. To do this, we let $x \in C_b$. Then, the continuity of $\Im x$ on the interval $[-\eta, 0]$ follows from hypothesis (H1). Further, the continuity of the nonlinear map h, the nonlinear operators ψ, φ and the linear operator D follows from the hypothesis (HH), the assumption (Q1) and the Lemma 1. This implies the continuity of $\Im x$ on the interval $[0, b]$. Hence, $\Im x \in C_b$. Moreover, for each point $r \in I$, we have

$$\begin{aligned}\|\Im x(r)\| &\leq \|\phi\|_0 + \frac{K^2}{\alpha}b^\alpha(1+b^\beta)\|P(0)\phi(0)\| + \widehat{N}K_1 + \widehat{K}M_5 + \widehat{N}\widehat{M}M_3(\|x_1\| + a_0 + \widehat{K}\|x\|_b) \\ &= a_0M_6 + (M_6-1)\|x_1\| + \widehat{K}M_6\|x\|_b\end{aligned}$$

with $M_6 = 1 + \widehat{N}\widehat{M}M_3$. Now, we select $r_1 \geq \frac{a_0M_6 + (M_6-1)\|x_1\|}{1-\widehat{K}M_6}$ and define a set $\mathbb{B}_{r_1} := \{x \in C_b : x(0) = \phi(0) \text{ and } \|x\|_b \leq r_1\}$. It is easy to see that the set $\mathbb{B}_{r_1}$ is closed, bounded and convex subset of C_b for each $r_1 \geq 0$ and $\Im(\mathbb{B}_{r_1}) \subseteq \mathbb{B}_{r_1}$.

Next, we define two operators $\Im_1, \Im_2$ on C_b such that $\Im_1 x(r) = \phi(r)$ and $\Im_2 x(r) = 0$ for each $r \in [-\eta, 0]$, and for every point $r \in I$,

$$(\Im_1 x)(r) = \phi(0) + \int_0^r \psi(r-\theta,\theta)U(\theta)P(0)\phi(0)d\theta + \int_0^r \psi(r-\theta,\theta)Du_x(\theta)d\theta + \int_0^r\int_0^\theta \psi(r-\theta,\theta)\varphi(\theta,\mu)Du_x(\mu)d\mu d\theta, \tag{21}$$

$$(\Im_2 x)(r) = \int_0^r \psi(r-\theta,\theta)h(\theta,x(\theta),x_\theta)d\theta + \int_0^r\int_0^\theta \psi(r-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)d\mu d\theta. \tag{22}$$

The definitions of operators $\Im_1, \Im_2$ imply that $(\Im x)(r) = (\Im_1 x)(r) + (\Im_2 x)(r)$ for $r \in [-\eta, b]$. Also, for the functions x, y in $\mathbb{B}_{r_1}$ and the point $r \in I$, we get

$$\|(\Im_1 x + \Im_2 y)(r)\| \le a_0 M_6 + (M_6 - 1)\|x_1\| + \widehat{K} M_6 r_1.$$

Note that $r_1 \ge \frac{a_0 M_6 + (M_6-1)\|x_1\|}{1-\widehat{K}M_6}$ which implies $a_0 M_6 + (M_6-1)\|x_1\| \le r_1(1-\widehat{C}M_6)$. Also, $(\Im_1 x + \Im_2 y)(0) = \phi(0)$. Therefore, it is easy to follow from above inequality that $\Im_1 x + \Im_2 y \in \mathbb{B}_{r_1}$ for every functions $x, y \in \mathbb{B}_{r_1}$.

Next, we show that the operator $\Im_1$ is contraction. For proving this, let $x,\ y \in \mathbb{B}_{r_1}$ and $r \in I$. The hypothesis (HH) along with the Lemmas 3 and 1 give

$$\left\|u_x(r) - u_y(r)\right\| \le 2KM_4M_3\|x-y\|_b = \widehat{K}M_3\|x-y\|_b, \quad \text{for } r \in I. \tag{23}$$

Therefore, from the definition of $\Im_1$, the inequality (23), and the Lemmas 3 and 1, we have

$$\|\Im_1 x(r) - \Im_1 y(r)\| \le \widehat{N}\widehat{M}M_3\widehat{K}\|x-y\|_b, \text{ for every } r \in I.$$

Also, $\|(\Im_1 x)(r) - (\Im_1 y)(r)\| = 0$ for $r \in [-\eta, 0]$. Further, the condition $\widehat{K}\left[1 + \widehat{N}\widehat{M}M_3\right] < 1$ implies that $\widehat{N}\widehat{M}M_3\widehat{K} < 1$, therefore, $\Im_1$ defines a contraction map on $\mathbb{B}_{r_1}$.

Now, we establish that the operator $\Im_2$ is completely continuous on the set $\mathbb{B}_{r_1}$. In particular, we show that:

(i)The operator $\Im_2$ is continuous on $\mathbb{B}_{r_1}$,
(ii)$\Im_2(\mathbb{B}_{r_1}) \subseteq C_b$ is equicontinuous and
(iii)$\Im_2(\mathbb{B}_{r_1})(r)$ is relatively compact for each $r \in I$.

To prove the fact in (i), we select a sequence $\{x^{(n)}\}$ from the set $\mathbb{B}_{r_1}$ such that $x^{(n)} \to x$ with $x \in \mathbb{B}_{r_1}$. From the hypothesis (HH) along with the Lemma 1, we have:

$$\begin{aligned}
\psi(.-\mu,\mu)h(\mu,x^{(n)}(\mu),x_\mu^{(n)}) &\to \psi(.-\mu,\mu)h(\mu,x(\mu),x_\mu), \quad \text{a.e. } \mu \in I,\\
\psi(.-\theta,\theta)\varphi(\theta,\mu)h(\mu,x^{(n)}(\mu),x_\mu^{(n)}) &\to \psi(.-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu), \quad \text{a.e. } \mu \in I,\\
\left\|\psi(.-\theta,\theta)\left(h(\theta,x^{(n)}(\theta),x_\theta^{(n)}) - h(\theta,x(\theta),x_\theta)\right)\right\| &\le 4rK(.-\theta)^{\alpha-1}L_h(\theta) \in L^1(I,\mathbb{R}^+),\\
\left\|\psi(.-\theta,\theta)\varphi(\theta,\mu)\left(h(\mu,x^{(n)}(\mu),x_\mu^{(n)}) - h(\mu,x(\mu),x_\mu)\right)\right\| &\le 4rK^2L_h(\theta)(.-\theta)^{\alpha-1}(\theta-\mu)^{\beta-1} \in L^1(I,\mathbb{R}^+).
\end{aligned}$$

Following the Lebesgue dominated convergence theorem, one may show that

$$\int_0^r \left\|\psi(r-\theta,\theta)\left(h(\theta,x^{(n)}(\theta),x_\theta^{(n)}) - h(\theta,x(\theta),x_\theta)\right)\right\| d\theta \to 0 \text{ as } n \to \infty,$$

$$\int_0^r\int_0^\theta \left\|\psi(r-\theta,\theta)\varphi(\theta,\mu)\left(h(\mu,x^{(n)}(\mu),x_\mu^{(n)}) - h(\mu,x(\mu),x_\mu)\right)\right\| d\mu d\theta \to 0 \text{ as } n \to \infty.$$

Hence, the above result and the hypothesis (HH) imply the following

$$\left\|\Im_2 x^{(n)} - \Im_2 x\right\|_b \le \sup_{r\in I}\left[\int_0^r \left\|\psi(r-\theta,\theta)\left(h(\theta,x^{(n)}(\theta),x_\theta^{(n)}) - h(\theta,x(\theta),x_\theta)\right)\right\| d\theta + \int_0^r\int_0^\theta \left\|\psi(r-\theta,\theta)\varphi(\theta,\mu)\left(h(\mu,x^{(n)}(\mu),x_\mu^{(n)}) - h(\mu,x(\mu),x_\mu)\right)\right\| d\mu d\theta\right] \to 0 \text{ as } n \to \infty.$$

This implies the continuity of $\Im_2$ on $\mathbb{B}_{r_1}$. In order to prove the fact in (ii), we let $0 < r < r+\varepsilon \leq b$ and $x \in \mathbb{B}_{r_1}$. Then, we get

$$(\Im_2 x)(r+\varepsilon) - (\Im_2 x)(r) = I_1 + I_2 + I_3 + I_4, \tag{24}$$

where

$$\begin{cases} I_1 = \int_0^r [\psi(r+\varepsilon-\theta,\theta) - \psi(r-\theta,\theta)]\, h(\theta,x(\theta),x_\theta)d\theta, \\ I_2 = \int_0^r \int_0^\theta [\psi(r+\varepsilon-\theta,\theta) - \psi(r-\theta,\theta)]\, \varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)d\mu d\theta, \\ I_3 = \int_r^{r+\varepsilon} \psi(r+\varepsilon-\theta,\theta)h(\theta,x(\theta),x_\theta)d\theta, \\ I_4 = \int_r^{r+\varepsilon} \int_0^\theta \psi(r+\varepsilon-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)d\mu d\theta. \end{cases}$$

For $i = 1,2,3,4$, we will prove that $\|I_i\| \to 0$ whenever $\varepsilon \to 0$. From the hypothesis (HH), the Lemma 1, and the inequality in (19), we get

$$\begin{aligned} &\|[\psi(r+\varepsilon-\theta,\theta) - \psi(r-\theta,\theta)]\, h(\theta,x(\theta),x_\theta)\| \leq K\left[(r-\theta)^{\alpha-1}(1+\varepsilon^\gamma) - (r+\varepsilon-\theta)^{\alpha-1}\right][2L_h(\theta)[r_1+\|\phi\|_0]+K_1], \\ &\left\|[\psi(r+\varepsilon-\theta,\theta) - \psi(r-\theta,\theta)]\, \varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)\right\| \\ &\qquad \leq K^2\left[(r-\theta)^{\alpha-1}(1+\varepsilon^\gamma) - (r+\varepsilon-\theta)^{\alpha-1}\right](\theta-\mu)^{\beta-1}[2L_h(\mu)[r_1+\|\phi\|_0]+K_1], \\ &\|\psi(r+\varepsilon-\theta,\theta)h(\theta,x(\theta),x_\theta)\| \leq K(r+\varepsilon-\theta)^{\alpha-1}[2L_h(\theta)[r_1+\|\phi\|_0]+K_1], \\ &\left\|\psi(r+\varepsilon-\theta,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)\right\| \leq K^2(r+\varepsilon-\theta)^{\alpha-1}(\theta-\mu)^{\beta-1}[2L_h(\mu)[r_1+\|\phi\|_0]+K_1]. \end{aligned}$$

Also, for each $\theta \in (0,r)$ and $r \in I$, the Hölder's inequality along with the hypothesis (HH) imply

$$\int_0^\theta (\theta-\mu)^{\beta-1}L_h(\mu)d\mu \leq \frac{L_1 b^{(m_1+1)(1-q_1)}}{(m_1+1)^{(1-q_1)}} := M_7$$

with $m_1 = \frac{\beta-1}{1-q_1} \in (-1,0)$. Therefore, from the hypothesis (HH), the Lemma 1, and the fact that $| a^\sigma - b^\sigma | \leq (b-a)^\sigma$ for $0 < \sigma \leq 1$, $0 < a \leq b$, we can deduce the following estimates:

$$\begin{aligned} \|I_1\| &\leq 2K(r_1+\|\phi\|_0)\left[\frac{L_1\{(2\varepsilon^{(m+1)})^{(1-q_1)} + \varepsilon^\gamma b^{(m+1)(1-q_1)}\}}{(m+1)^{(1-q_1)}}\right] + \frac{K}{\alpha}K_1[b^\alpha\varepsilon^\gamma + 2\varepsilon^\alpha], \\ \|I_2\| &\leq K^2(2\varepsilon^\alpha + b^\alpha\varepsilon^\gamma)\left[\frac{2M_7(r_1+\|\phi\|_0)}{\alpha} + \frac{b^\beta K_1}{\alpha\beta}\right], \\ \|I_3\| &\leq 2K(r_1+\|\phi\|_0)\left[\frac{L_1\varepsilon^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}}\right] + \frac{K\varepsilon^\alpha}{\alpha}K_1, \\ \|I_4\| &\leq \frac{K^2\varepsilon^\alpha}{\alpha}\left[2M_7(r_1+\|\phi\|_0) + \frac{b^\beta K_1}{\beta}\right]. \end{aligned}$$

It is easy to follow from the above inequalities that $\|I_i\| \to 0$ as $\varepsilon \to 0$, for $i = 1,2,3,4$. Hence, the equation (24) implies that $\|(\Im_2 x)(r+\varepsilon) - (\Im_2 x)(r)\| \to 0$ whenever $\varepsilon \to 0$, which establishes (ii).

Finally, we prove the fact in (iii). In particular, we show that the set $\Pi(r) := \{(\Im_2 x)(r) : x \in \mathbb{B}_{r_1}\}$ is relatively compact in the Banach space X for each $r \in I$. It is easy to check that $\Pi(0)$ is a compact set. For the case $r \in (0,b]$, we define the set $\Pi_{l,\vartheta}(r) = \left\{(\Im_{2,l,\vartheta}x)(r) : x \in \mathbb{B}_{r_1}\right\}$ for arbitrary $l \in (0,r)$, a positive constant ϑ and the function $x \in \mathbb{B}_{r_1}$, where $\Im_{2,l,\vartheta}x$ is given by

$$\Im_{2,l,\vartheta}x(r) = \int_0^{r-l}\int_\vartheta^\infty \mathscr{F}(v,r,\theta)h(\theta,x(\theta),x_\theta)dv d\theta + \int_0^{r-l}\int_0^\theta\int_\vartheta^\infty \mathscr{F}(v,r,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)dv d\mu d\theta.$$

From the assumption (Q2), it follows that, for arbitrary $l \in (0,r)$ and the positive constant ϑ, the set $(\Im_{2,l,\vartheta}x)(r)$ is relatively compact since $\mathbb{B}_{r_1} \subseteq C_b$ is bounded. Also, we have

$$(\Im_2 x)(r) - (\Im_{2,l,\vartheta}x)(r) = A_1 + A_2 + A_3 + A_4, \tag{25}$$

where

$$A_1 = \int_0^{r-l}\int_0^{\vartheta} \mathscr{F}(v,r,\theta)h(\theta,x(\theta),x_\theta)dvd\theta,$$
$$A_2 = \int_0^{r-l}\int_0^{\theta}\int_0^{\vartheta} \mathscr{F}(v,r,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)dvd\mu d\theta,$$
$$A_3 = \int_{r-l}^{r}\int_0^{\infty} \mathscr{F}(v,r,\theta)h(\theta,x(\theta),x_\theta)dvd\theta,$$
$$A_4 = \int_{r-l}^{r}\int_0^{\theta}\int_0^{\infty} \mathscr{F}(v,r,\theta)\varphi(\theta,\mu)h(\mu,x(\mu),x_\mu)dvd\mu d\theta.$$

The Hölder's inequality along with the Lemma 1 imply

$$\begin{aligned}
\|A_1\| &\leq \alpha K\left(\int_0^{\vartheta} vp_\alpha(v)dv\right)\left[2(r_1+\|\phi\|_0)\left(\frac{L_1 b^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}}\right)+\frac{K_1 b^\alpha}{\alpha}\right],\\
\|A_2\| &\leq \alpha K^2\left(\int_0^{\vartheta} vp_\alpha(v)dv\right)\left[\frac{2b^\alpha}{\alpha}M_7(r_1+\|\phi\|_0)+\frac{K_1 b^{\alpha+\beta}}{\alpha\beta}\right],\\
\|A_3\| &\leq \alpha K\left(\int_0^{\infty} vp_\alpha(v)dv\right)\left[2(r_1+\|\phi\|_0)\left(\frac{L_1 l^{(m+1)(1-q_1)}}{(m+1)^{(1-q_1)}}\right)+\frac{K_1 l^\alpha}{\alpha}\right],\\
\|A_4\| &\leq \alpha K^2\left(\int_0^{\infty} vp_\alpha(v)dv\right)\left[\frac{2l^\alpha}{\alpha}M_7(r_1+\|\phi\|_0)+\frac{K_1 b^\beta l^\alpha}{\alpha\beta}\right].
\end{aligned}$$

The derived inequalities imply that, for $i=1,2,3,4,$, $\|A_i\|\to 0$ whenever $l,\vartheta\to 0$. This further implies that $\left\|(\Im_2 x)(r)-(\Im_{2,l,\vartheta}x)(r)\right\|\to 0$ whenever $l,\vartheta\to 0$. Therefore, we deduce that the relatively compact sets $\Pi_{l,\vartheta}(r)$ approximates the set $\Pi(r)$ arbitrarily which will establish the fact in (iii), i.e., the set $(\Im_2\mathbb{B}_{r_1})(r)\subseteq X$ is relatively compact in the Banach space X for each $r\in I$. Hence, the defined operator $\Im_2$ is completely continuous. Then, the Krasnoselskii's fixed-point theorem implies that the operator $\Im$ has a fixed-point, say x^u, in the set $\mathbb{B}_{r_1}$. Further, the definition of the operator $\Im$ implies that x^u will be a mild solution of the fractional dynamical system (14). Also, x^u is such that $x^u(b)=x_1$. Therefore, the fractional dynamical system (14) is completely controllable on I.

5 Application

Consider the following nonlocal fractional retarded dynamical system:

$$\begin{cases} {}^C\partial_r^\alpha z(r,\mu)+P(r,\mu,D_\mu)z(r,\mu)=\mathscr{F}(r,\mu,z,z_r)+kw(r,\mu), \text{ in } I\times\Gamma,\\ D_\mu^m z(r,\mu)=0, |\, m \,|< n, \text{ on } I\times\partial\Gamma,\\ \frac{1}{\eta}\int_{-\eta}^{0} e^{2\xi}z(\xi,\mu)d\xi=\kappa_0(\mu),\ \ \mu\in\Gamma, \end{cases} \tag{26}$$

where $\eta,k\geq 0$ be constants, ${}^C\partial_r^\alpha$ is the $\alpha-$th order Caputo's derivative with $\alpha\in(0,1]$, $I=[0,b]$, Γ denotes a bounded domain whose boundary $\partial\Gamma$ is smooth in $\mathbb{R}^n$, $n\in\mathbb{N}$, and

$$\mathscr{F}(r,\mu,z,z_t)=\left(\int_\Omega z(r,\mu)d\mu\right)z(r,\mu)+\left(\int_{r-\eta}^{r}\vartheta(r-s)z(s,\mu)ds\right),$$
$$P(r,\mu,D_\mu):=\sum_{|m|\leq 2n} a_m(r,\mu)D_\mu^m$$

with $m=(m_1,m_2,....,m_n)$, $m_i\in\mathbb{R}^+$ for $i=1,2,...,n$, known as multi-index. Also, $|\, m\,|=\sum_{i=1}^{n} m_i$ and $\mu^m=\mu_1^{m_1}\mu_2^{m_2}....\mu_n^{m_n}$ for $\mu=(\mu_1,\mu_2,.....,\mu_n)\in\mathbb{R}^n$. Further, $D_\mu=(D_1,D_2,....,D_n)$, where each $D_s=\partial/\partial\mu_s$.

The family of linear operators $P(r,\mu,D_\mu)$ and the maps w,ϑ satisfy the following conditions:

(i)The linear operators $P(r,\mu,D_\mu)$ are uniformly strongly-elliptic in Γ, that is, there exists a constant $K_1>0$ such that

$$(-1)^n \text{Re} \sum_{|m|\le 2n} a_m(r,\mu)\nu^m \ge K_1|\,\nu\,|^{2n} \quad \text{for every } \mu\in\overline{\Gamma},\ r\in I \text{ and } \nu\in\mathbb{R}^n.$$

(ii)For each point $r\in I$, the functions $a_m(r,\mu)$ are smooth with respect to the second variable $\mu\in\overline{\Omega}$. Further,

$$|\,a_m(r,\mu)-a_m(t,\mu)\,|\le K\,|\,r-t\,|^\beta \quad \text{for } \mu\in\overline{\Gamma},\ r,t\in I, \text{ and } |\,m\,|\le 2n.$$

We assume that the constants $K>0$ and $\beta\in(0,1)$ are independent in r.

(iii)The map $\vartheta:[0,\eta]\to\mathbb{R}$ is continuous and $w:I\times\Gamma\to\Gamma$ is a continuous function with respect to the first variable $r\in I$.

Let $X=L^q(\Gamma)$, $1<q<\infty$. For each $r\in I$, a family of linear operators $P_q(r)$ (unbounded) on X is defined as:

$$P_q(r)v=P(r,\mu,D)v \quad \text{for } v\in D(P_q(r))=D(P)=W^{2n,q}(\Gamma)\cap W_0^{n,q}(\Gamma).$$

Since $C_0^\infty(\Gamma)\subseteq D(P)$, therefore, $\overline{D(P)}=X$. Note that $C_0^\infty(\Gamma)$ denotes the space of all infinite times continuously-differentiable real-valued functions. Following [24], it is easy to verify that the linear operator $P_q(r)$ is closed for each $r\in I$.

Next, we define $x(r)(\mu)=z(r,\mu)$ and $x_r=z_r(s,.)$, i.e., $(x(r+s))(\mu)=z(r+s,\mu)$ for $r\in I,\ \mu\in\Gamma$ and $s\in[-\eta,0]$. Further, $H:I\times X\times C_0\to X$, $Du:I\to X$ and $\tilde{g}:C_0\to X$ are defined as:

$$H(r,x(r),x_r)(\mu)=\left(\int_\Omega x(r)(\mu)d\mu\right)x(r)(\mu)+\left(\int_{-\eta}^0 \vartheta(-s)x_r(s)(\mu)ds\right),$$
$$(Du)(r)(\mu)=ku(r)(\mu)=kw(r,\mu),$$
$$\tilde{g}(\phi)(\mu)=\frac{1}{\eta}\int_{-\eta}^0 e^{2\xi}\phi(\xi)(\mu)d\xi$$

for $r\in I$ and $\mu\in\Gamma$. Also, we will define $g(x_0)(s)\equiv\tilde{g}(x_0)$ for $x_0\in C_0,\ s\in[-\eta,0]$ and $\kappa(s)\equiv\kappa_0$ for $s\in[-\eta,0]$.

Following [24], one can find a constant $l_1\ge 0$ so-that the family of operators $\{P_q(r)+l_1E\}_{r\in I}$ will satisfy the conditions (P1) - (P2) provided the points (i)-(ii) holds true, where E denotes the identity operator on the Banach space X. We select and fix such constant l_1. Then, the abstract formulation of the considered dynamical system (26) will be

$$\begin{cases} {}^C D_r^\alpha[x(r)]+(P_q(r)+l_1E)x(r)=h(r,x(r),x_r)+Du(r), & r\in I,\\ g(x_0)=\kappa, \end{cases} \tag{27}$$

where $h(r,x(r),x_r)=H(r,x(r),x_r)+l_1x(r)$. Due to the definition of $D(P)$ and the requirement that $x(r)\in D(P)$ for every $r\in I$, the boundary condition disappears. Following Minkowskii's inequality and Schwarz inequality, we will get

$$\|H(r,s_1,w_1)-H(r,s_2,s_2)\|_q\le(\|s_1\|_q+\|s_2\|_q)\|s_1-s_2\|_q+\rho\eta\|w_1-w_2\|_0,$$

where $\rho=\sup_{s\in[0,\eta]}\vartheta(s)$. It is easy to verify that the hypotheses (H2) and (HH) hold true for the nonlinear map h in a closed ball $\mathbb{B}_{r_1}$ with $L_h=\max\{2r_1,M,\rho\eta\}$. Further, if we consider $\phi(r)=\frac{1}{l_2}\kappa_0$ on $[-\eta,0]$ with $l_2=\frac{1}{\eta}\int_0^\eta e^{-2\xi}d\xi\neq 0$, then the hypothesis (H1) holds.

For the case $k=0$ in (26), then the operator $D=0$ in (27). Also, from above discussion the hypotheses (H1) and (H2) hold true. Therefore, one can apply the results derived in the Section 3 to establish the existence-uniqueness and regularity of mild solution to (27).

In the case when $k>0$, we have $\|D\|=k=\widehat{M}$. Moreover, the linear operator Q is defined as

$$(Qu)(\mu)=k\left[\int_0^b\psi(b-\theta,\theta)w(\theta,\mu)d\theta+\int_0^b\int_0^\theta\psi(b-\theta,\theta)\varphi(\theta,\xi)w(\xi,\mu)d\xi d\theta\right], \text{ for } \mu\in\Omega,$$

which shall satisfy the assumption (Q1) mentioned in the Theorem 4. Further, we make the assumption that the set $\Lambda_{l,\vartheta}(r)$, defined as in assumption (Q2) of the Theorem 4, is relatively compact in X with $r\in I$. Now the constants $\widehat{N},M_4$ and $\widehat{K}$ can be computed. Then, the Theorem 4 can be applied to establish the controllability of mild solution of (26) provided $\widehat{K}\left[1+\widehat{N}\widehat{M}M_3\right]<1$.

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