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A New Gompertz-Shanker Distribution with Applications

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Abstract: This work proposed a new probability distribution called Gompertz-Shanker distribution derived from Gompertz-G family of distributions. Some useful properties of the distribution were presented include: moment and its related measures like the expected value, variance, coefficient of variation, skewness, moment generating, survival, hazard, reversed hazard function and order statistics. The maximum likelihood estimation method is used to estimate the model parameters. Then, we apply the model to two real data sets to reveal its flexibility and applicability in real life situations over other competing distributions and simulation also considered in the research.

Keywords: Gompertz-G, Moments, Order Statistics, Reversed Hazard, Simulation,

1 Introduction

Different probability distributions have been proposed by many researchers and scholars in statistics to model in several areas like insurance, finance, economics, engineering, biology, to mention but few. Recently, usefulness and importance of robust distributions have led researchers developed several distributions in statistical literature using some notable approaches (such as generator approach, exponentiation, logit of beta function etc) to generate more flexible distributions to model complex and data sets with excessive skewness. Few of the works done by some authors using those methods stated above were: [1] convoluted beta with normal distribution, [2] they initiated Lomax generator of distributions. [3] introduced beta-loglogistic and some of its properties. Then, [4] came up with an alternative distribution called exponentiated exponential family to gamma and Weibull distributions, [5] studied and extended Lindley distribution, [2, 6] worked on exponentiated generalized class of distributions; [7] did justice on gamma-exponentiated exponential distribution, [8] investigated on generalized beta generated distribution, [9, 10, 11, 12, 13] have worked on modification, flexibility, generalization, skewness and kurtosis; and extended several distributions among others. Meanwhile, Gompertz-G (G-G) family of distributions is introduced by [14] where they developed a new generator with two additional parameters that led to Gompertz-G generator. [15, 16] investigated on Gompertz Inverse Exponential and Gompertz Frechet distributions; WHILE [17] worked on Gompertz Lindley distribution with application to real life data sets. [18] extensively discussed Shanker distribution in his work, compared with Lindley and exponential distributions where he discovered that the distribution has better fitting and suggested for modeling different lifetime data sets across various fields. In this article, a new distribution called Gompertz Shanker (GoSh) distribution with applications is propose. The main motivation for the proposed distribution is its flexibility and capability for capturing excessive kurtosis and skewness of some lifetime data sets.

2 Material and Methods

The cumulative and density function (CDF and PDF) of Gompertz-G distribution given by [14] are as follows:

$$G(y) = \int_0^{-\log[1-F(x)]} ae^{bu} e^{-\frac{a}{b}(e^{bu}-1)du} = 1 - e^{\frac{a}{b}\{1-[1-F(y)]^{-b}\}} \tag{1}$$

and

$$g(y) = a[1-F(y)]^{-b-1} e^{\frac{a}{b}\{1-[1-F(y)]^{-b}\}}.f(y) \tag{2}$$

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Furthermore, CDF and PDF of Shanker distribution with one parameter proposed by [18] are given in (3) and (4) below:

$$F(y) = 1 - \left[1 + \frac{\tau y}{\tau^2+1}\right] e^{-\tau y} \tag{3}$$

$$f(y) = \frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y} \tag{4}$$

However, we obtain the CDF and PDF of GoSh distribution by substituting expressions (3) and (4) into (1) and (2), gives equations (5) and (6).

2.1 Proposed Model

A random variable X follows a Gompertz-Shanker distribution with three parameters $a, b, \tau > 0$ where, a and b are shape parameters and τ is a scale parameter; if cdf and pdf are given by

$$G(y) = 1 - e^{\frac{a}{b}\left\{1-\left[1-\left[1-\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]\right]^{-b}\right\}}$$

This can be re-written as

$$G(y) = 1 - e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}} \tag{5}$$

and

$$g(y) = \frac{dG(y)}{dy} = a\left[\left[1 + \frac{\tau y}{\tau^2+1}\right] e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[1-\left[1-\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]\right]^{-b}\right\}} . \frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y}$$

Then, it is re-written as

$$(y) = a\left[\left[1 + \frac{\tau y}{\tau^2+1}\right] e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}} . \frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y} \tag{6}$$

where, a , b are shape and τ scale parameters. The proof of GoSh distribution is derived below:

A true and valid pdf of any distribution must equal to 1. Below is the proof of the pdf of the GoSh distribution and is equal to 1. The proof goes thus:

$$\int_{-\infty}^{\infty} g(y)dy = 1$$

$$\int_{-\infty}^{\infty} g(y)dy = \int_{-\infty}^{\infty} a\left[\left[1 + \frac{\tau y}{\tau^2+1}\right] e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}} . \frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y} dy$$

we simplify using transformation method.

setting $w = e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}$, if $k = \frac{a}{b}\left\{1 - \left[\left[1 + \frac{\tau y}{\tau^2+1}\right] e^{-\tau y}\right]^{-b}\right\}$, $w = e^k$,

when $u = \left[1 + \frac{\tau y}{\tau^2+1}\right] e^{-\tau y}$. This implies that $k = \frac{a}{b}\{1 - [u]^{-b}\}, k = \frac{a}{b} - \frac{a}{b} u^{-b}$. By substituting the transformation into expression below, we have

$$\int_{-\infty}^{\infty} g(y)dy = \frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y} u^{-(b+1)} e^k dy$$

$$\frac{dw}{dk} = \frac{dw}{dk} x \frac{dk}{du} x \frac{du}{dy}$$

where, $\frac{dw}{dk} = e^k, \frac{dk}{du} = a u^{-(b+1)}$ and $\frac{du}{dy} = -\frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y}$

$\frac{dw}{dk} = e^k . a u^{-(b+1)} . -\frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y}, \frac{dw}{dy} - \frac{\tau^2(\tau+y)}{\tau^2+1} e^{-\tau y} a u^{-(b+1)} . e^k$ and

$$dy = \frac{(\tau^2+1)dw}{a\tau^2(\tau+y)e^{-\tau y}u^{-(b+1)}e^{k}}$$

Also, by substituting for dy in equation below, we get

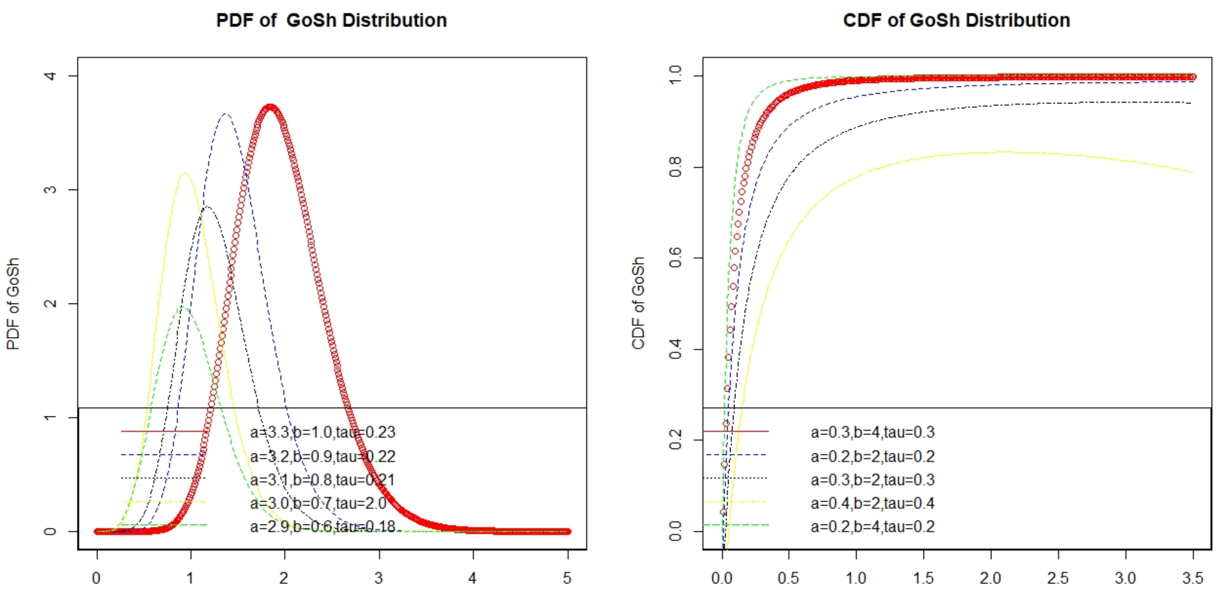
$$\int_{-\infty}^{\infty} g(y)dy = -\int_{-\infty}^{\infty} dw = -[w]_{-\infty}^{\infty} = -\, e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}\Bigg|_{0}^{\infty}$$

$$= -\left[e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau(\infty)}{\tau^2+1}\right]e^{-\tau(\infty)}\right]^{-b}\right\}} - e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau(0)}{\tau^2+1}\right]e^{-\tau(0)}\right]^{-b}\right\}}\right]$$

$$\int_{-\infty}^{\infty} g(y)dy = -[0-1] = 1$$

Hence, the last expression is the proof that (6) above is a valid pdf.

Hence, both density and cumulative functions plots generate from equations (5) and (6) is depict in figure 1 a and b.



(a): The Probability Density Function of the GoSh at different parameter values

(b): The Cumulative Density Function of the GoSh at different parameter values

Fig. 1: The density and Distribution Function Plots of GoSh distribution

3 Other Statistics Characteristics

3.1 Moments and its Related Measures

$$\mu_s = E(Y^s) = \int_0^{\infty} y^s g(y)dy \qquad (7)$$

Putting the pdf of the GoSh distribution in (6) into the equation (7), we get

$$\mu_s = E(Y^s) = \int_0^{\infty} y^s a\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}.\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}dy \qquad (8)$$

By taking the exponential part in (8) for expansion with power series, this leads to

$$e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}} = \sum_{i=0}^{\infty}\frac{a^p}{p!b^p}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}^{p} \qquad (9)$$

Then, putting (9) into (8) yields (10)

$$g(y) = a\frac{\tau^2}{\tau^2+1}\sum_{i=0}^{\infty}\frac{a^p}{p!b^p}(\tau+y)e^{-\tau y}\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}^p \quad (10)$$

Expression (10) can be re-written using binomial expansion as

$$\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}^p = \sum_{q=0}^{\infty}(-1)^l\binom{p}{q}\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-bq}$$

(10) gives

$$g(y) = a\frac{\tau^2}{\tau^2+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{a^p(-1)^q}{p!\,b^p}\binom{p}{q}(\tau+y)e^{-\tau y}\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b(q+1)+1)}$$

$$g(y) = a\frac{\tau^2}{\tau^2+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{a^p(-1)^q}{p!b^p}\binom{i}{l}(\tau+y)e^{b\tau(q+1)y}\left[\left[1+\frac{\tau y}{\tau^2+1}\right]^{b(q+1)+1}\right]^{-1} \quad (11)$$

To simplify this, we use power series expansion in (11) and it becomes

$$\left[1+\frac{\tau y}{\tau^2+1}\right]^{b(q+1)+1} = \sum_{r=0}^{\infty}\binom{b(q+1)+1}{r}\left(\frac{\tau}{\tau^2+1}\right)^r y^r \quad (12)$$

By inserting (12) into (11), we have

$$g(y) = a\frac{\tau^2}{\tau^2+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{a^p(-1)^q}{p!b^q}\binom{p}{q}\left\{\sum_{r=0}^{\infty}\binom{b(q+1)+1}{r}\left(\frac{\tau}{\tau^2+1}\right)^r\right\}^{-1} y^{-r}(\tau+y)e^{b\tau(q+1)y} \quad (13)$$

Let $\varphi_{p,q,r} = a\frac{\tau^2}{\tau^2+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{a^p(-1)^q}{p!b^q}\binom{p}{q}\left\{\sum_{r=0}^{\infty}\binom{b(q+1)+1}{r}\left(\frac{\tau}{\tau^2+1}\right)^r\right\}^{-1}$ for simplification

$$g(y) = \varphi_{p,q,r}y^{-r}(\tau+y)e^{b\tau(q+1)y}$$

Fortunately,

$$\mu_s = E(Y^s) = \int_0^{\infty} y^s g(y) = \varphi_{p,q,r}\left\{\int_0^{\infty} y^{s-r}\,e^{b\tau(q+1)y}dy + \int_0^{\infty} y^{s-r+1}\,e^{b\tau(q+1)y}dy\right\}$$

Therefore, the s-th moment of GoSh distribution is obtaining as

$$\mu_s = E(Y^s) = \varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{s-r+1}\Gamma(s-r+1) + \left(-\frac{1}{b\tau(q+1)}\right)^{s-r+2}\Gamma(s-r+2)\right\} \quad (14)$$

In this vein, the expectation, variance, kurtosis and the skewness of the GoSh distribution are obtaining below:

3.2 The Mean

The expected value of GoSh distribution is obtaining by replacing s with 1 in (14)

$$E(Y^1) = \varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{2-r}\Gamma(2-r) + \left(-\frac{1}{b\tau(q+1)}\right)^{3-r}\Gamma(3-r)\right\} \quad (15)$$

Hence, corresponding variance is

set,

$$\gamma_1 = E(Y^2) = \varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{3-r}\Gamma(3-r) + \left(-\frac{1}{b\tau(q+1)}\right)^{4-r}\Gamma(4-r)\right\}$$

and

$$\gamma_2 = E(Y^1) = \left[\varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{2-r}\Gamma(2-r) + \left(-\frac{1}{b\tau(q+1)}\right)^{3-r}\Gamma(3-r)\right\}\right]^2$$

$$Var(Y) = \gamma_1 - \gamma_2 \quad (16)$$

In addition, we have

$$E(Y^3) = \varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{4-r}\Gamma(4-r) + \left(-\frac{1}{b\tau(q+1)}\right)^{5-r}\Gamma(5-r)\right\}$$

Coefficient of variation (CV)

$$CV(Y) = \frac{\sqrt{\gamma_1-\gamma_2}}{\varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{2-r}\Gamma(2-r)+\left(-\frac{1}{b\tau(q+1)}\right)^{3-r}\Gamma(3-r)\right\}} \quad (17)$$

and

Skewness (SK)

If,

$$\omega_1 = \varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{4-r}\Gamma(4-r) + \left(-\frac{1}{b\tau(q+1)}\right)^{5-r}\Gamma(5-r)\right\} - 3(\gamma_1-\gamma_2).$$

$$\varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{2-r}\Gamma(2-r) + \left(-\frac{1}{b\tau(q+1)}\right)^{3-r}\Gamma(3-r)\right\}$$

$$\omega_2 = \left[\varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{2-r}\Gamma(2-r) + \left(-\frac{1}{b\tau(q+1)}\right)^{3-r}\Gamma(3-r)\right\}\right]^3$$

$$SK(Y) = \frac{\omega_1-\omega_2}{[\gamma_1-\gamma_2]^{1.5}} \quad (18)$$

3.3 Moment Generating Function (MGF)

$$M_y{}^{(t)} = E(e^{ty}) = \int_0^\infty e^{ty} g(y)dy$$

$$M_y{}^{(t)} = E(e^{ty}) = \int_0^\infty e^{ty}\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}.\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}dy \quad (19)$$

Simplifying the integral in (19) yields

$$M_y{}^{(t)} = E(e^{ty}) = E\left\{\sum_{s=0}^{\infty}\frac{(ty)^s}{s!}\right\} = \sum_{s=0}^{\infty}\frac{t^s}{s!}\mu_s$$

$$M_y{}^{(t)} = \sum_{s=0}^{\infty}\frac{t^s}{s!}\left[\varphi_{p,q,r}\left\{\left(-\frac{1}{b\tau(q+1)}\right)^{s-r+1}\Gamma(s-r+1) + \left(-\frac{1}{b\tau(q+1)}\right)^{s-r+2}\Gamma(s-r+2)\right\}\right] \quad (20)$$

Equation (20) becomes the MGF of GoSh distribution.

3.3.1 Survival Function

$$Surf(y) = 1 - Pr(Y \le y) = 1 - G(y)$$

$$= 1 - G(y) = 1 - \left[1 - e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}\right]$$

$$Surf(y) = e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}} \quad (21)$$

3.3.2 Hazard Rate Function

$$hazf)y) = \frac{g(y)}{1-G(y)} = \frac{g(y)}{Surf(y)}$$

This implies that expression divided by expression

$$hazf(y) = \frac{a\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}.\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}}{e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}}$$

$$hazf(y) = a\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)}.\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y} \quad (22)$$

3.3.3 Reversed Hazard Rate Function

$$Rharf(y) = \frac{g(y)}{G(y)}$$

$$Rhazf(y) = \frac{a\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}.\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}}{1-e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}}$$

$$Rhazf(y) = a\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)}.\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y} \quad (23)$$

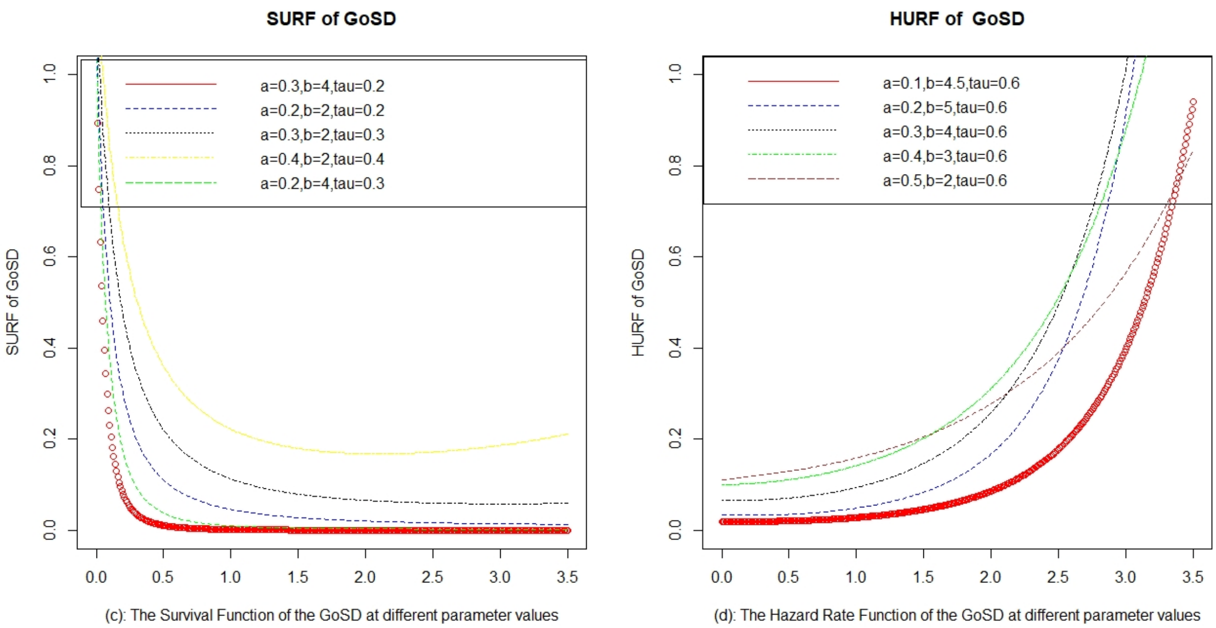


Figure 2: Plots of Survival and Hazard Rate Function of GoSh distribution

3.3.4 Quantile Function

Let the quantile function of $GoSh(a, b, \tau)$ be $Q(u)$ and defined by $G(Q(u)) = u$, [19]. We can obtain $Q(u)$ as the root of the following equation

$$-(\tau^2 + 1 + (\tau)Q(u))e^{-(\tau)Q(u)} = (u-1)(\tau^2+1)$$

for $0 < s < 1$, substituting $K(u) = -(\tau^2 + 1 + (\tau)Q(u))e^{-(\tau)Q(u)}$, this can be rewritten as

$$K(u)e^{K(u)} = (u-1)(\tau^2+1)e^{-(\tau)}$$

Therefore, $K(u)$ is

$$K(u) = W\left((u-1)(\tau^2+1)e^{-(\tau)}\right)$$

where, $W[.]$ Is the Lambert function and using the function, we obtain

$$Q(u) = -\frac{\tau^2+1}{\tau} - \frac{\frac{blog}{a}W\left((u-1)^{-\frac{1}{b}}(\tau^2+1)e^{-(\tau^2+1)}\right)}{\tau} \tag{24}$$

Meanwhile, the: 1st quartile, median and 3rd quartile can be obtained by substituting $u = 0.25, 0.5$ and 0.75 respectively in the above equation.

3.4 Order Statistics

Generally, Order statistics is an important instrument used in solving complex problems in statistical methods. Some of the problems might be goodness of fit tests, detection of outliers, entropy estimation etc. [9, 12, 13] defined it in their works. However, given that $Y_1, \ldots, Y_n$ be a random sample from a distribution with pdf $g(y)$ and $Y_{1:n} <, \ldots, < Y_{p:n}$ denote the corresponding order statistics from the random sample. Then, the pdf $g_{p:n}(y)$ of the kth order statistic is expressed as

$$g_{p:n}(y) = \frac{n!}{(p-1)!(n-p)!}\sum_{i=0}^{n-p}(-1)^i\binom{n-p}{i}g(y)G(y)^{\,i+p-1} \tag{25}$$

where, $g(y)$ and $G(y)$ are the pdf and cdf of GoSh distribution. Here, putting expressions (5) and (6) into (25) gives the pdf of the p-th order statistics $Y_{p:n}$ given as

$$g_{p:n}(y) = \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{n-p}(-1)^i\binom{n-p}{i}a\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}$$

$$\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}} \tag{26}$$

Then, the pdf for minimum and maximum $(Y_{(1)})$ and $(Y_{(n)})$ order statistic of the GoSh distribution are expressed by

$$f_{1:n}(y) = n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}a\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)}$$

$$\left[e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}\right]^i \tag{27}$$

and

$$g_{n:n}(y) = na\frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)}\left[e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}}\right]^{n-1} \tag{28}$$

3.5 Parameters Estimation

Let $Y_1, Y_2, \ldots, Y_n$ be a sample size of n independently and identically distributed (iid) random variables from the Gompertz-Shanker Distribution (GoSh) with unknown parameters a, b and τ which have been defined above. Hence, the *pdf* of the *GoSh* is given in (6) as:

$$g(y) = a\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]^{-b}\right\}} \cdot \frac{\tau^2(\tau+y)}{\tau^2+1}e^{-\tau y}$$

Using the MLE method, the likelihood function is given by

$$LF(Y|a,b,\tau) = \frac{(a\tau^2)^n \prod_{i=1}^{n}\left\{(\tau+y)e^{-\tau y}\prod_{i=1}^{n}\left\{\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right\}^{-(b+1)} e^{\frac{a}{b}\left\{1-\left[\left[1+\frac{\tau y}{\tau^2+1}\right]e^{-\tau y}\right]\right\}^{-b}}\right\}}{\tau^2+1} \tag{29}$$

and the log-likelihood function $logL = LL(Y|a,b,\tau)$ is written as

$$LL = nloga + 2nlog\tau - nlog(\tau^2+1) + \sum_{i=1}^{n} log(\tau+y_i) - \tau\sum_{i=1}^{n} y_i - (b+1)$$

$$\sum_{i=1}^{n} log\left\{\left[1+\frac{\tau y_i}{\tau^2+1}\right]e^{-\tau y_i}\right\} + \frac{a}{b}\sum_{i=1}^{n}\left\{\left[\left[1+\frac{\tau y_i}{\tau^2+1}\right]e^{-\tau y_i}\right]^{-b}\right\} \tag{30}$$

Also, differentiating (30) partially with respect to a, b and τ will give the MLEs of the parameters in the model. The estimates can be obtained through R programming software with data sets.

4 Model Validation and Analysis

The application of GoSh distribution to two different datasets is presented here and compared with the Gompertz Lindley, Gompertz and Shanker distributions. We used R (Programming) software for the analysis.

The values of the model selection criteria with better fit therefore, the lower the values of this criteria, the better fit it is.

First Analysis. The first analysis based on secondary data of survival times of some patients suffering from Head and Neck cancer disease and was treated with combination of radiotherapy and chemotherapy used by [21].

Table 1: Summary of Descriptive Statistics of First Analysis

Min	Max	Mean	1st Quat	Med	3rd Quat	Sks	Kts
12.20	1779.00	223.48	67.21	128.50	219.00	3.38	16.56

Table 2: Result from First Analysis

Parameter/Model	$\hat{\tau}$	$\hat{a}$	$\hat{b}$	Statistic	Model Ranking
GoSh	93.7734 (70.1470)	10.6153 (13.1866)	0.1643 (0.0285)	$W* = 0.1233$ $A = 0.8675$ $KS = 0.1207$ $AIC = 586.72$ $CAIC = 587.33$ $HQIC = 588.67$ $BIC = 592.00$	1
GoLd	48.7503 (32.8593)	18.4811 (39.0023)	0.1835 (0.0312)	$W* = 0.1463$ $A = 1.0170$ $KS = 0.1280$ $AIC = 588.78$ $CAIC = 589.39$ $HQIC = 590.72$	4

				$BIC = 594.10$	
Gompertz	--	5.8370 (6.8385)	0.1781 (0.0315)	$W *= 0.1353$ $A = 0.9464$ $KS = 0.1225$ $AIC = 587.81$ $CAIC = 588.42$ $HQIC = 589.75$ $BIC = 593.09$	2
Shanker	54.4503 (33.6196)	--	--	$W *= 0.1417$ $A = 0.9877$ $KS = 0.1273$ $AIC = 588.38$ $CAIC = 588.99$ $HQIC = 590.33$ $BIC = 593.66$	3

Second Analysis. The second data is on queue (waiting time) before service of 100 Bank customers. It was studied and applied by [22] in their work also, used by [18].

Table 3: Summary of Descriptive Statistics for Second Analysis

Min	Max	Mean	1st Quat	Med	3rd Quat	Sks	Kts
0.80	38.50	9.88	4.68	8.10	13.03	1.47	5.54

Table 4: Result from Second Analysis

Parameter/Model	$\hat{\tau}$	$\hat{a}$	$\hat{b}$	Statistic	Model Ranking
GoSh	2.6734 (1.6446)	0.4364 (0.1221)	0.9059 (0.2575)	$W *= 0.0175$ $A = 0.1272$ $KS = 0.0365$ $AIC = 640.07$ $CAIC = 640.32$ $HQIC = 643.23$ $BIC = 647.88$	1
GoLd	4.081 (2.1135)	0.2225 (8.1435)	0.8418 (0.1452)	$W *= 0.2300$ $A = 1.1732$ $KS = 0.2522$ $AIC = 643.81$ $CAIC = 644.05$ $HQIC = 645.38$	2

				$BIC = 649.01$	
Gompertz	--	4.5641 (1,8791)	0.9321 (0.5912)	$W *= 1.3234$ $A = 1.7334$ $KS = 1.4321$ $AIC = 645.11$ $CAIC = 645.85$ $HQIC = 647.67$ $BIC = 651.10$	3
Shanker	4.4786 (0.9867)	--	--	$W *= 1.5443$ $A = 2.1132$ $KS = 1.3342$ $AIC = 647.61$ $CAIC = 648.77$ $HQIC = 650.53$ $BIC = 654.45$	4

4.1 Simulation

Simulation is required to ascertain the estimation performances, most especially their Mean square errors and biases for different n sample sizes. We consider different n sample size when $n = 20, 30, 50, 100$ respectively and initial values of the model parameters ($a = 3.00, b = 2.00$ and $\tau = 0.90$. R programming software is used to conduct a numerical analysis and the result is displayed in Table 5 below.

Table 5. MLEs, Biases and MSEs for some parameter values

n	$Parameter$	Initial Value	MLE	$Bias$	MSE
20	a	3.00	3.7437	0.1727	3.0953
	b	2.00	5.3792	0.1098	7.3356
	τ	0.90	-0.2262	2.4727	0.0091
30	a	3.00	3.5246	0.1239	0.0000
	b	2.00	2.8913	2.2239	2.3146
	τ	0.90	0.1846	0.0058	0.0000
50	a	3.00	1.0377	0.0495	0.0659
	b	2.00	0.9632	0.0170	0.0361
	τ	0.90	1.4348	2.1495	0.0432
100	a	3.00	4.4050	0.0196	0.0000
	b	2.00	3.7823	0.0260	6.4727
	τ	0.90	0.1013	2.1196	0.0000

5 Discussion

Tables 1 and 3 contain the descriptive Statistics of the data sets studied and considered while the results on Tables 2 and 4 consist the values of: estimation, standard error (in parenthesis), model selection and goodness of fit. Then, according to the values in Tables 2 and 4, the proposed (GoSh) model recorded lowest values for both selection criteria and goodness of fit. On this note, the GoSh distribution has proved its superiority over other competing distributions which happens to be the best model to fit the datasets compared to others. The plots below are the fitted pdfs: GoSh, GoLd, Gomperz and shanker distributions with survival and queuing data considered here. Fortunately, GoSh distribution has better representation of the two data sets than other competing distributions.

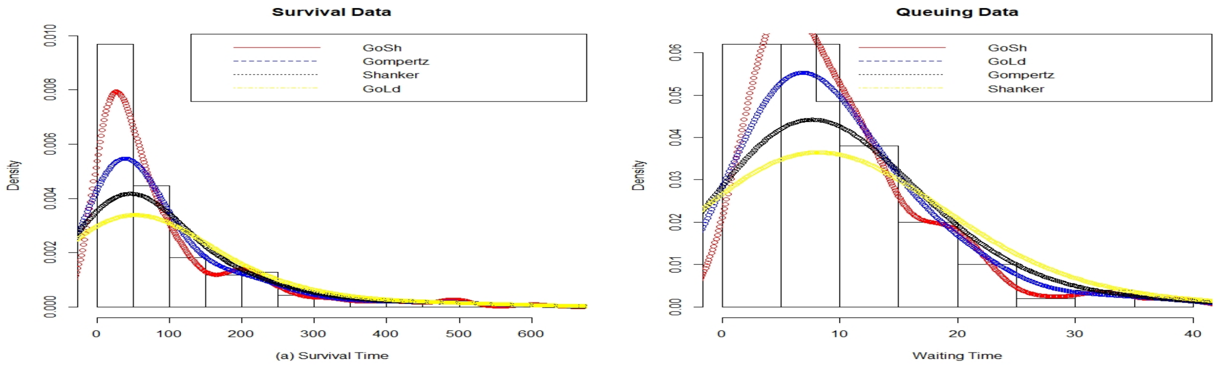


Figure 3: Fitted densities to survival and queuing data

5.1 Conclusion

We have been successfully studied and introduced GoSh distribution whereby some of its useful properties are obtained. The pdf of the model equal to one (1) which means it is a true and valid distribution and it's in the Appendix. Also, the model is capable of fitting real life phenomenon for it has unimodal failure rate. Therefore, due to its performance over Gompertz Lindley, Gompertz and Shanker distributions the model is tractable and flexible based on model selection criteria and goodness of fit tests values of these models. Then, without doubt the proposed model is a competitive model, and can be used in several fields such as: insurance, finance, economics, engineering, biology. The work can be extended to regression model However, the histogram, empirical cdf and some properties of the proposed model which were not considered here can be examined and even simulation study can also carry out.

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