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The q-pell Hyperbolic Functions

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Abstract: In this paper, since the standard hyperbolic functions have countless uses in modern science we present a mathematical study concerning an extension of the Pell sine and cosine hyperbolic functions. The main properties exhibited by the Pell hyperbolic functions [3], for which they can be obtained from k-Fibonacci hyperbolic functions when $k = 2$, are reviewed. The new class of functions, i.e., the q-analogues of Pell hyperbolic functions, are introduced and defined based on the q-analogues of the Pell numbers and the q-analogue of the "Silver ratio", where module q is a number defined between zero and the unity.A battery of properties is demonstrated: q-Pell Pythagorean Theorem; q-Pell sum and difference; q-Pell double argument; q-Pell half argument; q-Pell Catalan's identity; q-Pell Cassini's identity and q-Pell d'Ocagne's identity.

Keywords: Pell numbers; Pell hyperbolic functions; q-calculus; q-analogue.

1 Introduction

In recent years, Fibonacci numbers and their generalization of k-Fibonacci numbers have had many interesting properties and applications to almost every fields of science and art than mathematical areas (for instance, [1,9] and [2] *et al)*. For example, Fibonacci numbers and the Golden mean are popular in many scientific disciplines, from quasi-crystals through models of DNA sequences, phyllotaxis, to the research on brain activity (e.g. EEG signals). Similarly, the Pell numbers can be obtained from k-Fibonacci numbers for $k = 2$ [7, 8]. The sequence of Pell numbers starts with 0 and 1, and then each Pell number is the sum of twice the previous Pell number and the Pell number before that. It is interesting to emphasize that the ratio of two consecutive Pell numbers converges to the Silver ratio [4].

The k-Fibonacci hyperbolic functions were introduced and studied [10,11,12,13,14]. For these functions, some equalities were given like Pythagorean theorem, sum and difference, half argument etc. in the same paper.

First formula in what we now call q-calculus were obtained by Euler in the eighteenth century. Many remarkable results (like Jakobi's triple product identity and the theory of q-hypergeometric functions) were obtained in the nineteenth century. A q-analogue, also called a q-extension or q-generalization, is a mathematical expression parameterized by a quantity q that generalized a known expression and reduces to the known expression in the limit $q \to 1^-$. There are q-analogues for the fractional, binomial coefficient, derivative, integral, Fibonacci numbers and so on.

A sequence of polynomials $P_n(q)$ is defined as follows:

$$P_n(q) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ (1+q^{n-1})P_{n-1}(q) + q^{n-2}P_{n-2}(q) & \text{if } n \geq 2. \end{cases}.$$

Obviously $P_n(q)$ is the q-analogue of the Pell numbers. Arithmetic properties of q-Pell numbers were given [6].

For $0 < |q| < 1$, the q-analogue of the derivative of the function $f(x)$, denoted by $D_q f(x)$ is defined as follows:

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.$$

If $f'(0)$ exists, then $D_q f(0) = f'(0)$. The q-derivative reduces to the usual derivative as $q \to 1^-$.

Before we continue, let us introduce some notation that is used in the remainder of the paper [5]. For any real number α,

$$[\alpha] := \frac{q^\alpha - 1}{q-1}.$$

In particular, if $n \in \mathbb{Z}^+$, it is denoted as follows:

$$[n] = \frac{q^n - 1}{q-1} = q^{n-1} + ... + q + 1.$$

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In this paper, q-analogues of $\sin P_q h$ and $\cos P_q h$ Pell hyperbolic functions are defined. Some basic properties are studied of these functions. Also, q-analogues of some equalities like Pythagorean theorem, sum and difference, half argument etc. are obtained.

Now, we introduce basic definitions which will be used throughout the paper.

2 Pell Hyperbolic Functions

The Pell numbers are obtained from these continuous versions of the k-Fibonacci numbers which are the k-Fibonacci functions. These functions arise naturally from the k-Fibonacci numbers and can be seen as a new type of hyperbolic functions, where the Silver ratio $\delta = \sigma_2$, or more generally σ_k for each real number k, plays an analogue role as the number e into the classical hyperbolic function.

In addition, Binet's formula is well known in the Fibonacci numbers theory [13]. In this way, Binet's formula allows to express the k-Fibonacci numbers. The characteristic equation associated with the recursive definition of the k-Fibonacci numbers is $\sigma^2 = k\sigma + 1$ where σ_1 and σ_2 are roots of the equation as follows. That is, if $k = 2$, for the Pell sequence, $\sigma_1 = 1 - \sqrt{2} = \beta$ and $\sigma_2 = 1 + \sqrt{2} = \delta$ are obtained. δ is known as the Silver ratio and also denoted by σ_2 [4].

Now, the classical Pell sequence is given as follows:

Definition 1.([4]) *For any positive real number k, the Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{Z}}$ is defined recurrently by*

$F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$with initial conditions
$F_{k,0} = 0$; $F_{k,1} = 1$.

Particular case of the k-Fibonacci sequence is if $k = 2$, the classical Pell sequence appears

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 1.$$

For the Pell equation is $\lim\limits_{n\to\infty} \frac{P_n}{P_{n-1}} = \delta$. Also, the Pell sequence, it is obtained

$$\sigma_2 = \delta = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

Besides, for the Pell sequence, the Silver ratio is written as

$$\sigma_2 = \delta = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + \dots}}}}$$

Silver ratio may be written as positive root of the Pell equation in the following form:

$$P_n = \begin{cases} \frac{\delta^{2k+1} + \delta^{-(2k+1)}}{\sqrt{8}}, & n = 2k+1 \\ \frac{\delta^{2k} + \delta^{-2k}}{\sqrt{8}} & n = 2k \end{cases}$$

where the discrete variable k takes its values from the set $0, \pm 1, \pm 2, \pm 3, \dots$

Definition 2.([3]) *Let δ be Silver ratio. The functions $\sin Ph(x)$ and $\cos Ph(x)$ are defined as follows:*

$$\sin Ph(x) = \frac{\delta^{2x} - \delta^{-2x}}{\sqrt{8}},$$

$$\cos Ph(x) = \frac{\delta^{2x+1} + \delta^{-(2x+1)}}{\sqrt{8}},$$

and are called Pell hyperbolic functions.

By making a change of variable $2x + 1 = t$ in functions $\sin Ph(x)$ and $\cos Ph(x)$ and also using $\delta + \delta^{-1} = \sqrt{8}$, another representations of Pell hyperbolic *sine* and *cosine* functions are, respectively, given as follows [3].

Definition 3.([3]) *Let δ be Silver ratio. The functions $\sin Ph(x)$ and $\cos Ph(x)$ are defined as:*

$$\sin Ph(x) = \frac{\delta^{x} - \delta^{-x}}{\delta + \delta^{-1}},$$

$$\cos Ph(x) = \frac{\delta^{x} + \delta^{-x}}{\delta + \delta^{-1}}.$$

and are called Pell hyperbolic functions.

Now, equalities introduced [3] can naturally be related with the q-Pell hyperbolic functions studied before.

The following correlations that are similar to the equation $[\cos h(x)]^2 - [\sin h(x)]^2 = 1$ are valid for the Pell hyperbolic functions:

$$[\cos Ph(x)]^2 - [\sin Ph(x)]^2 = \frac{1}{2}. \tag{1}$$

Equality (2.1) is called Pythagorean Theorem.

Sum and difference formulas are as follows:

$$\begin{aligned}\cos Ph(x+y) = \sqrt{2}(&\cos Ph(x).\cos Ph(y) \\ &+ \sin Ph(x).\sin Ph(y))\end{aligned} \tag{2}$$

$$\begin{aligned}\cos Ph(x-y) = \sqrt{2}(&\cos Ph(x).\cos Ph(y) \\ &- \sin Ph(x).\sin Ph(y))\end{aligned} \tag{3}$$

$$\begin{aligned}\sin Ph(x+y) = \sqrt{2}(&\sin Ph(x).\cos Ph(y) \\ &+ \sin Ph(y).\cos Ph(x))\end{aligned} \tag{4}$$

$$\begin{aligned}\sin Ph(x-y) = \sqrt{2}(&\sin Ph(x).\cos Ph(y) \\ &- \sin Ph(y).\cos Ph(x))\end{aligned} \tag{5}$$

By doing $x = y$ in (2) and (4) formulas, we have the following equations.

$$\cos Ph(2x) = \sqrt{2}((\cos Ph(x))^2 + (\sin Ph(x))^2) \tag{6}$$

$$\sin Ph(2x) = 2\sqrt{2}(\sin Ph(x).\cos Ph(x)). \tag{7}$$

Equations mentioned above give the relationship between double argument and argument. Also, from equations (1) and (6) it is deduced, respectively, by summing up and subtracting:

$$[\cos Ph(x)]^2 = \frac{1}{2\sqrt{2}}\left(\cos Ph(2x) + \frac{\sqrt{2}}{2}\right)$$

$$[\sin Ph(x)]^2 = \frac{1}{2\sqrt{2}}\left(\cos Ph(2x) - \frac{\sqrt{2}}{2}\right).$$

For any integer r, Catalan's Identity can be given as in the following equation:

$$\cos Ph(x-r).\cos Ph(x+r) - (\cos Ph(x))^2 = (\sin Ph(r))^2$$

The following results can be obtained in a similar way.

$$\cos Ph(x-r).\cos Ph(x+r) - (\sin Ph(x))^2 = (\cos Ph(r))^2$$

$$\sin Ph(x-r).\sin Ph(x+r) - (\sin Ph(x))^2 = -(\sin Ph(r))^2$$

$$\sin Ph(x-r).\sin Ph(x+r) - (\cos Ph(x))^2 = -(\cos Ph(r))^2.$$

For $r = 1$,

$$\cos Ph(x-1).\cos Ph(x+1) - (\sin Ph(x))^2 = 1$$

$$\sin Ph(x-1).\sin Ph(x+1) - (\cos Ph(x))^2 = -1$$

can be obtained. These identities are called Cassini's or Simson's Identity. Furthermore, d'Ocagne's Identity is given in the following:

$$\cos Ph(x).\cos Ph(y+r) - \sin Ph(x+r).\sin Ph(y) = \cos Ph(r).\cos Ph(x-y) \tag{8}$$

$$\cos Ph(x).\sin Ph(y+r) - \cos Ph(x+r).\sin Ph(y) = \sin Ph(r).\cos Ph(x-y). \tag{9}$$

Notice that by taking $r = 1$ in equation (8) the following identity:

$$\cos Ph(x).\cos Ph(y+1) - \sin Ph(x+1).\sin Ph(y) = \cos Ph(x-y).$$

is obtained.

3 q-Analogues of Pell Hyperbolic Functions

We start with the definition of q-Silver ratio which will be used in the sequal of the paper. The q-Silver ratio is positive root of characteristic equation which can be obtained by using recurrence of q-analogues of the Pell numbers.

Definition 4. *Let $P_n(q)$ be a sequence of polynomials and*

$$\frac{P_n(q)}{P_{n-1}(q)} = \delta_q,$$

that is,

$$\frac{P_n(q)}{P_{n-1}(q)} = \frac{(1+q^{n-1})P_{n-1}(q)}{P_{n-1}(q)} + q^{n-2}\frac{P_{n-2}(q)}{P_{n-1}(q)}, \quad n \geqslant 2.$$

From now on, $\delta_q^2 = \delta_q(1+q^{n-1}) + q^{n-2}$ $(n \geqslant 2)$ characteristic equation is obtained, where $\delta_{q_1} = \frac{(1+q^{n-1})+\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}$ and $\delta_{q_2} = \frac{(1+q^{n-1})-\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}$. Then, q-Silver ratio which is the positive root of characteristic equation denoted by δ_q, is defined as

$$\delta_q = \frac{(1+q^{n-1}) + \sqrt{(1+q^{n-1})^2 + 4q^{n-2}}}{2}, \ n \geqslant 2.$$

Now, we give basic definitions called $\sin P_qh(x)$ and $\cos P_qh(x)$, respectively, where $\sin P_qh(x)$ is q-analogue of *sine* Pell hyperbolic function and in a similar way, $\cos P_qh(x)$ is q-analogue of *cosine* Pell hyperbolic function.

Definition 5. *Let δ_q be Silver ratio, q-analogue of Silver ratio. Then*

$$\sin P_qh(x) = \frac{\delta_q^{2x} - \delta_q^{-2x}}{\sqrt{(1+q^{n-1})^2 + 4q^{n-2}}},$$

$$\cos P_qh(x) = \frac{\delta_q^{2x+1} + \delta_q^{-(2x+1)}}{\sqrt{(1+q^{n-1})^2 + 4q^{n-2}}}.$$

We will call $\sin P_qh(x)$ is q-analogue of sine Pell hyperbolic function and $\cos P_qh(x)$ is analogue of cosine Pell hyperbolic function. Another representations q-analogue of Pell hyperbolic sine and cosine functions are, respectively, given as follows.

Definition 6. *Let δ_q be Silver ratio, q-analogue of Silver ratio. Then*

$$\sin P_qh(x) = \frac{\delta_q^{x} - \delta_q^{-x}}{\delta_q + \delta_q^{-1}}, \tag{10}$$

$$\cos P_qh(x) = \frac{\delta_q^{x} + \delta_q^{-x}}{\delta_q + \delta_q^{-1}}, \tag{11}$$

since $\delta_q + \delta_q^{-1} = \sqrt{(1+q^{n-1})^2 + 4q^{n-2}}$, for $n \geq 2$.

In the sequel we present the main properties of these functions in a similar way in which similar properties of the Pell hyperbolic functions are usually presented. For $n \geq 2$, we have (12) which may be regarded as a version of the Pythagorean theorem.

Theorem 1.(*q-Pell Pythagorean theorem*)

$$[\cos P_q h(x)]^2 - [\sin P_q h(x)]^2 = \frac{4}{(1+q^{n-1})^2+4q^{n-2}}. \tag{12}$$

*Proof.*Using the definitions of q-analogue of Pell hyperbolic sine and cosine functions of (10) and (11), we get

$$\begin{aligned} &[\cos P_q h(x)]^2 - [\sin P_q h(x)]^2 \\ &= \left(\frac{\delta_q^x+\delta_q^{-x}}{\delta_q+\delta_q^{-1}}\right)^2 - \left(\frac{\delta_q^x-\delta_q^{-x}}{\delta_q+\delta_q^{-1}}\right)^2 \\ &= \frac{\delta_q^{2x}+2\delta_q^x\delta_q^{-x}+\delta_q^{-2x}-\delta_q^{2x}+2\delta_q^x\delta_q^{-x}-\delta_q^{-2x}}{(\delta_q+\delta_q^{-1})^2} \\ &= \frac{4}{(\delta_q+\delta_q^{-1})^2} = \frac{4}{(1+q^{n-1})^2+4q^{n-2}}. \end{aligned}$$

Theorem 2.(*q-Pell Sum and Difference) Let* $\sin P_q h(x)$ *and* $\cos P_q h(x)$ *be two functions of q-analogue of Pell hyperbolic function. For* $n \geq 2$, $x, y \in \mathbb{R}$

$$\begin{aligned} &\cos P_q h(x+y) \\ &= \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\cos P_q h(x)\cos P_q h(y) \\ &+ \sin P_q h(x)\sin P_q h(y)) \end{aligned} \tag{13}$$

$$\begin{aligned} &\cos P_q h(x-y) \\ &= \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\cos P_q h(x)\cos P_q h(y) \\ &- \sin P_q h(x)\sin P_q h(y)) \end{aligned} \tag{14}$$

$$\begin{aligned} &\sin P_q h(x+y) \\ &= \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\sin P_q h(x)\cos P_q h(y) \\ &+ \sin P_q h(y)\cos P_q h(x)) \end{aligned} \tag{15}$$

$$\begin{aligned} &\sin P_q h(x-y) \\ &= \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\sin P_q h(x)\cos P_q h(y) \\ &- \sin P_q h(y)\cos P_q h(x)). \end{aligned} \tag{16}$$

Proof.(13) the correlation similar to the equation $\cos Ph(x+y) = \sqrt{2}(\cos Ph(x).\cos Ph(y) + \sin Ph(x).\sin Ph(y))$ is valid for the q-analogue of the Pell hyperbolic functions. We only give the proof of the (13), because the proof of (14)-(16) is similar.

$$\begin{aligned} &\cos P_q h(x).\cos P_q h(y) + \sin P_q h(x).\sin P_q h(y) \\ &= \left(\frac{\delta_q^x+\delta_q^{-x}}{\delta_q+\delta_q^{-1}}\right)\left(\frac{\delta_q^y+\delta_q^{-y}}{\delta_q+\delta_q^{-1}}\right) + \left(\frac{\delta_q^x-\delta_q^{-x}}{\delta_q+\delta_q^{-1}}\right)\left(\frac{\delta_q^y-\delta_q^{-y}}{\delta_q+\delta_q^{-1}}\right) \\ &= \frac{\delta_q^{x+y}+\delta_q^{x-y}+\delta_q^{-x+y}+\delta_q^{-x-y}+\delta_q^{x+y}-\delta_q^{x-y}-\delta_q^{-x+y}+\delta_q^{-x-y}}{(\delta_q+\delta_q^{-1})^2} \\ &= \frac{2(\delta_q^{x+y}+\delta_q^{-x-y})}{(\delta_q+\delta_q^{-1})^2} = \frac{2}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}\cos P_q h(x+y). \end{aligned}$$

By setting $x = y$ in the (13) and (15), we have the following corollary.

Corollary 1.(*q-Pell Double Argument*)

$$\begin{aligned} \cos P_q h(2x) &= \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2} \\ &\times \left((\cos P_q h(x))^2 + (\sin P_q h(x))^2\right) \end{aligned} \tag{17}$$

$$\begin{aligned} &\sin P_q h(2x) \\ &= \sqrt{(1+q^{n-1})^2+4q^{n-2}}(\sin P_q h(x)\cos P_q h(x)) \end{aligned} \tag{18}$$

From (17) and (12), we obtain the following corollary.

Corollary 2.(*q-Pell Half Argument) For* $n \geq 2$

$$\begin{aligned} [\cos P_q h(x)]^2 &= \frac{1}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}\left(\cos P_q h(2x)\right. \\ &\left. + \frac{2}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}\right) \end{aligned} \tag{19}$$

$$\begin{aligned} [\sin P_q h(x)]^2 &= \frac{1}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}\left(\cos P_q h(2x)\right. \\ &\left. - \frac{2}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}\right). \end{aligned} \tag{20}$$

*Proof.*Similarly, we only give proof the identity (19), because the proof of the identity (20) is similar. We start with the identity (17)

$$\begin{aligned} \cos P_q h(2x) &= \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}((\cos P_q h(x))^2 \\ &+ (\sin P_q h(x))^2) \\ \cos P_q h(2x) &= \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\cos P_q h(x))^2 \\ &+ \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\sin P_q h(x))^2. \end{aligned}$$

$$\begin{aligned} &\frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\cos P_q h(x))^2 \\ &= \cos P_q h(2x) \\ &- \frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\sin P_q h(x))^2 \\ (\cos P_q h(x))^2 &= \frac{2}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}(\cos P_q h(2x) \\ &- (\sin P_q h(x))^2) \end{aligned}$$

Using the formula (12)

$$(\cos P_q h(x))^2 = \frac{2}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}} \cos P_q h(2x) - (\cos P_q h(x))^2 + \frac{4}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}$$

$$(\cos P_q h(x))^2 = \frac{1}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}} \cos P_q h(2x) + \frac{2}{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}$$

It is necessary that we give q-Catalan's recurrence to obtain q-Catalan's identity of q-Pell hyperbolic functions. However, for $n \in \mathbb{N}$ and $r \in \mathbb{Z}$ we first recall Catalan's identity for Pell numbers stated in [2].

$$P_{n-r}P_{n+r} - P_n^2 = (-1)^{n+1-r} P_r^2.$$

Lemma 1. *q-Catalan's identity for the q-Pell numbers is given by*

$$P_{n-r}(q)P_{n+r}(q) - P_n^2(q) = (-1)^{n+1-r} P_r^2(q).$$

For the q-Pell hyperbolic functions we have the following result:

Theorem 3. *(q-Pell Catalan's Identity)*

$$\cos P_q h(x-r) \cos P_q h(x+r) - (\cos P_q h(x))^2 = (\sin P_q h(r))^2. \tag{21}$$

Proof. By definitions (10) and (11) of q-Pell hyperbolic functions

$$\begin{aligned}
&\cos P_q h(x-r) \cos P_q h(x+r) - (\cos P_q h(x))^2 \\
&= \left(\frac{\delta_q^{x-r}+\delta_q^{-(x-r)}}{\delta_q+\delta_q^{-1}}\right)\left(\frac{\delta_q^{x+r}+\delta_q^{-(x+r)}}{\delta_q+\delta_q^{-1}}\right) - \left(\frac{\delta_q^{x}+\delta_q^{-x}}{\delta_q+\delta_q^{-1}}\right)^2 \\
&= \frac{\delta_q^{2x}+\delta_q^{-2r}+\delta_q^{2r}+\delta_q^{-2x}-\delta_q^{2x}-2-\delta_q^{-2x}}{\left(\delta_q+\delta_q^{-1}\right)^2} \\
&= \left(\frac{\delta_q^{r}-\delta_q^{-r}}{\delta_q+\delta_q^{-1}}\right)^2 \\
&= \frac{(\delta_q^{r}-\delta_q^{-r})^2}{(1+q^{n-1})^2+4q^{n-2}}, n \geqslant 2 \\
&= (\sin P_q h(r))^2.
\end{aligned}$$

The following identities can be obtained in a similar way.

Corollary 3.

$$\cos P_q h(x-r) \cos P_q h(x+r) - (\sin P_q h(x))^2 = (\cos P_q h(r))^2 \tag{22}$$

$$\sin P_q h(x-r) \sin P_q h(x+r) - (\sin P_q h(x))^2 = -(\sin P_q h(r))^2 \tag{23}$$

$$\sin P_q h(x-r) \sin P_q h(x+r) - (\cos P_q h(x))^2 = -(\cos P_q h(r))^2. \tag{24}$$

From Lemma 1, by setting $r = 1$ into q-Catalan's identity, it is straightforwardly obtained q-Cassini's or Simson's identity for the q-Pell numbers:

$$P_{n-1}(q)P_{n+1}(q) - P_n^2(q) = (-1)^n.$$

The corresponding identity for the q-Pell hyperbolic functions is as follows:

Proposition 1. *(q-Pell Cassini's or Simson's Identity)*

$$\cos P_q h(x-1) \cos P_q h(x+1) - (\sin P_q h(x))^2 = 1,$$

$$\sin P_q h(x-1) \sin P_q h(x+1) - (\cos P_q h(x))^2 = -1.$$

Proof. Setting $r = 1$ in (22) and (24), we have

$$\cos P_q h(x-1) \cos P_q h(x+1) - (\sin P_q h(x))^2 = 1$$

and

$$\sin P_q h(x-1) \sin P_q h(x+1) - (\cos P_q h(x))^2 = -1.$$

It is necessary that we give q-d'Ocagne's recurrence to obtain q-d'Ocagne's identity of q-Pell hyperbolic functions. However, for $m,n \in \mathbb{N}$, and $m > n$ we first provide d'Ocagne's identity for Pell numbers stated in [2].

$$P_m P_{n+1} - P_{m+1} P_n = (-1)^n P_{m-n}.$$

q-d'Ocagne's identity for the q-Pell number is

$$P_m(q)P_{n+1}(q) - P_{m+1}(q)P_n(q) = (-1)^n P_{m-n}(q)$$

Theorem 4. *(q-Pell d'Ocagne's Identity)*

$$\cos P_q h(x) \cos P_q h(y+r) - \sin P_q h(x+r) \sin P_q h(y) = \cos P_q h(r) \cos P_q h(x-y) \tag{25}$$

$$\cos P_q h(x) \sin P_q h(y+r) - \cos P_q h(x+r) \sin P_q h(y) = \sin P_q h(r) \cos P_q h(x-y) \tag{26}$$

Proof. We only give proof the identity (25), because the proof of the identity (26) is similar.

$$\begin{aligned}
&\cos P_q h(x)\cos P_q h(y+r) - \sin P_q h(x+r)\sin P_q h(y)\\
&= \left(\frac{\delta_q^x+\delta_q^{-x}}{\delta_q+\delta_q^{-1}}\right)\left(\frac{\delta_q^{y+r}+\delta_q^{-(y+r)}}{\delta_q+\delta_q^{-1}}\right)\\
&- \left(\frac{\delta_q^{x+r}-\delta_q^{-(x+r)}}{\delta_q+\delta_q^{-1}}\right)\left(\frac{\delta_q^{y}-\delta_q^{-y}}{\delta_q+\delta_q^{-1}}\right)\\
&= \frac{\delta_q^{x-y-r}+\delta_q^{-x+y+r}+\delta_q^{x-y+r}+\delta_q^{-x+y-r}}{\left(\delta_q+\delta_q^{-1}\right)^2}\\
&= \frac{\delta_q^{-r}(\delta_q^{x-y}+\delta_q^{-(x-y)})+\delta_q^{r}(\delta_q^{x-y}+\delta_q^{-(x-y)})}{\left(\delta_q+\delta_q^{-1}\right)^2}\\
&= \left(\frac{\delta_q^r+\delta_q^{-r}}{\delta_q+\delta_q^{-1}}\right)\left(\frac{\delta_q^{x-y}+\delta_q^{-(x-y)}}{\delta_q+\delta_q^{-1}}\right)\\
&= \frac{\left(\delta_q^r+\delta_q^{-r}\right)\left(\delta_q^{x-y}+\delta_q^{-(x-y)}\right)}{(\delta_q+\delta_q^{-1})^2}\\
&= \frac{\left(\delta_q^r+\delta_q^{-r}\right)\left(\delta_q^{x-y}+\delta_q^{-(x-y)}\right)}{(1+q^{n-1})^2+4q^{n-2}}, n \geqslant 2\\
&= \cos P_q h(r)\cos P_q h(x-y).
\end{aligned}$$

Notice that by setting $r = 1$ in equation (25), we obtain the following identity.

Corollary 4.

$$\cos P_q h(x)\cos P_q h(y+1) - \sin P_q h(x+1)\sin P_q h(y) = \cos P_q h(x-y).$$

4 Conclusion

The formulas of classical hyperbolic and Pell hyperbolic identities that are already present:

Classical Hyperbolic Functions
$[\cosh(x)]^2-[\sinh(x)]^2=1$
$\cosh(x+y)=\cosh(x)\cosh(y)+\sinh(x)\sinh(y)$
$\cosh(x-y)=\cosh(x)\cosh(y)-\sinh(x)\sinh(y)$
$\sinh(x+y)=\sinh(x)\cosh(y)+\sinh(y)\cosh(x)$
$\sinh(x-y)=\sinh(x)\cosh(y)-\sinh(y)\cosh(x)$
$\cosh(2x)=[\cosh(x)]^2+[\sinh(x)]^2$
$\sinh(2x)=2.\sinh(x)\cosh(x)$

Pell Hyperbolic Functions
$[\cos Ph(x)]^2-[\sin Ph(x)]^2=\frac{1}{2}$
$\cos Ph(x+y)=\sqrt{2}(\cos Ph(x)\cos Ph(y)+\sin Ph(x)\sin Ph(y))$
$\cos Ph(x-y)=\sqrt{2}(\cos Ph(x)\cos Ph(y)-\sin Ph(x)\sin Ph(y))$
$\sin Ph(x+y)=\sqrt{2}(\sin Ph(x)\cos Ph(y)+\sin Ph(y)\cos Ph(x))$
$\sin Ph(x-y)=\sqrt{2}(\sin Ph(x)\cos Ph(y)-\sin Ph(y)\cos Ph(x))$
$\cos Ph(2x)=\sqrt{2}([\cos Ph(x)]^2+[\sin Ph(x)]^2)$
$\sin Ph(2x)=2\sqrt{2}(\sin Ph(x).\cos Ph(x))$

Some identities given throughout the paper by benefitting from the formulas above have been listed as follows:

q-Pell Hyperbolic Functions (for $n \geq 2$)
$[\cos P_q h(x)]^2-[\sin P_q h(x)]^2=\frac{4}{(1+q^{n-1})^2+4q^{n-2}}$
$\cos P_q h(x+y)=\frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\cos P_q h(x)\cos P_q h(y)+\sin P_q h(x)\sin P_q h(y))$
$\cos P_q h(x-y)=\frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\cos P_q h(x)\cos P_q h(y)-\sin P_q h(x)\sin P_q h(y))$
$\sin P_q h(x+y)=\frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\sin P_q h(x)\cos P_q h(y)+\sin P_q h(y)\cos P_q h(x))$
$\sin P_q h(x-y)=\frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}(\sin P_q h(x)\cos P_q h(y)-\sin P_q h(y)\cos P_q h(x))$
$\cos P_q h(2x)=\frac{\sqrt{(1+q^{n-1})^2+4q^{n-2}}}{2}\left((\cos P_q h(x))^2+(\sin P_q h(x))^2\right)$
$\sin P_q h(2x)=\sqrt{(1+q^{n-1})^2+4q^{n-2}}(\sin P_q h(x)\cos P_q h(x))$

New q-Pell hyperbolic functions have been introduced and studied here. These new functions are naturally related with the k-Fibonacci sequences. Several properties of these functions have been deduced and related with Pell hyperbolic functions and q-analogues of them. In addition, the q-Pell hyperbolic functions are used to introduce the so-called Silver Ratio which extend the Golden Ratio.

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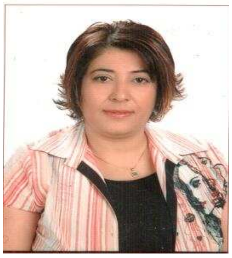

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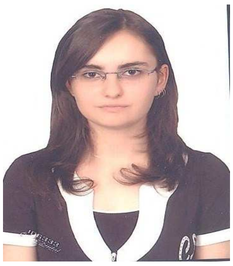

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