Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/050104

On Prime bi-k-Ideals of a Ternary Semiring

Swapnil Wani[1] *and Kishor Pawar*[2,*]

[1] University Institute of Chemical Technology, North Maharashtra University, Jalgaon - 425 001, India
[2] Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon - 425 001, India

Received: 14 Jun. 2017, Revised: 6 Jul. 2017, Accepted: 8 Jul. 2017
Published online: 1 Jan. 2018

Abstract: In this paper we define and study the concepts of prime bi-k-ideals, strongly prime bi-k-ideals, semiprime bi-k-ideals, irreducible bi-k-ideals and strongly irreducible bi-k-ideals of a ternary semiring using [2], [3], [7] and [8]. We generalize the concepts of fully idempotent semiring introduced by Ahsan [1] to a fully bi-k-idempotent ternary semiring using [2] and [7]. We also prove that if the set of bi-k-ideals of a ternary semiring S is totally ordered under the set inclusion, then S is fully bi-k-idempotent if and only if each bi-k-ideal of S is prime.

Keywords: Ternary Semiring, prime Ideal, bi-Ideal, quasi-Ideal, etc.

1 Introduction

In 2003, Dutta and Kar [4] introduced the notion of a ternary semiring which generalizes the notion of ternary ring. Later in 2005, they introduced the notions of prime ideal, semiprime ideal, irreducible ideal of a ternary semiring in [5] and [6]. In 2005, Kar [8] introduced the notions of quasi-ideal and bi-ideal in ternary semirings and study some properties of these two ideals. Then the notions of quasi-k-ideals and bi-k-ideals of a ternary semiring were introduced by Dubey [3] in 2011. The concepts of prime, strongly prime, semiprime, irreducible and strongly irreducible bi-ideals of a ternary semiring and their properties has been given in [2] by Bashir, Mehmood and Kamran in 2013. In 2014, Jagtap [7] defined prime, strongly prime, semiprime, irreducible and strongly irreducible k-bi-ideals of a gamma semiring and also study some of their properties. In 2016, Pawar and Wani [9] introduced the notion of full k-ideal of a ternary semiring and proved that the set of all full k-ideals of a ternary semiring is a complete lattice which is also modular. Also in 2017, they defined the notions of essential ideal, semi essential ideal, weak essential ideal of a ternary semiring and study their properties in [10].

In the present paper we define and study the concepts of prime, semiprime and irreducible bi-k-ideals of a ternary semiring using [2], [3], [7] and [8]. Also we generalize the concepts of fully idempotent semiring introduced by Ahsan [1] to a fully bi-k-idempotent ternary semiring using [2] and [7].

2 Preliminary Definitions

Definition 1. *[4] A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:*

(i) $(abc)de = a(bcd)e = ab(cde)$ *(Associative Law)*
(ii) $(a+b)cd = acd + bcd$ *(Right Distributive Law)*
(iii) $a(b+c)d = abd + acd$ *(Lateral Distributive Law)*
(iv) $ab(c+d) = abc + abd$ *(Left Distributive Law) for all* $a,b,c,d,e \in T$.

Example 1. [4] Let $\mathbb{Z}_0^-$ be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, $\mathbb{Z}_0^-$ forms a ternary semiring.

Definition 2. *[4] An additive subsemigroup T of a ternary semiring S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 3. *[4] An additive subsemigroup I of a ternary semiring S is called*

(i) A left ideal of S if $SSI \subseteq I$

* Corresponding author e-mail: kfpawar@nmu.ac.in

(ii) *A lateral ideal of S if $SIS \subseteq I$*
(iii) *A right ideal of S if $ISS \subseteq I$*
(iv) *A two sided ideal of S if I is both left and right ideal of S*
(v) *An ideal of S if I is a left, a right and a laterl ideal of S.*

Definition 4. *[4] A right ideal I of a ternary semiring S is said to be a right k-ideal if for $a \in I, a+b \in I, b \in S$ imply $b \in I$.*

Similarly we can define a left k-ideal and a lateral k-ideal of a ternary semiring S. If an ideal I is right, lateral and left k-ideal, then I is called as k-ideal of S.

Proposition 1. *[4] The intersection of an arbitrary collection of k-ideals of a ternary semiring S is again a k-ideal of S.*

Definition 5. *[4] Let A be an ideal of a ternary semiring S. Then the k-closure of A, denoted by $\overline{A}$, is define by $\overline{A} = \{a \in S : a+b = c for some b,\ c \in A\}$.*

Definition 6. *[4] A ternary semiring S is said to be regular if for each element a in S there exists an element x in S such that $a = axa$. If the element x is unique and satisfies $x = xax$, then S is called an inverse ternary semiring. An element x is called the inverse of a.*

Definition 7. *[4] An element a in a ternary semiring S is called an idempotent if $aaa = a$ that is $a^3 = a$. And if each element of S is idempotent, then S is called an idempotent ternary semiring.*

Definition 8. *[5] A proper ideal P of a ternary semiring S is called a prime ideal of S if $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for any three ideals A, B, C of S.*

Definition 9. *[6] A proper ideal Q of a ternary semiring S is called a semiprime ideal of S if $I^3 \subseteq Q$ implies $I \subseteq Q$ for any ideal I of S.*

Definition 10. *[6] A proper ideal I of a ternary semiring S is said to be strongly irreducible if for ideals H and K of S, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$.*

Definition 11. *[8] A ternary subsemiring B of a ternary semiring S is called a bi-ideal of S, if $BSBSB \subseteq B$.*

Definition 12. *[3] A ternary subsemiring B of a ternary semiring S is called a bi-k-ideal of S, if $\overline{BSBSB} \subseteq B$.*

Definition 13. *[2] A bi-ideal B of a ternary semiring S is called a prime bi-ideal of S, if $B_1B_2B_3 \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of S.*

Definition 14. *[2] A bi-ideal B of a ternary semiring S is called a strongly prime bi-ideal of S, if $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of S.*

Definition 15. *[2] A bi-ideal B of a ternary semiring S is called a semiprime bi-ideal of S, if $B_1^3 \subseteq B$ implies that $B_1 \subseteq B$ for any bi-ideal B_1 of S.*

Proposition 2. *[2] The intersection of any family of prime bi-ideals of a ternary semiring S is a semiprime bi-ideal of S.*

Definition 16. *[2] A bi-ideal B of a ternary semiring S is called an irreducible bi-ideal of S, if $B_1 \cap B_2 \cap B_3 = B$ implies that $B_1 = B$ or $B_2 = B$ or $B_3 = B$ for any bi-ideals B_1, B_2, B_3 of S.*

Definition 17. *[2] A bi-ideal B of a ternary semiring S is called a strongly irreducible bi-ideal of S, if $B_1 \cap B_2 \cap B_3 \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of S.*

3 Prime, Strongly Prime and Semiprime Bi-k-ideals of a Ternary Semiring

Definition 18. *A bi-k-ideal B of a ternary semiring S is called a prime bi-k-ideal of S, if $\overline{B_1B_2B_3} \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-k-ideals B_1, B_2, B_3 of S.*

Definition 19. *A bi-k-ideal B of a ternary semiring S is called a strongly prime bi-k-ideal of S, if $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-k-ideals B_1, B_2, B_3 of S.*

Definition 20. *A bi-k-ideal B of a ternary semiring S is called a semiprime bi-k-ideal of S, if $\overline{B_1B_1B_1} = \overline{B_1^3} \subseteq B$ implies that $B_1 \subseteq B$ for any bi-k-ideal B_1 of S.*

Proposition 3. *Every strongly prime bi-k-ideal of a ternary semiring S is a prime bi-k-ideal of S.*

Proposition 4. *Every prime bi-k-ideal of a ternary semiring S is a semiprime bi-k-ideal of S.*

Remark. A prime bi-k-ideal of a ternary semiring is not necessarily a strongly prime bi-k-ideal and a semiprime bi-k-ideal of ternary semiring is not necessarily a prime bi-k-ideal.

Proposition 5. *The intersection of any family of prime bi-k-ideals (semiprime bi-k-ideals) of a ternary semiring S is a semiprime bi-k-ideal of S.*

Corollary 1. *If B_1, B_2, B_3 be any prime bi-k-ideals of a regular ternary semiring S, then $B_1B_2B_3$ is a semiprime bi-k-ideal of S if and only if $\overline{B_1B_2B_3} = B_1B_2B_3$.*

4 Irreducible and Strongly Irreducible Bi-k-ideals of a Ternary Semiring

Definition 21. *A bi-k-ideal B of a ternary semiring S is called an irreducible bi-k-ideal of S, if $B_1 \cap B_2 \cap B_3 = B$ implies that $B_1 = B$ or $B_2 = B$ or $B_3 = B$ for any bi-k-ideals B_1, B_2, B_3 of S.*

Definition 22. *A bi-k-ideal B of a ternary semiring S is called a strongly irreducible bi-k-ideal of S, if $B_1 \cap B_2 \cap B_3 \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-k-ideals B_1, B_2, B_3 of S.*

Proposition 6. *Every strongly irreducible semiprime bi-k-ideal of a ternary semiring S is a strongly prime bi-k-ideal of S.*

Proposition 7. *Let B be any bi-k-ideal of a ternary semiring S such that $a \in S$ and $a \notin B$. Then there exists an irreducible bi-k-ideal I of S such that $B \subseteq I$ and $a \notin I$.*

Proposition 8. *Any proper bi-k-ideal B of a ternary semiring S is the intersection of all irreducible bi-k-ideals of S containing B.*

Proposition 9. *A prime bi-k-ideal of S is either a prime right k-ideal or a prime lateral k-ideal or a prime left k-ideal of S.*

Proposition 10. *A bi-k-ideal B of S is prime if and only if for a right k-ideal R, a lateral k-ideal M and a left k-ideal L of S, $\overline{RML} \subseteq B$ implies $R \subseteq B$ or $M \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that a bi-k-ideal B of S is prime. Let R be a right k-ideal, M be a lateral k-ideal and L be a left k-ideal of S such that $\overline{RML} \subseteq B$. Therefore R, M and L are itself bi-k-ideals of S. Hence $R \subseteq B$ or $M \subseteq B$ or $L \subseteq B$. Conversely, we have to show that a bi-k-ideal B of S is prime in S. Let B_1, B_2, B_3 be any three bi-k-ideals of S such that $\overline{B_1B_2B_3} \subseteq B$. For any $b_1 \in B_1, b_2 \in B_2$ and $b_3 \in B_3$, $\overline{(b_1)_r} \subseteq B_1$, $\overline{(b_2)_m} \subseteq B_2$ and $\overline{(b_3)_l} \subseteq B_3$, where $\overline{(b_1)_r}, \overline{(b_2)_m}$ and $\overline{(b_3)_l}$ denotes the right k-ideal, lateral k-ideal and left k-ideal generated by b_1, b_2 and b_3 respectively. Therefore $\overline{\overline{(b_1)_r}\,\overline{(b_2)_m}\,\overline{(b_3)_l}} \subseteq \overline{B_1B_2B_3} \subseteq B$. So by assumption $\overline{(b_1)_r} \subseteq B$ or $\overline{(b_2)_m} \subseteq B$ or $\overline{(b_3)_l} \subseteq B$. Thus either $b_1 \in B$ or $b_2 \in B$ or $b_3 \in B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is a prime bi-k-ideal of S.

Theorem 1. *Let S be a ternary semiring. Then the following statements are equivalent:*

1. *The set of bi-k-ideals of S is a totally ordered under the set inclusion.*
2. *Each bi-k-ideal of S is strongly irreducible.*
3. *Each bi-k-ideal of S is irreducible.*

Proof. (1) $\Rightarrow$ (2): Let the set of bi-k-ideals of S be a totally ordered under the set inclusion. Consider B be any bi-k-ideal of S. Let B_1, B_2, B_3 be any three bi-k-ideals of S such that $B_1 \cap B_2 \cap B_3 \subseteq B$. But by assumption, either $B_1 \cap B_2 \cap B_3 = B_1$ or $B_1 \cap B_2 \cap B_3 = B_2$ or $B_1 \cap B_2 \cap B_3 = B_3$. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly irreducible bi-k-ideal of S.

(2) $\Rightarrow$ (3): Let each bi-k-ideal of S be a strongly irreducible. Consider B be any bi-k-ideal of S. Let B_1, B_2, B_3 be any three bi-k-ideals of S such that $B_1 \cap B_2 \cap B_3 = B$. This implies $B \subseteq B_1, B \subseteq B_2$ and $B \subseteq B_3$. But by assumption, either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Therefore either $B_1 = B$ or $B_2 = B$ or $B_3 = B$. Hence B is an irreducible bi-k-ideal of S.

(3) $\Rightarrow$ (1): Let each bi-k-ideal of S be an irreducible. Let B_1, B_2 be any two bi-k-ideals of S. Then by proposition 5, $B_1 \cap B_2$ is irreducible bi-k-ideal of S. Now since $B_1 \cap B_2 \cap S = B_1 \cap B_2$ implies $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$ or $S = B_1 \cap B_2$ implies either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 = B_2 = S$. This shows that the set of bi-k-ideals of S is a totally ordered under the set inclusion.

5 Fully bi-k-idempotent Ternary Semiring

In this section we generalize the concepts of fully idempotent semiring introduced by Ahsan in [1] to a fully bi-k-idempotent ternary semiring using [2] and [7].

Definition 23. *A ternary semiring S is said to be fully bi-k-idempotent if every bi-k-ideal of S is k-idempotent. That is S is said to be fully bi-k-idempotent if for any bi-k-ideal B of S, $\overline{B^3} = \overline{BBB} = B$.*

Theorem 2. *Let S be a ternary semiring. Then the following statements are equivalent:*

1. *S is fully bi-k-idempotent.*
2. *$\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} = B_1 \cap B_2 \cap B_3$ for any bi-k-ideals B_1, B_2, B_3 of S.*
3. *Each bi-k-ideal of S is semiprime.*
4. *Each proper bi-k-ideal of S is the intersection of irreducible semiprime bi-k-ideals of S which contain it.*

Proof. (1) $\Rightarrow$ (2): Let B_1, B_2, B_3 be any three bi-k-ideals of S. Then by proposition 5, $B_1 \cap B_2 \cap B_3$ is also a bi-k-ideal of S. By assumption, we have

$$\begin{aligned} B_1 \cap B_2 \cap B_3 &= \overline{(B_1 \cap B_2 \cap B_3)^3} \\ &= \overline{(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)} \\ &\subseteq \overline{B_1B_2B_3}. \end{aligned}$$

Similarly,

$$B_1 \cap B_2 \cap B_3 \subseteq \overline{B_2B_3B_1}$$

and

$$B_1 \cap B_2 \cap B_3 \subseteq \overline{B_3B_1B_2}.$$

Therefore

$$B_1 \cap B_2 \cap B_3 \subseteq \overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2}.$$

Now $\overline{B_1B_2B_3}, \overline{B_2B_3B_1}$ and $\overline{B_3B_1B_2}$ are bi-k-ideals of S. So by proposition 5, $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2}$ is also a bi-k-ideal of S. Therefore by assumption,

$$\begin{aligned}\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} &= \overline{(\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2})^3}\\ &\subseteq \overline{(\overline{B_1B_2B_3})(\overline{B_3B_1B_2})(\overline{B_2B_3B_1})}\\ &\subseteq \overline{(B_1SS)(SB_1S)(SSB_1)}\\ &= \overline{B_1(SSS)B_1(SSS)B_1}\\ &= \overline{B_1SB_1SB_1}\\ &\subseteq B_1.\end{aligned}$$

Similarly,

$$\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B_2$$

and

$$\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B_3.$$

Thus

$$\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B_1 \cap B_2 \cap B_3.$$

Hence

$$\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} = B_1 \cap B_2 \cap B_3.$$

(2) $\Rightarrow$ (3): Let B be any bi-k-ideal of S. Suppose for any bi-k-ideal B_1 of S, $\overline{B_1^3} = \overline{B_1B_1B_1} \subseteq B$. By assumption, we have

$$\begin{aligned}B_1 &= B_1 \cap B_1 \cap B_1\\ &= \overline{B_1B_1B_1} \cap \overline{B_1B_1B_1} \cap \overline{B_1B_1B_1}\\ &= \overline{B_1B_1B_1}\\ &\subseteq B.\end{aligned}$$

Hence each bi-k-ideal of S is semiprime.

(3) $\Rightarrow$ (4): Let B be any proper bi-k-ideal of S. Therefore by proposition 8, B is the intersection of all irreducible bi-k-ideals of S containing B. By assumption, each bi-k-ideal of S is semiprime. Hence each proper bi-k-ideal of S is the intersection of irreducible semiprime bi-k-ideals of S which contain it.

(4) $\Rightarrow$ (1): Let B be any bi-k-ideal of S. If $\overline{B^3} = S$, then $S \subseteq \overline{B^3}$ implies $B \subseteq S \subseteq \overline{B^3}$. Also $\overline{B^3} \subseteq S$. Therefore $\overline{B^3} = B$, for each bi-k-ideal B of S. Now if $\overline{B^3}$ is proper bi-k-ideal of S, that is $\overline{B^3} \neq S$. Then by assumption, $\overline{B^3}$ is the intersection of irreducible semiprime bi-k-ideals of S which contain it. That is $\overline{B^3} = \cap\{B_i : B_i$ is irreducible semiprime bi-k-ideal of $S\}$. As each B_i is a semiprime bi-k-ideal of S, $B \subseteq B_i$, for all i. Therefore $B \subseteq \cap B_i = \overline{B^3}$. Also $\overline{B^3} \subseteq B$. Thus $\overline{B^3} = B$. Hence S is fully bi-k-idempotent.

Theorem 3. *If a ternary semiring S is fully bi-k-idempotent, then a bi-k-ideal B of S is strongly irreducible if and only if B is strongly prime.*

Proof. Let S be a fully bi-k-idempotent ternary semiring. Consider B be a strongly irreducible bi-k-ideal of S. Let B_1, B_2, B_3 be any three bi-k-ideals of S such that $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B$. By theorem 2, $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} = B_1 \cap B_2 \cap B_3$. Thus $B_1 \cap B_2 \cap B_3 \subseteq B$. But B is a strongly irreducible bi-k-ideal of S. Therefore either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly prime bi-k-ideal of S. Conversely suppose that B be a strongly prime bi-k-ideal of a fully bi-k-idempotent ternary semiring S. Let B_1, B_2, B_3 be any three bi-k-ideals of S such that $B_1 \cap B_2 \cap B_3 \subseteq B$. By theorem 2, $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} = B_1 \cap B_2 \cap B_3 \subseteq B$. As B is strongly prime bi-k-ideal of S, we have either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly irreducible bi-k-ideal of S.

Theorem 4. *Each bi-k-ideal of a ternary semiring S is strongly prime if and only if S is fully bi-k-idempotent and the set of bi-k-ideals of S is totally ordered under the set inclusion.*

Proof. Suppose that each bi-k-ideal of a ternary semiring S is strongly prime bi-k-ideal. Then each bi-k-ideal of S is a semiprime bi-k-ideal. Therefore by theorem 2, S is fully bi-k-idempotent. Now, let B_1 and B_2 be any two bi-k-ideals of S. Then by theorem 2, we have $B_1 \cap B_2 = B_1 \cap B_2 \cap S = \overline{B_1B_2S} \cap \overline{B_2SB_1} \cap \overline{SB_1B_2} = B_1 \cap B_2 \subseteq B_1 \cap B_2$. Also by assumption, $B_1 \cap B_2$ is strongly prime bi-k-ideal of S. Therefore either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ or $S \subseteq B_1 \cap B_2$. Thus $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-k-ideals of S is totally ordered under the set inclusion. Conversely, suppose that S be a fully bi-k-idempotent and the set of bi-k-ideals of S is totally ordered under the set inclusion. Let B be any bi-k-ideal of S and let B_1, B_2, B_3 be any three bi-k-ideals of S such that $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B$. By theorem 2, we have

$$B_1 \cap B_2 \cap B_3 = \overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B \qquad (1)$$

Since the set of bi-k-ideals of S is totally ordered under the set inclusion, so for B_1, B_2, B_3 we have following six possibilities:

(i) $B_1 \subseteq B_2 \subseteq B_3$
(ii) $B_1 \subseteq B_3 \subseteq B_2$
(iii) $B_2 \subseteq B_3 \subseteq B_1$
(iv) $B_2 \subseteq B_1 \subseteq B_3$
(v) $B_3 \subseteq B_1 \subseteq B_2$
(vi) $B_3 \subseteq B_2 \subseteq B_1$

In these cases we have

(i) $B_1 \cap B_2 \cap B_3 = B_1$
(ii) $B_1 \cap B_2 \cap B_3 = B_1$
(iii) $B_1 \cap B_2 \cap B_3 = B_2$
(iv) $B_1 \cap B_2 \cap B_3 = B_2$
(v) $B_1 \cap B_2 \cap B_3 = B_3$
(vi) $B_1 \cap B_2 \cap B_3 = B_3$

Thus equation 1 gives either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly prime bi-k-ideal of S.

Theorem 5. *If the set of bi-k-ideals of a ternary semiring S is totally ordered under the set inclusion, then S is fully bi-k-idempotent if and only if each bi-k-ideal of S is prime.*

Proof. Let the set of bi-k-ideals of a ternary semiring S is totally ordered under the set inclusion and suppose that S be a fully bi-k-idempotent. Let B be any arbitrary bi-k-ideal of S. And let B_1, B_2, B_3 be any three bi-k-ideals of S such that $B_1B_2B_3 \subseteq B$. Then by assumption, for B_1, B_2, B_3 we have the following six possibilities:

(i) $B_1 \subseteq B_2 \subseteq B_3$
(ii) $B_1 \subseteq B_3 \subseteq B_2$
(iii) $B_2 \subseteq B_3 \subseteq B_1$
(iv) $B_2 \subseteq B_1 \subseteq B_3$
(v) $B_3 \subseteq B_1 \subseteq B_2$
(vi) $B_3 \subseteq B_2 \subseteq B_1$

From i) and ii), we have $\overline{B_1^3} = \overline{B_1B_1B_1} \subseteq \overline{B_1B_2B_3} \subseteq B$ implies $B_1 \subseteq B$, implies $B_1 \subseteq B$, as B is bi-k-ideal of S. Similarly, we have $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is prime bi-k-ideal of S. Conversely, suppose that each bi-k-ideal of S is prime, so by proposition , it is semiprime bi-k-ideal of S. Therefore by theorem 2, S is fully bi-k-idempotent.

Theorem 6. *If the set of bi-k-ideals of a ternary semiring S is totally ordered under the set inclusion, then the concepts of primeness and strongly primeness coincide.*

Proof. Suppose that B be any arbitrary bi-k-ideal of S. Let B_1, B_2, B_3 be any three bi-k-ideals of S such that $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B$. Since the set of bi-$k$-ideals of a ternary semiring S is totally ordered under the set inclusion, then for B_1, B_2, B_3 we have the following six possibilities:

(i) $B_1 \subseteq B_2 \subseteq B_3$
(ii) $B_1 \subseteq B_3 \subseteq B_2$
(iii) $B_2 \subseteq B_3 \subseteq B_1$
(iv) $B_2 \subseteq B_1 \subseteq B_3$
(v) $B_3 \subseteq B_1 \subseteq B_2$
(vi) $B_3 \subseteq B_2 \subseteq B_1$

From i) and ii), we have $\overline{B_1^3} = \overline{B_1^3} \cap \overline{B_1^3} \cap \overline{B_1^3} \subseteq \overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B$, implies $B_1 \subseteq B$, as B is bi-k-ideal of S. Similarly, we have $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly prime bi-k-ideal of S. Conversely, suppose that B is strongly prime bi-k-ideal of S. Let B_1, B_2, B_3 be any three bi-k-ideals of S such that $B_1B_2B_3 \subseteq B$. This implies $\overline{B_1B_2B_3} \cap \overline{B_2B_3B_1} \cap \overline{B_3B_1B_2} \subseteq B$, implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is prime bi-k-ideal of S.

6 Conclusion

In this paper we have defined prime bi-k-ideal, strongly prime bi-k-ideal, semiprime bi-k-ideal, irreducible bi-k-ideal and strongly irreducible bi-k-ideal of a ternary semiring. Also we generalized the concepts of fully idempotent semiring to a fully bi-k-idempotent ternary semiring.

Acknowledgement

The authors would like to express their sincere thanks to the North Maharashtra University, Jalgaon for their financial support under the Vice Chancellor Research Motivation Scheme. The authors are very thankful to anonymous referee(s) for their helpful comments that improved this paper.

References

[1] Javed Ahsan, *Fully Idempotent semirings*, Proc. Japan Acad., **32** (series A) (1956), 185-188.
[2] Shahida Bashir, Jabran Mehmood, Muhammad Sarwar Kamran, *Prime bi-ideals in ternary semirings*, Annals of Fuzzy Mathematics and Informatics, **6** (1) (2013), 181-189.
[3] M. K. Dubey, *A note on Quasi-k-ideals and bi-k-ideals in ternary semirings*, Italian J. of Pure and App. Math., **28** (2011), 143-150.
[4] T. K. Dutta and S. Kar, *On regular ternary semirings*, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific, (2003), 343-355.
[5] T. K. Dutta and S. Kar, *On Prime Ideals And Prime Radical of Ternary Semirings*, Bull. Cal. Math. Soc., **97** (5) (2005), 445-454.
[6] T. K. Dutta and S. Kar, *On Semiprime Ideals And Irreducible Ideals Of Ternary Semirings*, Bull. Cal. Math. Soc., **97** (5) (2005), 467-476.
[7] R. D. Jagatap, *Prime k-Bi-ideals in Γ-Semirings*, Palestine Journal of Mathematics, **3** (1) (2014), 489494.
[8] S. Kar, *On quasi-ideals and bi-ideals in ternary semirings*, Int. J. Math. Math. Sci., **18** (2005), 3015-3023.
[9] Kishor Pawar and Swapnil Wani, *On full k-ideals of a Ternary Semirings*, Journal of Advances in Mathematics, **12(1)**, 5805-5807, (2016).
[10] Swapnil Wani and Kishor Pawar, *On Essential ideals of a Ternary Semirings*, Sohag Journal of Mathematics, **4(3)**, 1-4, (2017).

Swapnil Wani working as a Lecturer at University Institute of Chemical Technology, North Maharashtra University, Jalgaon, India. He is pursuing his PhD degree in Mathematics at Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon, India under the guidance of Dr. Kishor Pawar. His research area is ternary semiring, radical theory.

Kishor Pawar is an Associate Professor in Mathematics at Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon, India. He has received the PhD degree in Mathematics from North Maharashtra University, Jalgaon, India. He is a referee of national and international journals of pure and applied mathematics. His main research interest in the field of semirings, ternary semirings, radical theory, graph theory.