

Bessel Moment Polynomials

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Abstract: We study the family of even moments of squared Bessel functions of the first kind,

$$\mathcal{P}_n(x) = \sum_{k=0}^{\infty} k^{2n} J_k^2(2x), \quad n \geq 1.$$

By means of the classical generating-function identity $\sum_{k=-\infty}^{\infty} J_k^2(a) e^{iky} = J_0(2a \sin y/2)$, we prove that each $\mathcal{P}_n$ is an even polynomial of degree $2n$ in x , or equivalently a polynomial $p_n(u)$ of degree n in the variable $u = x^2$. We derive an explicit formula for the coefficients $a_{n,m}$ and establish a three-term recurrence relation. The first few polynomials are

$$\begin{aligned} p_1(u) &= u, \\ p_2(u) &= 3u^2 + u, \\ p_3(u) &= 10u^3 + 15u^2 + u, \\ p_4(u) &= 35u^4 + 140u^3 + 63u^2 + u, \\ p_5(u) &= 126u^5 + 1050u^4 + 1470u^3 + 255u^2 + u. \end{aligned}$$

All formulas are verified numerically with tight truncation-error bounds.

Keywords: Schrödinger equation; polynomials; generalized Bessel functions; special functions; nonlinear coherent states.

1 Introduction

Bessel functions of the first kind, $J_k(z)$, are among the most ubiquitous special functions in mathematical physics. They emerge naturally in problems possessing circular or cylindrical symmetry, in the theory of diffraction, in signal processing, and in the quantum mechanics of periodic systems [1,2,3,4]. In particular, the propagation of classical electromagnetic fields in weakly coupled optical waveguide arrays is governed by discrete Schrödinger-type equations whose eigenfunctions are precisely Bessel functions of integer order [5,6]. This observation has fueled a productive analogy between classical light propagation and quantum electrodynamics: under appropriate conditions, the interaction between

quantized light and matter can be modelled classically by suitably engineered optical lattices [7,8,9,10,11].

A prominent example is the Jaynes-Cummings model [12,13] and its generalisations (the Rabi model, the Dicke model, anisotropic and driven variants, etc.), all of which can be simulated experimentally by arrays of evanescently coupled waveguides [14,15,16,17,18,19,20,21,22,23]. Atom-field interactions when only counter rotating terms are considered, *i.e.*, the anti-Jaynes-Cummings model [24,25] also may be models. Furthermore, time-dependent couplings, which correspond to bent or curved waveguide arrays, further broaden the scope of this analogy and allow one to mimic non-autonomous quantum evolutions [26,27].

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In the simplest infinite-array geometry with constant nearest-neighbour coupling, the amplitudes of the propagating modes are given exactly by Bessel functions [5,6]. When the array is semi-infinite the analysis becomes substantially more involved, often requiring phase-operator techniques [28,29]. A natural extension, in which light can hop to several neighbours, leads to generalised Bessel functions that have been studied extensively by Dattoli and co-workers [30,31,32] and more recently in the context of multi-site waveguide couplers [33].

On the quantum-optical side, nonlinear coherent states whose expansion coefficients are Bessel functions have attracted recent attention [34,35,36]. In order to characterise the statistical properties of such states, for instance, to compute the Mandel Q -parameter [37] one needs closed-form expressions for moments of the number operator, which reduce to evaluating sums of the type

$$\sum_{k=0}^{\infty} k^n J_k^2(2x). \quad (1)$$

For $n = 0$ and $n = 1$ the sums are elementary consequences of Bessel's closure identity and a classical recurrence. The cases $n = 2$ and $n = 3$ can also be computed explicitly, but already at $n = 3$ the result is a cumbersome combination of Bessel functions of different orders, and no clear pattern is discernible for higher odd moments.

The purpose of the present paper is to show that a discrete family of even moments, namely

$$\mathcal{P}_n(x) = \sum_{k=0}^{\infty} k^{2n} J_k^2(2x), \quad n = 1, 2, 3, \dots, \quad (2)$$

possesses a remarkably simple structure: each $\mathcal{P}_n(x)$ is an even polynomial of degree $2n$ in x . Setting $u = x^2$ we write $p_n(u) = \mathcal{P}_n(x)$ and refer to the family $\{p_n(u)\}_{n \geq 1}$ as *Bessel moment polynomials*. Our main contributions are:

- a closed formula for the coefficients $a_{n,m}$ of $p_n(u)$;
- a three-term recurrence relation linking p_{n+1} to p_n ; and
- a generating function for Bessel moment polynomials.

The leading coefficients 1, 3, 10, 35, 126, ... are one half of the central binomial coefficients $\frac{1}{2} \binom{2n}{n}$; to the best of our knowledge have not appeared previously in the literature on Bessel moments [38,39,40,41].

2 First Moments and Motivation

Nonlinear coherent states generated by the action of exponential Susskind–Glogower–London operators on the vacuum take the form [35,36]

$$|x\rangle_{\text{SGL}} = \frac{1}{x} \sum_{n=0}^{\infty} (n+1) J_{n+1}(2x) |n\rangle, \quad (3)$$

where $|n\rangle$ denotes the Fock (number) state with n photons. The normalisation and moment sums of this state involve precisely the series (1). For $n = 0$ the closure identity for squared Bessel functions,

$$J_0^2(2x) + 2 \sum_{k=1}^{\infty} J_k^2(2x) = 1, \quad (4)$$

gives immediately

$$\sum_{k=0}^{\infty} J_k^2(2x) = \frac{1 - J_0^2(2x)}{2}. \quad (5)$$

The first moment ($n = 1$) is a classical identity:

$$\sum_{k=0}^{\infty} k J_k^2(2x) = 2x^2 [J_0^2(2x) + J_1^2(2x)] - x J_0(2x) J_1(2x). \quad (6)$$

For $n = 2$ the sum can be extracted from the normalisation of (3):

$$\sum_{k=0}^{\infty} k^2 J_k^2(2x) = x^2. \quad (7)$$

The case $n = 3$ is already considerably more involved [35]:

$$\begin{aligned} \sum_{k=0}^{\infty} k^3 J_k^2(2x) &= x^2 \left[(6x^2 + 1) J_0^2(2x) + (6x^2 - 1) J_1^2(2x) \right] \\ &\quad - \left[2x J_0(2x) J_1(2x) \right] x^2 + \frac{2x^4}{3} \left[J_0(2x) J_2(2x) + J_1(2x) J_3(2x) \right]. \end{aligned}$$

Figure 1 displays the logarithm of both sides of this identity, showing perfect agreement. The important observation is that odd moments do *not* appear to close polynomially; instead they remain transcendental combinations of Bessel functions. It is the restriction to *even* moments that reveals the hidden polynomial structure.

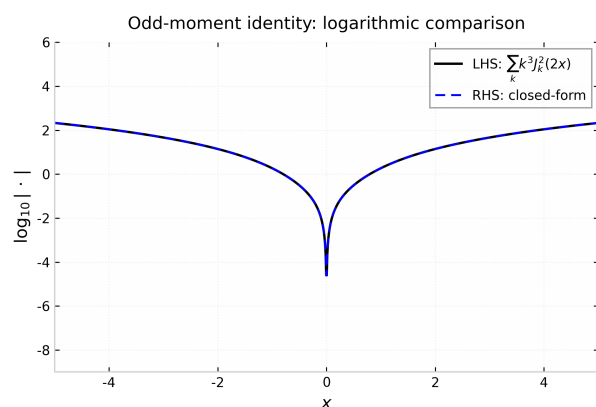


Fig. 1: Logarithmic plot of the left-hand side (solid line) and right-hand side (dashed line) of the odd-moment identity for $n = 3$, showing perfect agreement.

$$\sum_{k=0}^{\infty} k^5 J_k^2(2x) = x^2 \left(\frac{50x^4}{3} + \frac{52x^2}{3} + 1 \right) J_0^2(2x) + x^2 \left(\frac{50x^4}{3} + 20x^2 - \frac{29}{3} \right) J_1^2(2x) \quad (8)$$

$$- x^2 \left(\frac{28x^3}{3} - \frac{4x}{3} \right) J_0(2x) J_1(2x) \quad (9)$$

$$+ \frac{2x^6}{5} [J_0(2x) J_4(2x) + J_1(2x) J_5(2x)]. \quad (10)$$

Henceforth, we concentrate on the even-moment family (2). It is convenient to work in the quadratic variable

$$u = x^2, \quad (11)$$

and to define the associated polynomials

$$p_n(u) = \mathcal{P}_n(x) = \sum_{m=1}^n a_{n,m} u^m, \quad n \geq 1. \quad (12)$$

The first few are

$$p_1(u) = u,$$

$$p_2(u) = 3u^2 + u,$$

$$p_3(u) = 10u^3 + 15u^2 + u,$$

$$p_4(u) = 35u^4 + 140u^3 + 63u^2 + u,$$

$$p_5(u) = 126u^5 + 1050u^4 + 1470u^3 + 255u^2 + u.$$

2.1 Relation to Existing Literature

Moments of the form $\int_0^\infty v^n J_v(t) dv$ were investigated by Fényes [38], who showed that they decompose into a polynomial part plus oscillatory corrections. Kapteyn series [39, 1], i.e. sums of the type $\sum a_n J_{v+n}((v+n)z)$, appear in orbital mechanics and radiation problems. The Lommel polynomials $R_{m,v}(z)$ satisfy a three-term recurrence and are orthogonal with respect to a discrete measure [40, 3]. The Bessel polynomials $y_n(x) = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k$ and the standard orthogonal families (Legendre, Hermite, Laguerre) are likewise distinct from the present family. We therefore believe that the explicit results developed below are new.

3 Generating Function and Polynomial Structure

The starting point of our analysis is the well-known identity [1, 2]

$$\sum_{k=-\infty}^{\infty} J_k^2(a) e^{iky} = J_0(2a \sin \frac{y}{2}), \quad y \in \mathbb{R}. \quad (13)$$

For $a = 2x$ this becomes

$$\sum_{k=-\infty}^{\infty} J_k^2(2x) e^{iky} = J_0(4x \sin \frac{y}{2}). \quad (14)$$

Because $J_{-k}(z) = (-1)^k J_k(z)$, we have $J_{-k}^2(2x) = J_k^2(2x)$; hence the terms with k and $-k$ are equal and the sum is an even function of y .

Proposition 1. For every integer $n \geq 1$ the sum (2) is a polynomial in $u = x^2$ of degree exactly n . In particular, the representation (12) holds for suitable rational coefficients $a_{n,m}$.

Proof. Differentiate (14) $2n$ times with respect to y and set $y = 0$. Since $J_0(4x \sin y/2)$ is an even, analytic function of y , all odd derivatives vanish at the origin. Using $\frac{d^{2n}}{dy^{2n}} e^{iky} \Big|_{y=0} = (-1)^n k^{2n}$, we obtain

$$\sum_{k=-\infty}^{\infty} k^{2n} J_k^2(2x) = (-1)^n \left[\frac{d^{2n}}{dy^{2n}} J_0(4x \sin \frac{y}{2}) \right]_{y=0}. \quad (15)$$

The right-hand side is a polynomial in x because $J_0(z)$ is an even entire function of z , so $J_0(4x \sin y/2)$ is an even entire function of the product $x \sin y/2$. After differentiation at $y = 0$, only even powers of x survive. Moreover, the highest power of x that can appear after $2n$ differentiations is x^{2n} ; consequently the degree in $u = x^2$ is exactly n .

For $n \geq 1$ the $k = 0$ term on the left-hand side of (15) vanishes, and the symmetry $J_{-k}^2 = J_k^2$ gives

$$2 \sum_{k=1}^{\infty} k^{2n} J_k^2(2x) = (-1)^n \left[\frac{d^{2n}}{dy^{2n}} J_0(4x \sin \frac{y}{2}) \right]_{y=0}.$$

Adding the $k = 0$ term (which is zero for $n \geq 1$) yields the fundamental representation

$$\sum_{k=0}^{\infty} k^{2n} J_k^2(2x) = \frac{(-1)^n}{2} \left[\frac{d^{2n}}{dy^{2n}} J_0(4x \sin \frac{y}{2}) \right]_{y=0}. \quad (16)$$

This proves the proposition.

Equation (16) reduces the evaluation of the infinite series to the computation of a finite derivative of a composite entire function. In the next section we extract the coefficients explicitly.

3.1 Explicit Coefficients

The closed formula for the coefficients $a_{n,m}$ appearing in (12) is

$$a_{n,m} = \frac{1}{(m!)^2} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} (m-j)^{2n}. \quad (17)$$

For $m > n$ we have $a_{n,m} = 0$.

It is not difficult to show that the leading and linear coefficients are

$$a_{n,n} = \frac{1}{2} \binom{2n}{n}, \quad a_{n,1} = 1 \quad (n \geq 1).$$

Moreover, for every $n \geq 2$,

$$a_{n,2} = 4^{n-1} - 1.$$

Table 1: Coefficients $a_{n,m}$ for $1 \leq n \leq 5$.

$n \backslash m$	1	2	3	4	5
1	1				
2	1	3			
3	1	15	10		
4	1	63	140	35	
5	1	255	1470	1050	126

In the table above we show the first coefficients obtained from (17) which agree with the ones

Remark. The coefficients $a_{n,m}$ are strictly positive for all $1 \leq m \leq n$. This follows from (17) because the alternating sum is precisely the $2n$ -th forward difference of the convex sequence $j \mapsto (m-j)^{2n}$ evaluated at $j=0$, which is known to be positive. Consequently $p_n(u) > 0$ for all $u > 0$, a property that is consistent with the ones given in the abstract definition (2) as a sum of positive terms.

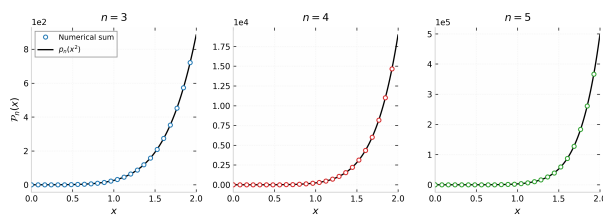


Fig. 2: Comparison of the numerical sum $\sum_{k=0}^{200} k^{2n} J_k^2(2x)$ (markers) and the polynomial $p_n(x^2)$ (solid curves) for $n = 3, 4, 5$ and $0 \leq x \leq 2$.

3.2 Recurrence Relation

The explicit formula (17) is well suited for symbolic computation, but it is also desirable to have a recurrence that generates the sequence $\{p_n(u)\}$ directly. It is not difficult to show that the Bessel moment polynomials satisfy the three-term recurrence

$$p_{n+1}(u) = \left(u \frac{d}{du}\right)^2 p_n(u) + 4u p_n(u) - 2 \int_0^u p_n(t) dt$$

for $n \geq 1$ with $p_1(u) = u$.

4 Conclusions

The polynomials $p_n(u) = \sum_{k=0}^{\infty} k^{2n} J_k^2(2\sqrt{u})$ constitute a new family of moment polynomials generated by the Bessel function J_0 . They are not among the classical orthogonal polynomial families, but they enjoy several pleasant properties:

- explicit coefficients given by a finite alternating sum of binomial terms;
- a compact three-term recurrence;
- a direct combinatorial interpretation through the generating function $J_0(4\sqrt{u} \sin y/2)$.

Because the underlying moments arise naturally in the study of nonlinear coherent states with Bessel coefficients, we expect that the formulas obtained here will prove useful in future investigations of nonclassical light statistics and in the design of waveguide-array analogues of quantum optical systems.

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