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Derivations on ranked bigroupoids

N. O. Alshehri[1], *H. S. Kim*[2] *and J. Neggers*[3]

[1] Department of Mathematics, King Abdulaziz University, Faculty of science for girls, Jeddah, KSA
[2] Department of Mathematics, Hanyang University, Seoul, 133-791, Korea
[3] Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U.S.A

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Abstract: In this paper, we introduce the notion of ranked bigroupoids and we define as well as discuss $(X, *, \omega)$-self-(co)derivations. In addition we define rankomorphisms and $(X, *, \omega)$-scalars for ranked bigroupoids, and we consider some properties of these as well.

Keywords: ranked bigroupoids, bigroupoids, derivations

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]). J. Neggers and H. S. Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs ([8]). H. S. Kim and J. Neggers ([7]) introduced the notion of $Bin(X)$ and obtained a semigroup structure. E. Posner [9] discussed derivations in prime rings, and H. E. Bell and L. C. Kappe [2] studied rings in which derivations satisfy certain algebraic conditions. Y. B. Jun and X. L. Xin [6] discussed derivations in BCI-algebras, and N. O. Alshehri [1] applied the notion of derivations in incline algebras. In this paper, we introduce the notion of ranked bigroupoids and discuss $(X, *, \omega)$-self-(co)derivations. $(X, *, \omega)$-scalars in ranked bigroupoids will be discussed as well.

2. Preliminaries

An *d-algebra* ([8]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(A)$x * x = 0$,
(B)$0 * x = 0$,
(C)$x*y = 0$ and $y*x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra is a d-algebra X satisfying the following additional axioms:

(D)$((x * y) * (x * z)) * (z * y) = 0$,
(E)$(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and $(X, \bullet)$ of $Bin(X)$, define a product "$\square$" on these groupoids as follows:

$$(X, *) \,\square\, (X, \bullet) = (X, \boxtimes), \tag{1}$$

where

$$x \boxtimes y = (x * y) \bullet (y * x), \tag{2}$$

for any $x, y \in X$. Using that notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.1. ([7]) *$(Bin(X), \square)$ is a semigroup, i.e., the operation "$\square$" as defined in general is associative. Furthermore, the left- zero-semigroup is the identity for this operation.*

* Corresponding author: e-mail: nalshehri@kau.edu.sa, heekim@hanyang.ac.kr, jneggers@as.ua.edu

3. Ranked bigroupoids

A *ranked bigroupoid* is an algebraic system $(X,*,\bullet)$ where X is a non-empty set and "$*$" and "$\bullet$" are binary operations defined on X. We may consider the first binary operation $*$ as the *major operation*, and the second binary operation $\bullet$ as the *minor operation*.

Example 3.1. A K-algebra ([3]) is defined as an algebraic system $(G,\bullet,\odot)$ where $(G,\bullet)$ is a group and where $x\odot y := x\bullet y^{-1}$. Hence every K-algebra is a ranked bigroupoid.

Example 3.2. We construct a ranked bigroupoid from any BCK-algebra. In fact, given a BCK-algebra $(X,*,0)$, if we define a binary operation "$\wedge$" on X by $x\wedge y := x*(x*y)$ for any $x,y\in X$, then $(X,*,\wedge)$ is a ranked bigroupoid.

We introduce the notion of "ranked bigroupoids" to distinguish two bigroupoids $(X,*,\bullet)$ and $(X,\bullet,*)$. Even though $(X,*,\bullet)=(X,\bullet,*)$ in the sense of bigroupoids, the same is not true in the sense of ranked bigroupoids. This is analogous to the situation for sets, where $\{x,y\}=\{y,x\}$ but $<x,y>\neq<y,x>$ in general.

Given an element $(X,*)\in Bin(X)$, $(X,*)$ has a natural associated ranked bigroupoid $(X,*,*)$, i.e., the major operation and the minor operation coincide.

We denote the class of all ranked bigroupoids defined on a non-empty set X by $Rbbin(X)$, i.e., $Rbbin(X) := \{(X,*,\bullet)\,|\,(X,*,\bullet)$ is a ranked bigroupoid on $X\}$. We denote the class of all bigroupoids defined on a non-empty set X by $Bin^2(X)$, i.e., $Bin^2(X) := \{(X,*,\bullet)\,|\,*,\bullet$ are binary operations on $X\}$.

Theorem 3.3. *If we define $(X,\boxtimes,\xi) := (X,*,\omega)\Box(X,\bullet,\psi)$ for any $(X,*,\omega),(X,\bullet,\psi)\in Rbbin(X)$, then $(Rbbin(X),\Box)$ is a semigroup where $x\boxtimes y := (x*y)\bullet(y*x)$ and $x\xi y := (x\omega y)\psi(y\omega x)$ for all $x,y\in X$.*

Proof. The proof is similar to the case of Theorem 2.1 in [7], and we omit it.

Proposition 3.4. *If $(X,*)$ is a left-zero-semigroup, then $(X,*,*)$ is the identity element in $(Rbbin(X),\Box)$.*

Proof. Let $(X,*)$ be the left-zero-semigroup and let $(X,\bullet,\psi)\in Rbbin(X)$. If $(X,\boxtimes,\xi) := (X,*,*)\Box(X,\bullet,\psi)$, then, for all $x,y\in X$, we have $x\boxtimes y = (x*y)\bullet(y*x) = x\bullet y$ and $x\xi y = (x*y)\psi(y*x) = x\psi y$, since $(X,*)$ is the left-zero-semigroup, i.e., $(X,\boxtimes,\xi) = (X,\bullet,\psi)$. If $(X,\boxtimes,\xi) := (X,\bullet,\psi)\Box(X,*,*)$, then, for all $x,y\in X$, we have $x\boxtimes y = (x\bullet y)*(y\bullet x) = x\bullet y$ and $x\xi y = (x\psi y)*(y\psi x) = x\psi y$, since $(X,*)$ is the left-zero-semigroup, i.e., $(X,\boxtimes,\xi) = (X,\bullet,\psi)$. This proves that $(X,*,*)$ is the identity in $(Rbbin(X),\Box)$.

If $(X,*)$ is the right-zero-semigroup and if $(X,\boxtimes,\xi) := (X,*,*)\Box(X,\bullet,\psi)$, then it is easy to see that $x\boxtimes y = y\bullet x$ and $x\xi y = y\psi x$ for all $x,y\in X$. We denote by $x\bullet^{\rm op}y = y\bullet x$ and $x\psi^{\rm op}y = y\psi x$. It follows that $(X,*,*)\Box(X,\boxtimes,\xi) = (X,\bullet^{\rm op},\psi^{\rm op})$ and $(X,\boxtimes,\xi)\Box(X,*,*) = (X,\bullet^{\rm op},\psi^{\rm op})$.

Proposition 3.5. *If we define a map $E : Bin(X)\to Rbbin(X)$ by $E((X,*)) := (X,*,*)$, then it is an injective homomorphism of semigroups.*

Proof. Given $(X,*),(X,\bullet)\in Bin(X)$, if we let $(X,\Box) := (X,*)\Box(X,\bullet)$, then $(X,\Box,\Box) = E((X,\Box)) = E((X,*)\Box(X,\bullet))$. If we let $(X,\boxtimes,\xi) := (X,*,*)\Box(X,\bullet,\bullet)$, then $x\boxtimes y = (x*y)\bullet(y*x) = x\Box y$ and $x\xi y = (x*y)\bullet(y*x) = x\Box y$ for all $x,y\in X$. It follows that $(X,\boxtimes,\xi) = (X,\Box,\Box)$. Hence

$$\begin{aligned}E((X,*)\Box(X,\bullet)) &= E((X,\Box))\\ &= (X,\Box,\Box)\\ &= (X,*,*)\Box(X,\bullet,\bullet)\\ &= E((X,*))\Box E((X,\bullet)),\end{aligned}$$

proving the proposition.

A ranked bigroupoid (X,λ,ρ) is said to be *left-over-right* if for all $x,y\in X$, $x\lambda y = x$ and $x\rho y = y$. Similarly, a ranked bigroupoid (X,ρ,λ) is said to be *right-over-left* if for all $x,y\in X$, $x\rho y = y$ and $x\lambda y = x$.

Proposition 3.6. *For any $(X,*,\omega)\in Rbbin(X)$, we have the following:*

(i) $(X,\lambda,\rho)\Box(X,*,\omega) = (X,*,\omega)\Box(X,\lambda,\rho) = (X,*,\omega^{\rm op})$,
(ii) $(X,\rho,\lambda)\Box(X,*,\omega) = (X,*,\omega)\Box(X,\rho,\lambda) = (X,*^{\rm op},\omega)$.

Using the notion of two binary operations λ and ρ we construct an interesting table which is a copy of the Klein-4-group as follows:

$\Box$	(X,λ,λ)	(X,ρ,ρ)	(X,λ,ρ)	(X,ρ,λ)
(X,λ,λ)	(X,λ,λ)	(X,ρ,ρ)	(X,λ,ρ)	(X,ρ,λ)
(X,ρ,ρ)	(X,ρ,ρ)	(X,λ,λ)	(X,ρ,λ)	(X,λ,ρ)
(X,λ,ρ)	(X,λ,ρ)	(X,ρ,λ)	(X,λ,λ)	(X,ρ,ρ)
(X,ρ,λ)	(X,ρ,λ)	(X,λ,ρ)	(X,ρ,ρ)	(X,λ,λ)

4. Derivations in ranked bigroupoids

Given a ranked bigroupoid $(X,*,\omega)$, a map $d : X\to X$ is called an $(X,*,\omega)$-*self-derivation* if for all $x,y\in X$,

$$d(x*y) = (d(x)*y)\omega(x*d(y))$$

In the same setting, a map $d : X\to X$ is called an $(X,*,\omega)$-*self-coderivation* if for all $x,y\in X$,

$$d(x*y) = (x*d(y))\omega(d(x)*y)$$

Note that if (X,ω) is a commutative groupoid, then $(X,*,\omega)$-self-derivations are $(X,*,\omega)$-self-coderivations. A map $d : X \to X$ is called an *abelian-$(X,*,\omega)$-self-derivation* if it is both an $(X,*,\omega)$-self-derivation and an $(X,*,\omega)$-self-coderivation.

Proposition 4.1. *Let $(X,*,\omega)$ be a ranked bigroupoid such that $(X,\omega,0)$ is a d-algebra. For any $(X, *, \omega)$-self-derivation $d : X \to X$ if the identity mapping, then $X = \{0\}$.*

Proof. Consider $d(x*y) = (d(x)*y)\omega(x*d(y)) = (x*y)\omega(x*y) = 0$. Thus $x*y = y*x = 0$ and $x = y$, whence $|X| = 1$ and $X = \{0\}$.

For the case where $d : X \to X$ is an $(X,*,\omega)$-self-coderivation the same conclusion holds if d is the identity map. Indeed, $d(x*y) = (x*d(y))\omega(d(x)*y) = (x*y)\omega(x*y) = 0$, so that $x*y = y*x = 0$ implies $x = y$ and $|X| = 1$.

Proposition 4.2. *Let d be an $(X,*,\omega)$-self- derivation. If $(X,*)$ is a monoid with identity 1, then $d(1)$ is an idempotent in (X,ω).*

Proof. Since d is an $(X,*,\omega)$-self-derivation, $d(x) = d(x*1) = [d(x)*1]\omega[x*d(1)] = d(x)\omega[x*d(1)]$ for all $x \in X$. If we let $x := 1$, then $d(1) = d(1)\omega[1*d(1)] = d(1)\omega d(1)$, proving that $d(1)$ is an idempotent in (X,ω).

Proposition 4.3. *Let d be an $(X,*,\omega)$-self-derivation and let $(X,*)$ be a semigroup with zero 0. If d is regular, i.e., $d(0) = 0$, then 0 is an idempotent in (X,ω).*

Proof. Since d is an $(X,*,\omega)$-self-derivation, $d(0) = d(0*x) = [d(0)*x]\omega[0*d(x)] = [d(0)*x]\omega 0$, i.e., $d(0) = (d(0)*x)\omega 0$. If we let $x := 0$, then $0 = d(0) = (d(0)*0)\omega 0 = 0\omega 0$. Hence 0 is an idempotent in (X,ω).

Theorem 4.4. *Let $(X,*)$ be the left-zero-semigroup.*

(i) if d_1 is an $(X,*,\omega)$-self-derivation and if d_2 is an $(X,*,\omega)$-self-coderivation, then $(d_1 \circ d_2)(x*y) = d_1(x)*d_2(y)$ for all $x,y \in X$,
(ii) if d_1 is an $(X,*,\omega)$-self-coderivation and if d_2 is an $(X,*,\omega)$-self-coderivation, then $(d_1 \circ d_2)(x*y) = d_2(x)*d_1(y)$ for all $x,y \in X$,
(iii) if d_i are an $(X,*,\omega)$-self-coderivations $(i = 1,2)$, then $(d_1 \circ d_2)(x*y) = d_1(x)*d_2(y)$ for all $x,y \in X$,
(iv) if d_i are an $(X,*,\omega)$-self-derivations $(i = 1,2)$, then $(d_1 \circ d_2)(x*y) = (d_1 \circ d_2)(x)*y$ for all $x,y \in X$,

Proof. (i). Given $x,y \in X$, we have

$$\begin{aligned}(d_2 \circ d_2)(x*y) &= d_1(d_2(x*y))\\ &= d_1[(x*d_2(y))\,\omega\,(d_2(x)*y)]\\ &= d_1(x*d_2(y))\\ &= [d_1(x)*d_2(y)]\,\omega\,[x*d_1(d_2(y))]\\ &= d_1(x)*d_2(y)\end{aligned}$$

Other cases are similar to the case (i), and we omit the proofs.

We can obtain similar properties to Theorem 4.4 when we discuss the right-zero-semigroup.

Proposition 4.5. *If (X,λ,ρ) is a left-over-right ranked bigroupoid, then every (X,λ,ρ)-self-derivation μ is the identity map on X.*

Proof. For any $x,y \in X$, $\mu(x) = \mu(x\lambda y) = (\mu(x)\lambda y)$
ρ
$(x\lambda\mu(y)) = \mu(x)\rho x = x$.

Similarly we obtain the following proposition:

Proposition 4.5′. *If (X,ρ,λ) is a right-over-left ranked bigroupoid, then every (X,ρ,λ)-self-derivation μ is the identity map on X.*

Proposition 4.6. *If μ is an (X,λ,λ)-self-coderivation or an (X,ρ,ρ)-self-derivation, then it is the identity map on X.*

Proof. Given $x,y \in X$, if μ is an (X,λ,λ)-self-coderivation, then $\mu(x) = \mu(x\,\lambda\,y) = (x\lambda\mu(y))\lambda\,(\mu(x)\,\lambda\,y) = x\,\lambda\,\mu(x) = x$. If μ is an (X,ρ,ρ)-self-derivation, then $\mu(y) = \mu(x\,\rho\,y) = (x\,\rho\,\mu(y))\,\rho\,(\mu(x)\,\rho\,y) = y$.

Proposition 4.7. *Every map $\mu : X \to X$ is both an (X,λ,ρ)-self-coderivation and an (X,ρ,λ)-self-coderivation.*

Proof. Given $x,y \in X$, we have $\mu(x\lambda y) = \mu(x) = x\rho\,\mu(x) = (x\,\lambda\,\mu(y))\,\rho\,(\mu(x)\,\lambda\,y)$, proving that μ is an (X,λ,ρ)-self-coderivation. Moreover, we have $\mu(x\rho y) = \mu(y) = \mu(y)\lambda y = (x\rho\,\mu(y))\lambda(\mu(x)\rho y)$, proving that μ is an (X,ρ,λ)-self-coderivation.

Let $(X,*,\omega)$ be a ranked bigroupoid with distinguished element 0 and let d be an $(X,*,\omega)$-self-derivation. A right ideal I, i.e., $I*X \subseteq I$, of the groupoid $(X,*)$ is said to be *d-friendly* if $x*d(x) \in I$ for any $x \in X$. We denote by $Ker(d) = \{x \in X | d(x) = 0\}$ the *kernel* of d.

Proposition 4.8. *Let $(X,*)$ be a groupoid and let $0 \in X$ such that $0*x = x*x = 0$ for all $x \in X$. If d is an $(X,*,*)$-self-derivation, then $Ker(d)$ is a d-friendly right ideal of $(X,*)$.*

Proof. If $x \in Ker(d)$ and $y \in X$, then $d(x*y) = (d(x)*y)*(x*d(y)) = (0*y)*(x*d(y)) = 0$ and hence $x*y \in Ker(d)$, proving that $Ker(d)$ is a right ideal of $(X,*)$.

Given $x,y \in X$, since d is an $(X,*,*)$-self-derivation, we have $d(x*y) = (d(x)*y)*(x*d(y))$. If we let $y := d(x)$, then $d(x*d(x)) = (d(x)*d(x))*(x*d^2(x)) = 0$, which means that $x*d(x) \in Ker(d)$. This proves the proposition.

Corollary 4.9. *Let $(X,*,0)$ be a d/BCK-algebra. If d is an $(X,*,*)$-self-derivation, then $Ker(d)$ is a d-friendly right ideal of $(X,*)$.*

Proof. Every d/BCK-algebra contains $0 \in X$ such that $0*x = x*x = 0$ for all $x \in X$.

Let d be an $(X,*,*)$-self-derivation and let $Rad(d)$ be the intersection of all d-friendly right ideals of $(X,*)$.

Since $X \in Rad(d)$, $Rad(d)$ is always defined and $Rad(d) \subseteq Ker(d)$.

Proposition 4.10. *Let $(X, *)$ be a groupoid and let $0 \in X$ such that $0 * x = 0, x * 0 = x$ for all $x \in X$. If d is an $(X, *, *)$-self-derivation, then $Ker(d) = Rad(d)$.*

Proof. If $x \in Ker(d)$, then $x = x*0 = x*d(x) \in I_i$ for any d-friendly right ideal I_i of $(X, *)$, i.e., $x \in \cap I_i = Rad(d)$. Hence $Ker(d) \subseteq Rad(d)$.

Corollary 4.11. *Let $(X, *, 0)$ be a BCK-algebra. If d is an $(X, *, *)$-self-derivation, then $Ker(d) = Rad(d)$.*

Proof. The conditions $0*x = 0$ and $x*0 = x$ hold for any $x \in X$ in every BCK-algebra.

5. $(X, *, \omega)$-scalars in ranked bigroupoids

Let $(X, *, \omega)$ be a ranked bigroupoid and let $\xi \in X$. ξ is called an $(X, *, \omega)$-*scalar* if for any $x, y \in X$,

(3)$\xi * (x * y) = (\xi * x) * y = x * (\xi * y)$
(4)$\xi * (x\, \omega\, y) = (\xi * x)\, \omega\, (\xi * y)$.

For example, if $(R, \cdot, +)$ is a commutative ring, then every element is an $(R, \cdot, +)$-scalar.

Example 5.1. Let $(G, \bullet, \odot)$ be a K-algebra and let e_G be the identity of $(G, \bullet)$. Then e_G is the unique $(G, \bullet, \odot)$-scalar. In fact, if α is a $(G, \bullet, \odot)$-scalar, then $\alpha \bullet (a \odot b) = \alpha \bullet (ab^{-1})$ and $(\alpha \bullet a) \odot (\alpha \bullet b) = (\alpha \bullet a) \bullet (\alpha \bullet b)^{-1} = \alpha \bullet (a \bullet b^{-1}) \bullet \alpha^{-1}$. It follows that $\alpha \bullet (a \bullet b^{-1}) = \alpha \bullet (a \bullet b^{-1}) \bullet \alpha^{-1}$ and hence $\alpha^{-1} = e_G$, proving that $\alpha = e_G$.

Proposition 5.2. *Let d be an $(X, *, \omega)$-self-derivation and let ξ be an $(X, *, \omega)$-scalar. If we define a map $\xi * d : X \to X$ by $(\xi * d)(x) := \xi * d(x)$, then it is an $(X, *, \omega)$-self-derivation.*

Proof. Since $d(x*y) = (d(x)*y)\omega(x*d(y))$ for any $x, y \in X$, we have

$$\begin{aligned}(\xi * d)(x * y) &= \xi * [(d(x) * y)\omega(x * d(y))]\\ &= [\xi * (d(x) * y)]\omega[\xi * (x * d(y))]\\ &= [(\xi * d)(x) * y]\omega[x * (\xi * d(y))]\\ &= [(\xi * d)(x) * y]\omega[x * ((\xi * d)(y))],\end{aligned}$$

proving that $\xi * d$ is an $(X, *, \omega)$-self-derivation.

Proposition 5.3. *Let $(X, *) \in Bin(X)$. If $\xi \in X$ satisfies the condition (3), then ξ is both an $(X, *, \lambda)$-scalar and an $(X, *, \rho)$-scalar.*

Proof. For any $x, y \in X$, we have $\xi * [x\lambda y] = \xi * x = (\xi * x)\lambda(\xi * y)$ and $\xi * [x\rho y] = \xi * y = (\xi * x)\rho(\xi * y)$.

Proposition 5.4. *Let $(X, *, f)$ be a leftoid, i.e., $x * y = f(x)$, a function of x, for all $x, y \in X$. If a groupoid (X, ω) does not contain any idempotent, then the ranked bigroupoid $(X, *, \omega)$ does not contain any $(X, *, \omega)$-scalar.*

Proof. Assume that there is an $(X, *, \omega)$-scalar α in X. Then for any $x, y \in X$, we have $f(\alpha) = \alpha * [x\omega y] = (\alpha * x)\omega(\alpha * y) = f(\alpha)\omega f(\alpha)$, which implies that $f(\alpha)$ is an idempotent in X, a contradiction.

Proposition 5.5. *Let $(X, *, g)$ be a rightoid, i.e., $x * y = g(y)$, a function of y, for all $x, y \in X$. If $\alpha \in X$ is an $(X, *, \omega)$-scalar, then*

(i)$g^2(b) = g(b)$ for all $b \in X$,
(ii)$g : (X, \omega) \to (X, \omega)$ is a homomorphism.

Proof. (i). Let α be an $(X, *, \omega)$-scalar. Then $\alpha * (a * b) = \alpha * g(b) = g^2(b)$ and $(\alpha * a) * b = g(b)$ for all $a, b \in X$. Hence we obtain $g^2(b) = g(b)$ for all $b \in X$.

(ii). Given $a, b \in X$, we have $g(a\omega b) = \alpha * (a\omega b) = (\alpha * a)\, \omega\, (\alpha * b) = g(a)\omega g(b)$, proving that $g : (X, \omega) \to (X, \omega)$ is a homomorphism.

Proposition 5.6. *Let $(X, *, g)$ be a rightoid and let $g : (X, \omega) \to (X, \omega)$ be an idempotent homomorphism. Then every element of X is an $(X, *, \omega)$-scalar.*

Proof. For any $\alpha \in X$, we have $\alpha * (a * b) = \alpha * g(b) = g^2(b) = g(b) = (\alpha * a) * b$, since g is an idempotent map. Moreover, $a * (\alpha * b) = a * g(b) = g^2(b)$, proving the condition (3).

$\alpha * (a\omega b) = g(a\omega b) = g(a)\omega g(b) = (\alpha * a)\omega(\alpha * b)$, proving the condition (4).

Theorem 5.7. *Let ξ, μ be $(X, *, \omega)$-scalars. Then $\xi * \mu$ is also an $(X, *, \omega)$-scalar.*

Proof. Given $a, b \in X$, we have

$$\begin{aligned}(\xi * \mu)(a * b) &= \xi * [\mu * (a * b)] = \xi * [(\mu * a) * b]\\ &= [\xi * (\mu * a)] * b] = [(\xi * \mu) * a] * b,\end{aligned}$$

$$\begin{aligned}a * [(\xi * \mu) * b] &= a * [\xi * (\mu * b)]\\ &= \xi * [a * (\mu * b)] = \xi * [(\mu * a) * b]\\ &= \xi * [\mu * (a * b)] = (\xi * \mu)(a * b),\end{aligned}$$

proving the condition (3). Moreover, for any $a, b \in X$, we obtain

$$\begin{aligned}(\xi * \mu)(a\omega b) &= \xi * [\mu * (a\omega b)]\\ &= \xi * [(\mu * a)\omega(\mu * b)]\\ &= [\xi * (\mu * a)]\omega[\xi * (\mu * b)]\\ &= [(\xi * \mu) * a]\omega[(\xi * \mu) * b],\end{aligned}$$

proving the condition (4).

Up to this point we have discussed $(X, *, \omega)$-self-derivations and self-coderivations as mappings $d : X \to X$ with certain properties.

Given ranked bigroupoids $(X, *, \omega)$ and $(Y, \bullet, \psi)$ we shall be interested in defining what is meant by an $(X, *, \omega)$-derivation $\delta : X \to Y$ of which $(X, *, \omega)$-self-derivations form special cases.

In order to do so we need to introduce the concept of a rankomorphism, i.e., a morphism in the category of ranked bigroupoids.

Thus given ranked bigroupoids $(X, *, \omega)$ and $(Y, \bullet, \psi)$, a map $f : (X, *, \omega) \to (Y, \bullet, \psi)$ is called a *rankomorphism* if for all $x, y \in X$, $f(x * y) = f(x) \bullet f(y)$ and $f(x\omega y) = f(x)\,\psi\, f(y)$. If $f(x * y) = f(x)\psi f(y)$ and $f(x\omega y) = f(x) \bullet f(y)$, then $f : X \to Y$ is a morphism for $Bin^2(X)$, but it is not a rankomorphim since it mixes the rankings.

If by *Rbbin* we denote $\cup_X Rbbin(X)$, i.e., the class of all ranked bigroupoids $(X, *, \omega)$ for arbitrary set X with rankomorphisms as the morphisms for this class of objects then *Rbbin* becomes a category since the (function) composition of rankomorphisms is also a rankomorphism and, since the identity map on a set, naturally generates a corresponding rankomorphism as well.

Rankomorphisms can be studied in much greater detail certainly, but here the purpose is to introduce the following idea.

A map $\delta : (X, *, \omega) \to (Y, \bullet, \psi)$ is called an $(X, *, \omega)$-*derivation* if there exists a rankomorphism (not necessarily unique) $f : X \to Y$ such that $\delta(x * y) = (\delta(x) \bullet f(y))\psi(f(x) \bullet \delta(y))$ for all $x, y \in X$.

Note that the composition of a rankomorphism and an $(X, *, \omega)$-self-derivation forms an $(X, *, \omega)$-derivation. In fact, if $f : (X, *, \omega) \to (Y, \bullet, \psi)$ is a rankomorphism and $d : X \to X$ is an $(X, *, \omega)$-self-derivation, then for all $x, y \in X$,

$$\begin{aligned}(f \circ d)(x * y) &= f(d(x * y)) \\ &= f((d(x) * y)\omega(x * d(y)) = f(d(x) * y)\psi f(x * d(y)) \\ &= ((f \circ d)(x) \bullet f(y))\psi(f(x) \bullet (f \circ d)(y))\end{aligned}$$

so that $f \circ d : (X, *, \omega) \to (Y, \bullet, \psi)$ is an $(X, *, \omega)$-*derivation*.

Now suppose $f : (X, *, \omega) \to (Y, \bullet, \psi)$ is a rankomorphism and $d : Y \to Y$ is a $(Y, \bullet, \psi)$-self-derivation. Then $d(f(x * y)) = (d(f(x)) \bullet f(y))\,\psi\,(f(x) \bullet d(f(y))$ and $d \circ f : (X, *, \omega) \to (Y, \bullet, \psi)$ is an $(X, *, \omega)$-derivation.

Thus, if $f : (X, *, \omega) \to (X, *, \omega)$ is the identity map, then it is a rankomorphism and if $d : X \to X$ is an $(X, *, \omega)$-self-derivation, it is an $(X, *, \omega)$-derivation.

Suppose now that $f : (X, *, \omega) \to (Y, \bullet, \psi)$ is a rankomorphism and that $\delta : (Y, \bullet, \psi) \to (Z, \triangledown, \theta)$ is a $(Y, \bullet, \psi)$-derivation, i.e., for some rankomorphism $g : (Y, \bullet, \psi) \to (Z, \triangledown, \theta)$ we have $\delta(f(x*y)) = \delta(f(x) \bullet f(y)) = (\delta f(x) \triangledown g f(y))\theta(g f(x) \triangledown \delta f(y))$ where $(g\circ f)(x) = g(f(x))$ and $(g\circ f)(x*y) = g(f(x)\bullet f(y)) = g(f(x))\triangledown g(f(y))$, i.e., $g \circ f : (X, *, \omega) \to (Z, \triangledown, \theta)$ is a rankomorphism since $(g \circ f)(x\omega y) = g(f(x)\psi f(y)) = (g \circ f)(x)\theta(g \circ f)(y)$ as well. Hence $\delta \circ f : (X, *, \omega) \to (Z, \triangledown, \theta)$ is an $(X, *, \omega)$-derivation.

If $\delta : (X, *, \omega) \to (Y, \bullet, \psi)$ is an $(X, *, \omega)$-derivation and if $g : (Y, \bullet, \psi) \to (Z, \triangledown, \theta)$ is a rankomorphism where $\delta(x * y) = (\delta(x) \bullet f(y)\psi(f(x) \bullet \delta(y))$, then $(g\circ\delta)(x*y) = g((\delta(x)\bullet f(y))\psi(f(x)\bullet\delta(y))) = g(\delta(x)\bullet f(y))\theta g(f(x) \bullet \delta(y)) = (g(\delta(x))\triangledown g(f(y)))\theta (g(f(x))\triangledown g(\delta(y)))$. Since $x \longmapsto g(f(x))$ defines a rankomorphism $g\circ f : (X, *, \omega) \to (Z, \triangledown, \theta)$, $g\circ\delta : (X, *, \omega) \to (Z, \triangledown, \theta)$ is an $(X, *, \omega)$-derivation.

Among the $(X, *, \omega)$-derivations, $\delta : (X, *, \omega) \to (Y, \bullet, \psi)$ there are those which correspond to additive maps, i.e., those for which $\delta(x\omega y) = \delta(x)\psi\delta(y)$. More generally, we shall consider mappings $\alpha : X \to X$, $\beta : Y \to Y$ where $\alpha(x\omega y) = \alpha(x)\omega\alpha(y)$ and $\beta(u\psi v) = \beta(u)\psi\beta(v)$ in addition to δ to obtain $\beta\delta\alpha(x\omega y) = \beta\delta(\alpha(x)\omega\alpha(y)) = \beta(\delta\alpha(x)\psi\delta\alpha(y)) = (\beta\delta\alpha(x))\psi(\beta\delta\alpha(y))$. In particular, if α and β are rankomorphims, then $\beta\circ\delta\alpha$ is an $(X, *, \omega)$-derivation if δ is an $(X, *, \omega)$-derivation.

Example 5.8. (i). Suppose that $\mathbf{R}$ is the collection of all real numbers. Then we have ranked bigroupoids $(\mathbf{R}, \cdot, +)$ and $(\mathbf{R}, +, \cdot)$ where $+, \cdot$ are usual addition and multiplication on $\mathbf{R}$ respectively. If $f : (\mathbf{R}, \cdot, +) \to (\mathbf{R}, +, \cdot)$ is a rankomorphism, then $f(x \cdot y) = f(x) + f(y)$ and $f(x + y) = f(x) \cdot f(y)$. Hence $f(0) = f(x \cdot 0) = f(x) + f(0)$, showing that $f(x) = 0$ for all $x \in \mathbf{R}$. Thus the zero mapping is the only rankomorphism between $(\mathbf{R}, \cdot, +)$ and $(\mathbf{R}, +, \cdot)$.

(ii). Suppose that $\delta : (\mathbf{R}, \cdot, +) \to (\mathbf{R}, +, \cdot)$ is an $(\mathbf{R}, \cdot, +)$-derivation. Then for some rankomorphism $f : (\mathbf{R}, \cdot, +) \to (\mathbf{R}, +, \cdot)$ we have $\delta(x \cdot y) = (\delta(x) + f(y)) \cdot (f(x) + \delta(y)) = (\delta(x) + 0) \cdot (0 + \delta(y)) = \delta(x) \cdot \delta(y)$ since $f = 0$ is the only rankomorphism by (i). Hence $(\mathbf{R}, \cdot, +)$-derivations include multiplicative mappings on $\mathbf{R}$. If n is a positive integer, then $(x \cdot y)^n = x^n \cdot y^n$, i.e., $x \longmapsto x^n$ is then a multiplicative map.

Proposition 5.9. *If $f : (X, *, \omega) \to (Y, \bullet, \psi)$ is an onto rankomorphism and $\xi \in X$ is an $(X, *, \omega)$-scalar, then $f(\xi)$ is a $(Y, \bullet, \psi)$-scalar.*

Proof. Let $u = f(a), v = f(b)$ in Y. Then

$$\begin{aligned}f(\xi) \bullet (u \bullet v) &= f(\xi) \bullet (f(a) \bullet f(b)) \\ &= f(\xi) \bullet f(a * b) = f(\xi * (a * b)) \\ &= f(a * (\xi * b)) = f(a) \bullet [f(\xi) \bullet f(b)] \\ &= u \bullet [f(\xi) \bullet v],\end{aligned}$$

$$\begin{aligned}u \bullet [f(\xi) \bullet v] &= f(a * (\xi * b)) \\ &= f(\xi * (a * b)) = f(\xi * a) \bullet f(b) \\ &= [f(\xi) \bullet f(a)] \bullet f(b) \\ &= [f(\xi) \bullet u] \bullet v,\end{aligned}$$

proving the condition (3).

$$\begin{aligned}f(\xi) \bullet [u\,\psi\, v] &= f(\xi) \bullet [f(a)\psi f(b)] \\ &= f[(\xi * (a\omega b)] = f[(\xi * a)\omega(\xi * b)] \\ &= f(\xi * a)\psi f(\xi * b) \\ &= [f(\xi) \bullet u]\psi[f(\xi) \bullet v],\end{aligned}$$

proving the condition (4). This proves the proposition.

Example 5.10. Let K be a field. Define a binary operation "$*$" on K by $x * y := x(x - y)$ for all $x, y \in$

K. Let $(X, \omega) \in Bin(X)$. We show that there is no $(K, *, \omega)$-scalar in K. Assume that α is a $(K, *, \omega)$-scalar for some $(K, \omega) \in Bin(K)$. Then $\alpha * (a * b) = \alpha^2 - \alpha a^2 + \alpha ab$ and $(\alpha * a) * b = \alpha^4 - 2\alpha^3 a + \alpha^2(a^2 - b) + \alpha ab$ for any $a, b \in X$.

$$\alpha^2 - \alpha a^2 + \alpha ab = \alpha^4 - 2\alpha^3 a + \alpha^2(a^2 - b) + \alpha ab \quad (3)$$

If we let $a := 0$ in (5), then $\alpha^2 = \alpha^4 - b\alpha^2$ for any $b \in K$. If we let $b := -1$, then $\alpha^4 = 0$ and hence $\alpha = 0$. Hence we obtain $a^2 = a * 0 = a * (0 * b) = 0 * (a * b) = 0$ for all $a, b \in K$. It follows that $a = 0$, which shows that $|K| = 1$, a contradiction.

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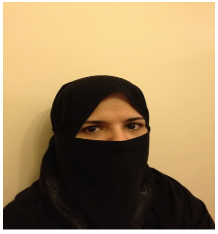
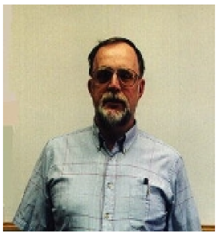


Noura Alshehri is working at Dept. of Mathematics, King Abdulaziz University as an associate professor. She received a Ph.D. (2008) in Mathematics from King Abdulaziz University. Her mathematical research areas are BCK-algebras, fuzzy algebras and Ring Theory.

Hee Sik Kim is working at Dept. of Mathematics, Hanyang University as a professor. He has received his Ph.D. at Yonsei University. He has published a book, Basic Posets with professor J. Neggers, and published 143 papers in several journals. His mathematical research areas are *BCK*-algebra, fuzzy algebras, poset theory and theory of semirings, and he is reviewing many papers in this areas. He has concerned on martial arts, photography and poetry also.



Joseph Neggers received a Ph.D. from the Florida State University in 1963. After positions at the Florida State University, the University of Amsterdam, King's college (London, UK), and the University of Puerto Rico, he joined the University of Alabama in 1967, where he is still engaged in teaching, research and writing poetry often in a calligraphic manner as well as enjoying friends and family through a variety of media both at home and aboard. He has reviewed over 500 papers in Zentralblatt Math., and published 76 research papers, and published a book, Basic Posets, with Professor Kim. He has studied several areas: poset theory, algebraic graph theory and combiniatorics and include topics which are of an applied as well as a pure nature.