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A modified fractional derivative

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A Modified Fractional Derivative and its Application to Fractional Vibration Equation

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Abstract: In this paper, a new modified definition of the fractional derivative is presented. The Laplace transform of the modified fractional derivative involves the initial values of the integer-order derivatives, but does not involve the initial values of the fractional derivatives as the Caputo fractional derivative. Using this new definition, Nutting's law of viscoelastic materials can be derived from the Scott-Blair stress-strain law as the Riemann-Liouville fractional derivative. Moreover, as the order α approaches n^- and $(n-1)^+$, the new modified fractional derivative ${}^{\dagger}D_t^{\alpha} f(t)$ approaches the corresponding integer-order derivatives $f^{(n)}(t)$ and $f^{(n-1)}(t)$, respectively. Therefore, the proposed modified fractional derivative preserves the merits of the Riemann-Liouville fractional derivative and the Caputo fractional derivative, while avoiding their demerits. By solving a fractional vibration equation, we confirm the advantages of the proposed fractional derivative.

Keywords: fractional derivatives and integrals, Mittag-Leffler function, Laplace transform

1 Introduction

Fractional calculus belongs to the field of mathematical analysis which involves the investigation and applications of integrals and derivatives of arbitrary order. Although fractional calculus has almost the same long history as the classical calculus, it was only in recent decades that its theory and applications have rapidly developed. Oldham and Spanier [1] published the first monograph in 1974. Ross [2] edited the first proceedings that was published in 1975. Thereafter theory and applications of fractional calculus have attracted much interest and have become a vibrant research area. Nowadays, the number of monographs and proceedings devoted to fractional calculus has reached several dozen, e.g. [3–15].

Fractional calculus provides an excellent mathematical description for modeling memory and hereditary properties of various materials and processes. It finds important applications in different areas of applied science including viscoelastic theory [10, 16–21], non-Newtonian fluid dynamics [22–25], anomalous diffusion [1, 26–32], dynamical systems [12, 33–38], control theory [7, 39–41], etc. Scientists and engineers have become well aware of the fact that the description of some phenomena is more accurate when the fractional derivative is used.

Viscoelasticity is one of the earliest and the most successful applied fields of the fractional calculus. The use of fractional calculus for the mathematical modelling of viscoelastic materials is quite natural. For viscoelastic materials the stress-strain constitutive relation can be more accurately described by introducing the fractional derivative [7, 10, 16–21, 42–44].

The Scott-Blair stress-strain law [16, 17] states that stress is proportional to the fractional derivative of strain. Such a fractional calculus element is said to constitute a spring-pot in [19]. Based on this idea, the fractional oscillation or vibration equations were introduced and discussed by Caputo [45], Bagley and Torvik [46], Beyer and Kempfle [47], Mainardi [48, 49], Gorenflo and Mainardi [50], and others [51, 52]. Thus fractional differential equations [4, 7, 8, 10–13, 32, 50, 53–55], a class of integro-differential equations with singularities, occur naturally.

In physical problems, the initial conditions are usually expressed in terms of a given number of bounded values of the field variable and its derivatives of integer order, no matter if the governing equation may be a generic integro-differential equation and therefore, in particular, a

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fractional differential equation. Unfortunately, the Riemann-Liouville fractional derivative leads to initial values of the Riemann-Liouville fractional integral and fractional derivatives. Since there is no known physical interpretation for such types of initial conditions, the applications of the Riemann-Liouville fractional derivative were restricted.

The Caputo fractional derivatives are often preferable in problems of physical interest because the corresponding initial conditions include integer-order derivatives having a conventional meaning. Nevertheless, the Caputo fractional derivative also has certain deflects in applications. For example, as the order $\alpha \to (n-1)^+$, the Caputo fractional derivative ${}^C D_t^\alpha f(t)$ does not approach $f^{(n-1)}(t)$ unless $f^{(n-1)}(0^+) = 0$.

In addition, when the Caputo fractional derivative is used to describe the constitutive equations of viscoelastic materials, say by the Scott-Blair stress-strain law, a constant strain ε implies that the stress $\sigma \equiv 0$ independent of the time t, instead of the temporal dependence by Nutting's law as deduced from experimental data [56–59] that $\sigma(t) \propto t^{-\alpha}$, where $0 < \alpha < 1$. Obviously this conclusion using the Caputo fractional derivative does not reflect the physical properties of real viscoelastic materials [60].

In this paper, we propose a new definition of the fractional derivative, denoted as ${}^\dagger D_t^\alpha f(t)$, which is a modification for the Riemann-Liouville and the Caputo fractional derivatives, where the new fractional derivative only involves the initial values of the integer-order derivatives as the Caputo fractional derivative. The proposed fractional derivative satisfies

$$\lim_{\alpha\to(n-1)^+} {}^\dagger D_t^\alpha f(t) = f^{(n-1)}(t),$$
$$\lim_{\alpha\to n^-} {}^\dagger D_t^\alpha f(t) = f^{(n)}(t). \tag{1}$$

Moreover, the fractional derivative of a constant yields the result of Nutting's law.

In the next section, we review the concepts of the Riemann-Liouville and Caputo fractional derivatives. We present the new definition in Section 3 and list its properties. In Section 4, we display the advantages of the new fractional derivative as contrasted with the Riemann-Liouville and Caputo fractional derivatives by investigating a fractional vibration equation.

2 A brief review of the Riemann-Liouville and Caputo fractional derivatives

Let $f(t)$ be piecewise continuous on $(0,+\infty)$ and integrable on any finite subinterval of $(0,+\infty)$. Then the Riemann-Liouville fractional integral of $f(t)$ of order β is defined as the convolution

$$J_t^\beta f(t) = \mathrm{D}_t^{-\beta} f(t) := \frac{t_+^{\beta-1}}{\Gamma(\beta)} * f(t) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau)\mathrm{d}\tau,$$

where $t > 0$, $\beta > 0$ and $\Gamma(\cdot)$ is Euler's gamma function. For complementarity, we define $J_t^0 = I$, the identity operator, i.e. we mean $J_t^0 f(t) = f(t)$.

It was strictly proved that [3, 61]

$$J_t^\beta f(t) \to f(t), \text{ as } \beta \to 0^+, \tag{2}$$

if $f(t)$ is continuous on the interval $[0,\varepsilon)$ for some $\varepsilon > 0$.

Nonlocal fractional derivatives have several different definitions. The Riemann-Liouville fractional derivative and the Caputo fractional derivative are two popular and often used definitions in the literature.

Let $n-1 < \alpha \le n$ and $n \in \mathbb{N}^+$. The Riemann-Liouville fractional derivative of $f(t)$ of order α is defined as

$${}^R D_t^\alpha f(t) := \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(J_t^{n-\alpha} f(t)\right), \; n-1 < \alpha \le n, \; n \in \mathbb{N}^+. \tag{3}$$

The Caputo fractional derivative of $f(t)$ of order α is defined as

$${}^C D_t^\alpha f(t) := J_t^{n-\alpha} f^{(n)}(t), \; n-1 < \alpha \le n, \; n \in \mathbb{N}^+. \tag{4}$$

We note that the Caputo fractional derivative is also referred to as the Gerasimov-Caputo fractional derivative [15, 18].

We also mention other definitions, such as the fractional derivative derived from the fractional difference [62] and the initialized fractional derivative [63].

The Laplace transform is one of the most commonly used methods for the analytic solutions of linear fractional differential equations. We list the Laplace transform formulas as follows. For the Riemann-Liouville fractional derivative, we have [1, 4, 7]

$$\mathscr{L}\left[{}^R D_t^\alpha f(t)\right] = s^\alpha \tilde{f}(s) - \sum_{k=0}^{n-1} s^k \left[{}^R D_t^{\alpha-k-1} f(t)\right]_{t=0^+}, \tag{5}$$

where $n-1 < \alpha \le n$. In particular,

$$\mathscr{L}\left[{}^R D_t^\alpha f(t)\right] = s^\alpha \tilde{f}(s) - \left[J_t^{1-\alpha} f(t)\right]_{t=0^+}, \; 0 < \alpha \le 1, \tag{6}$$

where $\tilde{f}(s)$ is the Laplace transform of the function $f(t)$.

The practical applicability of the Riemann-Liouville fractional derivative has been limited by the absence of any physical interpretation of the limiting values of fractional derivatives and integral at the lower limit $t = 0$.

If $f(t)$ is bounded on some small interval $(0,\varepsilon)$, i.e. $|f(t)| \le M$ for some positive number M, then the fractional integral satisfies

$$|J_t^\beta f(t)| \le \frac{Mt^\beta}{\Gamma(\beta+1)}, \; 0 < t < \varepsilon, \; \beta > 0, \tag{7}$$

which implies that the initial value of the fractional integral is zero, i.e. for $\beta > 0$,

$$J_t^\beta f(t) \to 0, \text{ as } t \to 0^+, \tag{8}$$

so Eq. (6) becomes

$$\mathscr{L}\left[{}^{R}D_t^{\alpha} f(t)\right] = \begin{cases} s^{\alpha}\tilde{f}(s), & 0<\alpha<1. \\ s\tilde{f}(s) - f(0^+), & \alpha = 1. \end{cases} \quad (9)$$

This can result in the solution of a fractional differential equation is not left-continuous at $\alpha = 1$ with respect to the order α; see Example 1.

For the Caputo fractional derivative, the Laplace transform formula is [7, 48, 50]

$$\mathscr{L}\left[{}^{C}D_t^{\alpha} f(t)\right] = s^{\alpha}\tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+), \quad (10)$$

where $n-1<\alpha\le n$. In particular,

$$\mathscr{L}\left[{}^{C}D_t^{\alpha} f(t)\right] = s^{\alpha}\tilde{f}(s) - s^{\alpha-1} f'(0^+),\ 0<\alpha\le 1. \quad (11)$$

From Eq. (4) or the Laplace transform (10), we easily observe the undesirable result

$$\lim_{\alpha\to(n-1)^+} {}^{C}D_t^{\alpha} f(t) = f^{(n-1)}(t) - f^{(n-1)}(0^+). \quad (12)$$

A problem also occurs when describing the stress-strain relation of viscoelastic materials by using the Caputo fractional derivative. For example, in the Scott-Blair stress-strain law [16, 17], a constant strain ε implies that the stress $\sigma \equiv 0$. This claim does not reflect the physical properties of real viscoelastic materials, i.e. Nutting's law [56, 57].

In next section, we present a modified definition of the fractional derivative to avoid all of these deflects.

3 A modification of the fractional derivative

Definition 1. Let $n\in\mathbb{N}^+$, $f^{(n)}(t)$ be piecewise continuous on $(0,+\infty)$ and integrable on any finite subinterval of $(0,+\infty)$, $f^{(n-1)}(0^+)$ exist and $n-1<\alpha\le n$. Then the modified fractional derivative of $f(t)$ of order α is defined as

$$\begin{aligned} {}^{\dagger}D_t^{\alpha} f(t) &:= J_t^{n-\alpha} f^{(n)}(t) + \frac{n-\alpha}{\Gamma(n-\alpha)} f^{(n-1)}(0^+) t^{n-1-\alpha}, \\ &= {}^{C}D_t^{\alpha} f(t) + \frac{n-\alpha}{\Gamma(n-\alpha)} f^{(n-1)}(0^+) t^{n-1-\alpha}, \quad (13) \\ &\quad n-1<\alpha\le n,\ n\in\mathbb{N}^+. \end{aligned}$$

The following propositions can be directly verified.

Proposition 1. The modified fractional derivative operator ${}^{\dagger}D_t^{\alpha}$ is linear, i.e. the following equalities hold

$${}^{\dagger}D_t^{\alpha}(c\,f(t)) = c\,{}^{\dagger}D_t^{\alpha} f(t), \quad (14)$$

$${}^{\dagger}D_t^{\alpha}(f(t)+g(t)) = {}^{\dagger}D_t^{\alpha} f(t) + {}^{\dagger}D_t^{\alpha} g(t). \quad (15)$$

Proposition 2. The Laplace transform of the modified fractional derivative is

$$\begin{aligned} \mathscr{L}\left[{}^{\dagger}D_t^{\alpha} f(t)\right] &= s^{\alpha}\tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+) \\ &+ (n-\alpha) s^{\alpha-n} f^{(n-1)}(0^+),\ n-1<\alpha\le n. \quad (16) \end{aligned}$$

Proposition 3. If $0<\alpha\le 1$, then

$${}^{\dagger}D_t^{\alpha} f(t) = J_t^{1-\alpha} f'(t) + \frac{1-\alpha}{\Gamma(1-\alpha)} f(0^+) t^{-\alpha}, \quad (17)$$

$$\mathscr{L}\left[{}^{\dagger}D_t^{\alpha} f(t)\right] = s^{\alpha}\tilde{f}(s) - \alpha s^{\alpha-1} f(0^+). \quad (18)$$

Proposition 4. The fractional derivative of a constant is

$${}^{\dagger}D_t^{\alpha} c = \begin{cases} \frac{c(1-\alpha)}{\Gamma(1-\alpha)} t^{-\alpha}, & 0<\alpha<1, \\ 0, & \alpha\ge 1. \end{cases} \quad (19)$$

Proposition 5. Suppose $n-1<\alpha<n$ and $n\in\mathbb{N}^+$, then the modified fractional derivative ${}^{\dagger}D_t^{\alpha} f(t)$ satisfies

$${}^{\dagger}D_t^{\alpha} f(t) \to f^{(n)}(t),\ \text{ as }\ \alpha\to n^-, \quad (20)$$

$${}^{\dagger}D_t^{\alpha} f(t) \to f^{(n-1)}(t),\ \text{ as }\ \alpha\to (n-1)^+. \quad (21)$$

We note that Proposition 2 can be verified by the Laplace transform of the Caputo fractional derivative and the Laplace formula

$$\mathscr{L}[t^{\nu}] = \frac{\Gamma(\nu+1)}{s^{\nu+1}},\ \mathrm{Re}(\nu) > -1. \quad (22)$$

The Laplace transform of the modified fractional derivative involves the initial values of the integer-order derivatives, but does not involve the initial values of the fractional derivatives. By the Scott-Blair stress-strain law [16, 17], $\sigma(t) = \eta\ {}^{\dagger}D_t^{\alpha}\varepsilon(t)$, Nutting's law of viscoelastic materials can be readily derived. Moreover, as the order α approaches n^- and $(n-1)^+$, the modified fractional derivative ${}^{\dagger}D_t^{\alpha} f(t)$ approaches the corresponding integer-order derivatives $f^{(n)}(t)$ and $f^{(n-1)}(t)$, respectively.

Therefore, the proposed modified fractional derivative preserves the merits of the Riemann-Liouville fractional derivative and the Caputo fractional derivative, while avoiding their demerits. It is more favorable and convenient for theoretical analysis and physical applications.

4 A comparative study in fractional vibration equation

In this section, we consider the fractional vibration equation with the Riemann-Liouville fractional derivative, the Caputo fractional derivative and the new modified fractional derivative, respectively, and compare their results. We will use the Mittag-Leffler function with two parameters [7, 64, 65]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha>0,\ \beta>0,\ z\in\mathrm{R}, \quad (23)$$

and its formula of the Laplace transform

$$\mathscr{L}\left[t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(-bt^{\alpha})\right] = \frac{s^{\beta}}{b+s^{\alpha}}, \quad (24)$$

$$b>0,\ \alpha>0,\ \alpha>\beta,\ \mathrm{Re}(s) > b^{1/\alpha}.$$

We list several special cases of the Mittag-Leffler function with two parameters [7, 48, 64–67]

$$E_{1,1}(t)=e^t,\ E_{2,1}(-t^2)=\cos(t),\ tE_{1,2}(t)=e^t-1,\\ tE_{2,2}(-t^2)=\sin(t),\ t^2E_{2,3}(-t^2)=1-\cos(t). \tag{25}$$

The solution of the initial value problem (IVP)

$$u''(t)+u(t)=0,\ u(0)=1,\ u'(0)=1, \tag{26}$$

is

$$u^{\langle 0\rangle}(t)=\cos(t)+\sin(t), \tag{27}$$

while the solution of the IVP

$$u''(t)+u'(t)=0,\ u(0)=1,\ u'(0)=1, \tag{28}$$

is

$$u^{\langle 1\rangle}(t)=2-e^{-t}. \tag{29}$$

One naturally expects that there should be a type of fractional derivative D_t^α such that the solution of the IVP

$$u''(t)+D_t^\alpha u(t)=0,\ u(0)=1,\ u'(0)=1, \tag{30}$$

continuously varies from $u^{\langle 0\rangle}(t)=\cos(t)+\sin(t)$ to $u^{\langle 1\rangle}(t)=2-e^{-t}$ as the order α increases from 0 to 1. We will show that the proposed fractional derivative ${}^{\dagger}D_t^\alpha$ can realize this aim, but the Riemann-Liouville fractional derivative or the Caputo fractional derivative cannot.

Example 1. We consider the IVP for the fractional differential equation with the Riemann-Liouville fractional derivative

$$u''(t)+{}^{R}D_t^\alpha u(t)=0,\ 0<\alpha<1,\\ u(0)=1,\ u'(0)=1. \tag{31}$$

Applying the Laplace transform we have

$$s^2\tilde{u}(s)-s-1+s^\alpha\tilde{u}(s)=0, \tag{32}$$

where we used the fact that the initial value of fractional integral is zero; see Eqs. (8) and (9). The Laplace transform of the function $u(t)$ is calculated as

$$\tilde{u}(s)=\frac{s^{1-\alpha}+s^{-\alpha}}{s^{2-\alpha}+1}. \tag{33}$$

Calculating the inverse Laplace transform yields the solution in terms of the Mittag-Leffler function as

$$u(t;\alpha)=E_{2-\alpha,1}(-t^{2-\alpha})+tE_{2-\alpha,2}(-t^{2-\alpha}). \tag{34}$$

The two limiting cases of the solution are

$$u(t;0^+)=\cos(t)+\sin(t), \tag{35}$$
$$u(t;1^-)=1. \tag{36}$$

In Fig. 1, we plot the curves of $u(t;\alpha)$ versus t for $\alpha=0.1, 0.36, 0.64$ and 0.9. The two dot lines correspond to the solutions in the integer-order cases, $u^{\langle 0\rangle}(t)=\cos(t)+\sin(t)$ and $u^{\langle 1\rangle}(t)=2-e^{-t}$.

Example 2. We consider the IVP for the fractional differential equation with the Caputo fractional derivative

$$u''(t)+{}^{C}D^\alpha u(t)=0,\ 0<\alpha<1,\\ u(0)=1,\ u'(0)=1. \tag{37}$$

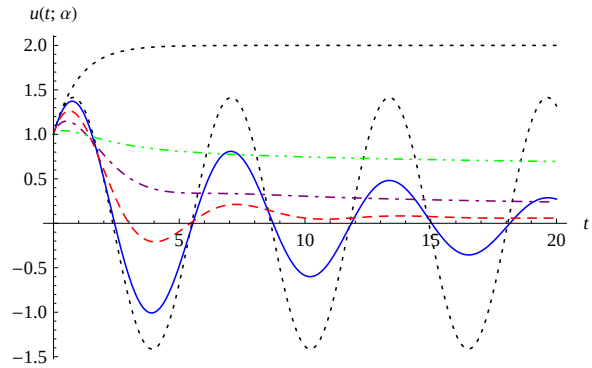

Fig. 1: Curves of $u(t;\alpha)$ versus t for $\alpha=0.1$ (solid line), $\alpha=0.36$ (dash line), $\alpha=0.64$ (dot-dash line) and $\alpha=0.9$ (dot-dot-dash line) in Example 1.

Applying the Laplace transform, we obtain

$$s^2\tilde{u}(s)-s-1+s^\alpha\tilde{u}(s)-s^{\alpha-1}=0, \tag{38}$$

which yields the expression

$$\tilde{u}(s)=\frac{s^{1-\alpha}+s^{-1}+s^{-\alpha}}{s^{2-\alpha}+1}. \tag{39}$$

By using the inverse Laplace transform, we obtain

$$u(t;\alpha)=E_{2-\alpha,1}(-t^{2-\alpha})+t^{2-\alpha}E_{2-\alpha,3-\alpha}(-t^{2-\alpha})\\ +tE_{2-\alpha,2}(-t^{2-\alpha}). \tag{40}$$

The two limiting cases of the solution are

$$u(t;0^+)=1+\sin(t), \tag{41}$$
$$u(t;1^-)=2-e^{-t}. \tag{42}$$

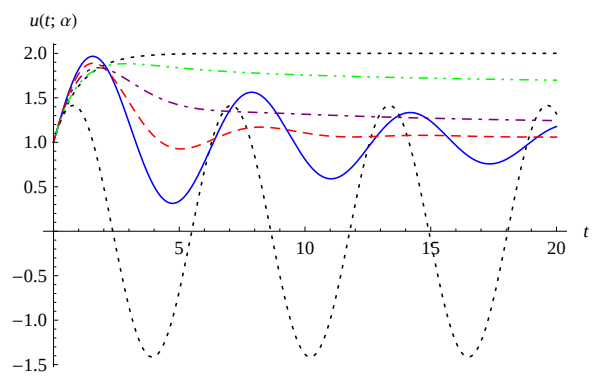

Fig. 2: Curves of $u(t;\alpha)$ versus t for $\alpha=0.1$ (solid line), $\alpha=0.36$ (dash line), $\alpha=0.64$ (dot-dash line) and $\alpha=0.9$ (dot-dot-dash line) in Example 2.

In Fig. 2, we plot the curves of $u(t;\alpha)$ versus t for $\alpha=0.1, 0.36, 0.64$ and 0.9. The two dot lines correspond

to the solutions in the integer-order cases, $u^{\langle 0\rangle}(t) = \cos(t) + \sin(t)$ and $u^{\langle 1\rangle}(t) = 2 - e^{-t}$.

Example 3. We consider the IVP for the fractional differential equation with the new modified fractional derivative

$$\begin{aligned} &u''(t) + {}^{\dagger}D_t^{\alpha} u(t) = 0,\ 0 < \alpha < 1, \\ &u(0) = 1,\ u'(0) = 1. \end{aligned} \tag{43}$$

Applying the Laplace transform, we have

$$s^2\tilde{u}(s) - s - 1 + s^{\alpha}\tilde{u}(s) - \alpha s^{\alpha-1} = 0, \tag{44}$$

which yields

$$\tilde{u}(s) = \frac{s^{1-\alpha} + \alpha s^{-1} + s^{-\alpha}}{s^{2-\alpha} + 1}. \tag{45}$$

The inverse Laplace transform of (45) leads to the solution

$$\begin{aligned} u(t;\alpha) = {} & E_{2-\alpha,1}(-t^{2-\alpha}) + \alpha t^{2-\alpha} E_{2-\alpha,3-\alpha}(-t^{2-\alpha}) \\ & + t E_{2-\alpha,2}(-t^{2-\alpha}). \end{aligned} \tag{46}$$

The two limiting cases of the solution are

$$u(t;0^+) = \cos(t) + \sin(t), \tag{47}$$

$$u(t;1^-) = 2 - e^{-t}, \tag{48}$$

both of which are just the solutions in the integer-order cases, $u^{\langle 0\rangle}(t) = \cos(t) + \sin(t)$ and $u^{\langle 1\rangle}(t) = 2 - e^{-t}$, respectively.

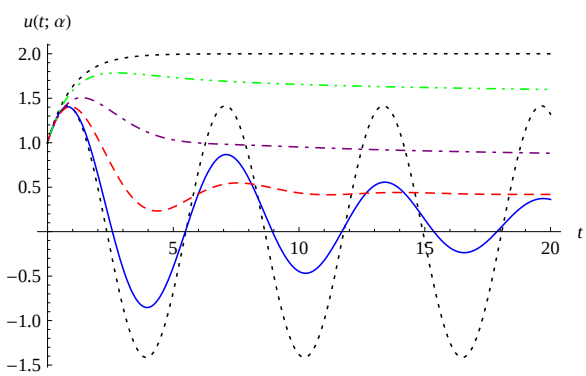

Fig. 3: Curves of $u(t;\alpha)$ versus t for $\alpha = 0.1$ (solid line), $\alpha = 0.36$ (dash line), $\alpha = 0.64$ (dot-dash line) and $\alpha = 0.9$ (dot-dot-dash line) in Example 3.

In Fig. 3, we plot the curves of $u(t;\alpha)$ versus t for $\alpha = 0.1, 0.36, 0.64$ and 0.9. The two dot lines correspond to the solutions in the integer-order cases, $u^{\langle 0\rangle}(t) = \cos(t) + \sin(t)$ and $u^{\langle 1\rangle}(t) = 2 - e^{-t}$.

We note that the proposed fractional derivative is especially compatible for a fractional differential equation where the order of the highest-order derivative is an integer more than 1.

5 Conclusions

In this paper, a new modified definition of the fractional derivative is presented and its properties are considered. By using the Laplace transform, the modified fractional derivative involves the initial values of the integer-order derivatives, and does not involve the initial values of the fractional derivatives such as the Caputo fractional derivative. By the new definition, Nutting's law of viscoelastic materials can be derived from the Scott-Blair stress-strain law, $\sigma(t) = \eta\, {}^{\dagger}D^{\alpha}\varepsilon(t)$, as the Riemann-Liouville fractional derivative. Moreover, as the order α approaches n^- and $(n-1)^+$, the modified fractional derivative ${}^{\dagger}D^{\alpha} f(t)$ approaches the corresponding integer-order derivatives $f^{(n)}(t)$ and $f^{(n-1)}(t)$, respectively. Therefore, the proposed modified fractional derivative preserves the merits of the Riemann-Liouville fractional derivative and the Caputo fractional derivative, while avoiding their demerits. It is more favorable and convenient for theoretical analysis and physical applications. Finally, we further demonstrate the advantages of the proposed fractional derivative by comparing the results of a fractional vibration equation.

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