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Characterization of Some Lacunary $\chi^2_{A_{uv}}-$ Convergence of Order α with $p-$ Metric Defined by mn Sequence of Moduli Musielak

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Abstract: We study some connections between lacunary strong $\chi^2_{A_{uv}}-$convergence with respect to a *mn* sequence of moduli Musielak and lacunary $\chi^2_{A_{uv}}-$ statistical convergence, where A is a sequence of four dimensional matrices $A(uv) = \left(a^{m_1\cdots m_r n_1\cdots n_s}_{k_1\cdots k_r \ell_1\cdots \ell_s}(uv)\right)$ of complex numbers.

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1 Introduction

Throughout w,χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m,n \in \mathbb{N}$, the set of positive integers. The notion of single sequence spaces properties are investigated by [6, 7, 8, 17]. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4] and Robison [27]. Later on, they were investigated by Hardy [15], Moricz [20], Moricz and Rhoades [21], Basarir and Solankan [2], Tripathy [30]-[39], W.H.Ruckle [28] Turkmenoglu [40], V.N. Mishra et al. [44]-[49] and many others.

We procure the following sets of double sequences:

$$\mathscr{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathscr{C}_p(t) := \left\{(x_{mn}) \in w^2 : p-lim_{m,n\to\infty} |x_{mn}-l|^{t_{mn}} = 1\ for\ some\ l \in \mathbb{C}\right\},$$

$$\mathscr{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p-lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathscr{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathscr{C}_{bp}(t) := \mathscr{C}_p(t) \bigcap \mathscr{M}_u(t) \text{ and } \mathscr{C}_{0bp}(t) = \mathscr{C}_{0p}(t) \bigcap \mathscr{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,n \in \mathbb{N}$ and $p-lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense . In the case $t_{mn} = 1$ for all $m,n \in \mathbb{N}; \mathscr{M}_u(t), \mathscr{C}_p(t), \mathscr{C}_{0p}(t), \mathscr{L}_u(t), \mathscr{C}_{bp}(t)$ and $\mathscr{C}_{0bp}(t)$ reduce to the sets $\mathscr{M}_u, \mathscr{C}_p, \mathscr{C}_{0p}, \mathscr{L}_u, \mathscr{C}_{bp}$ and $\mathscr{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gȯkhan and Colak [9, 10] have proved that $\mathscr{M}_u(t)$ and $\mathscr{C}_p(t), \mathscr{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-,\beta-,\gamma-$ duals of the spaces $\mathscr{M}_u(t)$ and $\mathscr{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [43] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen [23], Mursaleen and Edely [22, 24] and Tripathy [30] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [1] have defined the spaces $\mathscr{BS}, \mathscr{BS}(t), \mathscr{CS}_p, \mathscr{CS}_{bp}, \mathscr{CS}_r$ and $\mathscr{BV}$ of double sequences consisting of all double

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series whose sequence of partial sums are in the spaces $\mathscr{M}_u, \mathscr{M}_u(t), \mathscr{C}_p, \mathscr{C}_{bp}, \mathscr{C}_r$ and $\mathscr{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathscr{BS}, \mathscr{BV}, \mathscr{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathscr{CS}_{bp}$ and $\mathscr{CS}_r$ of double series. Basar and Sever [3] have introduced the Banach space $\mathscr{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathscr{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [19] as an extension of the definition of strongly Cesàro summable sequences. Connor [5] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [26] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [12]-[13], and [14] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all \;\; finite \;\; sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m,n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m,n \in \mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality) [See [16]] \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [18] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \; for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mn})(u) : u \geq 0\}, m,n = 1,2,\cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m,n \to \infty \right\},$$

where I_f is a convex modular [see [25,42,11]] defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{ inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1\right\}$$

If X is a sequence space, we give the following definitions:[see [41]]

(i)X' = the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \; for\, each\, x \in X \right\}$;

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent, \; for each\, x \in X \right\}$;

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \Sigma_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, for each x \in X \right\}$;

(v)$let\, X\, be an FK - space \supset \phi; then X^{f} = \left\{ f(\Im_{mn}) : f \in X' \right\}$;

(vi)$X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty,\ for each\, x \in X \right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (or Köthe - Toeplitz)$dual of $X, \beta - (or\, generalized - Köthe - Toeplitz)\, dual\, of X, \gamma - dual\, of\, X,\ \delta - dual\, of X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [16]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [1]. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and}$$
$$\|x\|_{bv_p} = \left(\Sigma_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2 Definition and Preliminaries

Let $mn (\geq 2)$ be an integer. A function $x : (M \times N) \times (M \times N) \times \cdots \times (M \times N)$. $(M \times N)(m \times n - factors) \to \mathbb{R}(\mathbb{C})$ is called a real complex $mn-$ sequence, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers and complex numbers respectively. Let $m_1, m_2, \cdots m_r, n_1, n_2, \cdots, n_s \in \mathbb{N}$ and X be a real vector space of dimension w, where $m_1, m_2, \cdots m_r, n_1, n_2, \cdots, n_s \leq w$. A real valued function $d_p(x_{11}, \ldots, x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}) = \|(d_1(x_{11},0), \ldots, d_n(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s},0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_{11},0), \ldots, d_n(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s},0))\|_p = 0$ if and and only if $d_1(x_{11},0), \ldots, d_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s},0)$ are linearly dependent,

(ii) $\|(d_1(x_{11},0), \ldots, d_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s},0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_{11},0), \ldots, d_{m_1,m_2,\cdots m_p,n_1,n_2,\cdots,n_q}(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s},0))\|_p = |\alpha| \|(d_1(x_{11},0), \ldots, d_n(x_{m_1,m_2,\cdots m_p,n_1,n_2,\cdots,n_q},0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p\left((x_{11},y_{11}),(x_{12},y_{12}) \cdots (x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}, y_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s})\right) = \left(d_X(x_{11},x_{12},\cdots x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_q})^p + d_Y(y_{11},y_{12},\cdots y_{m_1,m_2,\cdots m_p,n_1,n_2,\cdots,n_s})^p\right)^{1/p}$ $for 1 \leq p < \infty$; (or)

(v) $d\left((x_{11},y_{11}),(x_{12},y_{12}), \cdots (x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}, y_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s})\right) := \sup\left\{d_X(x_{11},x_{12},\cdots x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}), d_Y(y_{11},y_{12},\cdots y_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s})\right\}$,

for $x_{11}, x_{12}, \cdots x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s} \in X, y_{11}, y_{12}, \cdots y_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s} \in Y$ is called the $p-$ product metric of the Cartesian product of $m_1, m_2, \cdots m_r, n_1, n_2, \cdots, n_s$ metric spaces is the $p-$ norm of the $m \times n$-vector of the norms of the $m_1, m_2, \cdots m_r, n_1, n_2, \cdots, n_s$ subspaces.

A trivial example of p product metric of $m_1, m_2, \cdots m_r, n_1, n_2, \cdots, n_s$ metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_{11},0), \ldots, d_n(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s},0))\|_E = sup\left(|det(d_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}\left(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}\right))|\right) =$$

$$sup\left(\left|\begin{matrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}\left(x_{1,n_1,n_2,\cdots,n_s},0\right) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}\left(x_{2,n_1,n_2,\cdots,n_s},0\right) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ d_{m1n1}(x_{m1n1},0) & d_{m2n2}(x_{m2n2},0) & \ldots & d_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s}\left(x_{m_1,m_2,\cdots m_r,n_1,n_2,\cdots,n_s},0\right) \end{matrix}\right|\right)$$

where $x_i = (x_{i1}, \cdots x_{i,n_1,n_2,\cdots,n_s}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots m_1, m_2 \cdots m_r$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

By a lacunary sequence $\theta = (m_r n_s)$, where $m_0 n_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_{rs} = m_r n_s - m_{r-1} n_{s-1} \to \infty$ as $r, s \to \infty$. The intervals determined by θ will be denoted by $I_{rs} = (m_{r-1} n_{s-1}, m_r n_s]$.

Let $F = (f_{mn})$ be a $mn-$ sequence of moduli musielak such that $lim_{u \to 0^+} sup_{mn} f_{mn}(u) = 0$. Throughout this paper $\chi^2_{A_{uv}} -$ convergence of $p-$ metric of $mn-$ sequence of musielak modulus function determinated by F will be denoted by $f_{mn} \in F$ for every $m, n \in \mathbb{N}$.

The purpose of this paper is to introduce and study a concept of lacunary strong $\chi^2_{A_{uv}} -$ convergence of $p-$ metric with respect to a $mn-$ sequence of moduli musielak. We now introduce the generalizations of lacunary strongly $\chi^2_{A_{uv}} -$ convergence of $p-$ metric with respect a $mn-$ sequence of musielak modulus function and investigate some inclusion relations.

Let A denote a sequence of the matrices $A^{uv} = \left(a^{m_1 \cdots m_r n_1 \cdots n_s}_{k_1 \cdots k_r \ell_1 \cdots \ell_s}(uv)\right)$ of complex numbers. We write for any sequence $x = (x_{mn}), y_{ij}(uv) = A^{uv}_{ij}(x) = \Sigma^{\infty}_{m_1 \cdots m_r} \Sigma^{\infty}_{n_1 \cdots n_s} \left(a^{m_1 \cdots m_r n_1 \cdots n_s}_{k_1 \cdots k_r \ell_1 \cdots \ell_s}(uv)\right)\left((m_1 \cdots m_r + n_1 \cdots n_s)! \left|x_{m_1 \cdots m_r n_1 \cdots n_s}\right|\right)^{1/m_1 \cdots m_r + n_1 \cdots n_s}$ if it exits for each i and uv. We $A^{uv}(x) = \left(A^{uv}_{ij}(x)\right)_{ij}, Ax = (A^{uv}(x))_{uv}$.

2.1 Definition

Let $F = \left(f^{ij}_{m_1\cdots m_r n_1 \cdots n_s}\right)$ be a $mn-$ sequence of moduli musielak, A denote the sequence of four dimensional infinte matrices of complex numbers and X be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous semi norms η and $\left(X, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right)$ be a $p-$metric space, $q = (q_{ij})$ be double analytic sequence of strictly positive real numbers. By $w^2(p-X)$ we denote the space of all sequences defined over $\left(X, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right)$. In the present paper we define the following sequence spaces:

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11}\right), d\left(x_{12}\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}}\right)\right)\right\|_p\right] =$$

$$lim_{rs}\left\{\left[f_{ij}\left(\left\|N^{\alpha}_{\theta}(x),, \left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right)\right]^{q_{ij}} = 0\right\},$$

where $N^{\alpha}_{\theta}(x) =$

$$\frac{1}{h^{\alpha}_{rs}}\Sigma_{i\in I_{rs}}\Sigma_{j\in I_{rs}}\left(\eta\left(A^{uv}_{ij}\left(\left(\left(m_1\cdots m_r+n_1\cdots n_s\right)!\left|x_{m_1\cdots m_r n_1\cdots n_s}\right|\right)^{1/m_1\cdots m_r+n_1\cdots n_s}\right)\right)\right),$$

uniformly in uv

$$\left[\Lambda^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] =$$

$$sup_{rs}\left\{\left[f_{uv}\left(\left\|N^{\alpha}_{\theta}(x), \left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right)\right]^{q_{ij}} < \infty\right\}$$

$$\text{where } e = \begin{pmatrix} 1\ 1\ \ldots 1 \\ 1\ 1\ \ldots 1 \\ . \\ . \\ . \\ 1\ 1\ \ldots 1 \end{pmatrix}.$$

3 Main Results

3.1 Proposition

$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ and $\left[\Lambda^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted.

The inclusion relation between $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ and $\left[\Lambda^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$.

3.2 Theorem

Let A be a $mn-$ sequence the four dimensional infinite matrices $A^{uv} = \left(a^{m_1\cdots m_r n_1\cdots n_s}_{k_1\cdots k_r \ell_1\cdots \ell_s}(uv)\right)$ of complex numbers and $F = \left(f^{ij}_{mn}\right)$ be a $mn-$ sequence of moduli musielak. If $x = (x_{mn})$ lacunary strong $A_{uv}-$ convergent of orer α to zero then $x = (x_{mn})$ lacunary strong $A_{uv}-$ convergent of order α to zero with respect to $mn-$ sequence of moduli musielak, (i.e)

$$\left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \subset \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$$

Proof: Let $F = \left(f^{ij}_{mn}\right)$ be a $mn-$ sequence of moduli musielak and put $sup f^{ij}_{mn}(1) = T$. Let$x = (x_{mn}) \in \left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $f^{ij}_{mn}(u) < \varepsilon$ for every u with $0 \le u \le \delta \, (i, j \in \mathbb{N})$. We can write

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] =$$

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] +$$

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$$

where the first part is over $\le \delta$ and second part is over $> \delta$. By definition of Musielak modulus f^{ij}_{mn} for every ij, we have

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \le \varepsilon^{H_2} + \left(2T\delta^{-1}\right)^{H_2}\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right].$$

Therefore $x = (x_{mn}) \in \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$.

3.3 Theorem

Let A be a $mn-$ sequence of the four dimensional infinite matrices $A^{uv} = \left(a^{m_1\cdots m_r n_1\cdots n_s}_{k_1\cdots k_r \ell_1\cdots \ell_s}(uv)\right)$ of complex numbers, $q = (q_{ij})$ be a $mn-$ sequence of positive real numbers with $0 < inf q_{ij} = H_1 \le sup q_{ij} = H_2 > \infty$ and $F = \left(f^{ij}_{mn}\right)$ be a $mn-$ sequence of moduli Musielak. If $lim_{u,v\to\infty} inf f_{ij}\frac{f_{ij}(uv)}{uv} > 0$, then

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] = \left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right].$$

Proof: If $lim_{u,v\to\infty} inf f_{ij}\frac{f_{ij}(uv)}{uv} > 0$, then there exists a number $\beta > 0$ such that $f_{ij}(uv) \ge \beta u$ for all $u \ge 0$ and $i, j \in \mathbb{N}$. Let $x = (x_{m_1}\cdots m_r n_1\cdots n_s) \in \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$. Clearly

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \ge \beta\left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right].$$

Therefore $x = (x_{m_1}\cdots m_r n_1\cdots n_s) \in \left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$. By using Theorem 3.2, the proof is complete.

We now give an example to show that $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \neq$

$\left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ in the case when $\beta = 0$. Consider $A = I$, unit matrix, $\eta(x) = ((m_1 \cdots m_r + n_1 \cdots n_s)! |x_{m_1\cdots m_r n_1\cdots n_s}|)^{1/m_1\cdots m_r+n_1\cdots n_s}$, $q_{ij} = 1$ for every $i,j \in \mathbb{N}$ and $f^{ij}_{mn}(x) = \frac{|x_{m_1\cdots m_r n_1\cdots n_s}|^{1/((m_1\cdots m_r+n_1\cdots n_s)(i+1)(j+1))}}{((m_1\cdots m_r+n_1\cdots n_s)!)^{1/m_1\cdots m_r+n_1\cdots n_s}}$ $(i,j \geq 1, x > 0)$ in the case $\beta > 0$. Now we define $x_{ij} = h^{\alpha}_{rs}$ if $i,j = m_r n_s$ for some $r,s \geq 1$ and $x_{ij} = 0$ otherwise. Then we have,

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \to 1 \text{ as } r,s \to \infty$$

and so $x = (x_{m_1\cdots m_r n_1\cdots n_s}) \notin \left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$

The inclusion Relation between $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ and $\left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$.

In this section we introduce natural relationship between lacunary $A^{uv}-$ statistical convergence of order α and lacunary strong $A^{uv}-$ convergence of order α with respect to $mn-$ sequence of moduli Musielak.

3.4 Definition

Let θ be a lacunary $mn-$ sequence. Then a $mn-$ sequence $x = (x_{m_1\cdots m_r n_1\cdots n_s})$ is said to be lacunary statistically convergent of order α to a number zero if for every $\varepsilon > 0, lim_{rs\to\infty} h^{-\alpha}_{rs} |K_{\theta}(\varepsilon)| = 0$, where $|K_{\theta}(\varepsilon)|$ denotes the number of elements in $K_{\theta}(\varepsilon) = \left\{i,j \in I_{rs} : ((m_1\cdots m_r + n_1\cdots n_s)! |x_{m_1\cdots m_r n_1\cdots n_s} - 0|)^{1/m_1\cdots m_r+n_1\cdots n_s} \geq \varepsilon\right\}$. The set of all lacunary statistical convergent of order α of $mn-$ sequences is denoted by S^{α}_{θ}.

Let $A^{uv} = \left(a^{m_1\cdots m_r n_1\cdots n_s}_{k_1\cdots k_r \ell_1\cdots \ell_s}(uv)\right)$ be an four dimensional infinite matrix of complex numbers. Then a $mn-$ sequence $x = (x_{m_1\cdots m_r n_1\cdots n_s})$ is said to be lacunary $A-$ statistically convergent of order α to a number zero if for every $\varepsilon > 0, lim_{rs\to\infty} h^{-\alpha}_{rs} |KA_{\theta}(\varepsilon)| = 0$, where $|KA_{\theta}(\varepsilon)|$denotes the number of elements in $KA_{\theta}(\varepsilon) = \left\{i,j \in I_{rs} : ((m_1\cdots m_r + n_1\cdots n_s)! |x_{m_1\cdots m_r n_1\cdots n_s} - 0|)^{1/m_1\cdots m_r+n_1\cdots n_s} \geq \varepsilon\right\}$.The set of all lacunary $A-$ statistical convergent of order α of $mn-$ sequences is denoted by $S^{\alpha}_{\theta}(A)$.

3.5 Definition

Let A be a $mn-$ sequence of the four dimensional infinite matrices $A^{uv} = \left(a^{m_1\cdots m_r n_1\cdots n_s}_{k_1\cdots k_r \ell_1\cdots \ell_s}(uv)\right)$ of complex numbers and let $q = (q_{ij})$ be a $mn-$ sequence of positive real numbers with $0 < inf q_{ij} = H_1 \leq sup q_{ij} = H_2 < \infty$. Then a $mn-$ sequence $x = (x_{m_1\cdots m_r n_1\cdots n_s})$ is said to be lacunary $A^{uv}-$ statistically convergent of order α to a number zero if for every $\varepsilon > 0, lim_{rs\to\infty} h^{-\alpha}_{rs} |KA_{\theta\eta}(\varepsilon)| = 0$, where $|KA_{\theta\eta}(\varepsilon)|$ denotes the number of elements in $KA_{\theta\eta}(\varepsilon) = \left\{i,j \in I_{rs} : ((m_1\cdots m_r + n_1\cdots n_s)! |x_{m_1\cdots m_r n_1\cdots n_s} - 0|)^{1/m_1\cdots m_r+n_1\cdots n_s} \geq \varepsilon\right\}$. The set of all lacunary $A_{\eta}-$ statistical convergent of order α of $mn-$ sequences is denoted by $S^{\alpha}_{\theta}(A,\eta)$.

The following theorems give the relations between lacunary $A^{uv}-$ statistical convergence of order α and lacunary strong $A^{uv}-$ convergence of order α with respect to a $mn-$ sequence of moduli Musielak.

3.6 Theorem

Let $F = (f_{ij})$ be a $mn-$ sequence of moduli Musielak. Then $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \subseteq \left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ if and only if $lim_{ij\to\infty} f_{ij}(u) > 0, (u > 0)$.

Proof: Let $\varepsilon > 0$ and $x = (x_{m_1\cdots m_r n_1\cdots n_s}) \in \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$. If $lim_{ij\to\infty} f_{ij}(u) > 0, (u > 0)$, then there exists a number $d > 0$ such that $f_{ij}(\varepsilon) > d$ for $u > \varepsilon$ and $i,j \in \mathbb{N}$. Let $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \geq h^{-\alpha}_{rs} d^{H_1} KA_{\theta\eta}(\varepsilon)$. It follows that $\left[\chi^{2\eta}_{AfS^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$. Conversely, suppose that $lim_{ij\to\infty} f_{ij}(u) > 0$ does not hold, then there is a number $t > 0$ such that $lim_{ij\to\infty} f_{ij}(t) = 0$. We can select a lacunary $mn-$ sequence $\theta = (m_1\cdots m_r n_1\cdots n_s)$ such that $f_{ij}(t) < 2^{-rs}$ for any $i > m_1\cdots m_r, j > n_1\cdots n_s$. Let $A = I$, unit matrix, define the $mn-$ sequence x by putting $x_{ij} = t$ if $m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1} < i,j < \frac{m_1,m_2,\cdots m_r n_1,n_2,\cdots n_s + m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}}{2}$ and $x_{ij} = 0$ if $\frac{m_1,m_2,\cdots m_r n_1,n_2,\cdots n_s + m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}}{2} \leq i,j \leq m_1,m_2,\cdots m_r n_1,n_2,\cdots n_s$. We have $x = (x_{m_1\cdots m_r n_1\cdots n_s}) \in \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ but $x \notin \left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$.

3.7 Theorem

Let $F = (f_{ij})$ be a $mn-$ sequence of moduli Musielak. Then $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \supseteq \left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ if and only if $sup_u sup_{ij} f_{ij}(u) < \infty$.

Proof: Let $x \in \left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \left\|\left(d(x_{11},0), d(x_{12},0), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$. Suppose that $h(u) = sup_{ij} f_{ij}(u)$ and $h = sup_u h(u)$. Since $f_{ij}(u) \leq h$ for all i,j and $u > 0$, we have for all u,v,

$\left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \left\| \left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right] \leq h^{H_2} h_{rs}^{-\alpha} \left|KA_{\theta\eta}\left(\varepsilon\right)\right| + \left|h\left(\varepsilon\right)\right|^{H_2}$. It follows from $\varepsilon \to 0$ that $x \in \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\| \left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$.

Conversely, suppose that $sup_u sup_{ij} f_{ij}\left(u\right) = \infty$. Then we have $0 < u_{11} < \cdots < u_{r-1s-1} < u_{rs} < \cdots$, such that $f_{m_r n_s}\left(u_{rs}\right) \geq h^{\alpha}_{rs}$ for $r,s \geq 1$. Let $A = I$, unit matrix, define the $mn-$ sequence x by putting $x_{ij} = u_{rs}$ if $i,j = m_1 m_2 \cdots m_r n_1 n_2 \cdots n_s$ for some $r,s = 1,2,\cdots$ and $x_{ij} = 0$ otherwise. Then we have $x \in \left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \left\| \left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$ but $x \notin \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \left\| \left(d\left(x_{11},0\right), d\left(x_{12},0\right), \cdots, d\left(x_{m_1,m_2,\cdots m_{r-1}n_1,n_2,\cdots n_{s-1}},0\right)\right)\right\|_p\right]$.

4 Conclusion

We study characterization of certain lacunary strong $\chi^2_{A_{uv}}-$convergence with respect to a mn sequence of moduli Musielak and lacunary $\chi^2_{A_{uv}}-$ statistical convergence, where A is a sequence of four dimensional matrices $A\left(uv\right) = \left(a^{m_1\cdots m_r n_1\cdots n_s}_{k_1\cdots k_r \ell_1\cdots \ell_s}\left(uv\right)\right)$ and also inclusion results are discuss about in above sequence spaces.

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