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Entire Labeling of Plane Graphs

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Abstract: A face irregular entire k-labeling $\varphi : V \cup E \cup F \to \{1,2,\dots,k\}$ of a 2-connected plane graph $G = (V,E,F)$ is a labeling of vertices, edges and faces of G in such a way that for any two different faces f and g their weights $w_\varphi(f)$ and $w_\varphi(g)$ are distinct. The *weight* of a face f under a k-labeling φ is the sum of labels carried by that face and all the edges and vertices incident with the face. The minimum k for which a plane graph G has a face irregular entire k-labeling is called the entire face irregularity strength. We investigate a face irregular entire labeling as a modification of the well-known vertex irregular and edge irregular total labelings of graphs. We obtain some estimations on the entire face irregularity strength and determine the precise values for graphs from three families of plane graphs.

Keywords: face irregular entire k-labeling, entire face irregularity strength, plane graph

1 Introduction

We consider finite undirected graphs without loops and multiple edges. Denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively.

Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba in [5] introduced labelings of the edges of a graph G with positive integers such that the sum of the labels of edges incident with a vertex is different for all the vertices. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [3,6,7,12,14].

Motivated by irregular assignments Bača, Jendroľ, Miller and Ryan in [2] defined a *vertex irregular total k-labeling* of a graph $G = (V,E)$ to be a labeling of the vertices and edges of G

$$\phi : V \cup E \to \{1,2,\dots,k\}$$

such that the *total vertex-weights*

$$wt_\phi(x) = \phi(x) + \sum_{xy \in E} \phi(xy)$$

are different for all vertices, that is, $wt_\phi(x) \neq wt_\phi(y)$ for all different vertices $x,y \in V$. Furthermore, they defined the *total vertex irregularity strength*, $tvs(G)$, of G as the minimum k for which G has a vertex irregular total k-labeling.

It is easy to see that irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G. Moreover, for graphs with no component of order ≤ 2, $tvs(G) \leq s(G)$.

In [2] several bounds and exact values of tvs were determined for different types of graphs (in particular for stars, cliques and prisms). These results were then improved by Przybylo [18], Anholcer *et al.* [1] and Nurdin *et al.* [15,16].

Furthermore, in [2] the authors defined an *edge irregular total k-labeling* of a graph $G = (V,E)$ to be a labeling $\phi : V \cup E \to \{1,2,\dots,k\}$ such that the edge weights $wt_\phi(xy) = \phi(x) + \phi(xy) + \phi(y)$ are different for all edges, that is, $wt_\phi(xy) \neq wt_\phi(x'y')$ for all edges

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$xy, x'y' \in E$ with $xy \neq x'y'$. They defined the *total edge irregularity strength*, $tes(G)$, of G as the minimum k for which G has an edge irregular total k-labeling and determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs.

Recently Ivančo and Jendroľ [9] posed a conjecture that for arbitrary graph G different from K_5 and maximum degree $\Delta(G)$,

$$tes(G) = \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}.$$

This conjecture has been verified for complete graphs and complete bipartite graphs in [10] and [11], for the Cartesian product of two paths in [13], for generalized Petersen graphs in [8], for corona product of a path with certain graphs in [17] and for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [4].

Motivated by total irregularity strengths and a recent paper on entire colouring of plane graphs [19] we study irregular labelings of plane graphs with restrictions placed on the weights of faces. A *plane* graph is a particular drawing of a planar graph on the Euclidean plane. Suppose that $G = (V, E, F)$ is a 2-connected plane graph with face set F.

Now, for a plane graph $G = (V, E, F)$ we define a labeling $\varphi : V \cup E \cup F \to \{1, 2, \ldots, k\}$ to be an entire k-labeling. The *weight* of a face f under an entire k-labeling φ, $w_\varphi(f)$, is the sum of labels carried by that face and all the edges and vertices surrounding it. An entire k-labeling φ is defined to be a *face irregular entire k-labeling* of the plane graph G if for every two different faces f and g of G there is $w_\varphi(f) \neq w_\varphi(g)$.

The *entire face irregularity strength*, denoted $efs(G)$, of a plane graph G is the smallest integer k such that G has a face irregular entire k-labeling.

The main aim of this paper is to obtain estimations on the parameter *efs* and determine the precise values of *efs* for some families of plane graphs.

2 Bounds for the entire face irregularity strength

Next theorem provides lower and upper bounds on parameter *efs*.

Theorem 2.1. Let $G = (V, E, F)$ be a 2-connected plane graph with n_i i-sided faces, $i \geq 3$. Let $a = \min\{i | n_i \neq 0\}$ and $b = \max\{i | n_i \neq 0\}$. Then

$$\left\lceil \frac{2a + n_3 + n_4 + \cdots + n_b}{2b+1} \right\rceil \leq efs(G) \leq$$

$$\leq \max\{n_i | 3 \leq i \leq b\} = m. \quad (1)$$

Proof. Let φ be a face irregular entire k-labeling of a 2-connected plane graph $G = (V, E, F)$ with $efs(G) = k$.

Clearly $2a + 1 \leq w_\varphi(f) \leq (2b+1)k$ for every face $f \in F$, and therefore $|F| = n_3 + n_4 + \cdots + n_b \leq (2b+1)k - 2a$. This implies

$$k = efs(G) \geq \left\lceil \frac{2a + |F|}{2b+1} \right\rceil.$$

To see the upper bound let us label the elements of G as follows:

$$\varphi(v) = l = \left\lfloor \frac{m}{2} \right\rfloor \text{ for every } v \in V,$$

$$\varphi(e) = h = \left\lceil \frac{m}{2} \right\rceil \text{ for every } e \in E.$$

Let $f_1^i, f_2^i, \ldots, f_{n_i}^i$ be i-sided faces of G, $i \geq 3$. Then we put

$$\varphi(f_j^i) = j \text{ for all } 1 \leq j \leq n_i, \text{ and all } 3 \leq i \leq b.$$

Clearly $\varphi(f_j^i) \leq m$.

Now we have to show that $w_\varphi(f) \neq w_\varphi(g)$ whenever $f \neq g$.

Let $f = f_r^i \neq f_s^t = g$ and, without loss of generality, $i < t$. Then $w_\varphi(f_r^i) = li + hi + r = mi + r \leq m(i+1) < mt + 1 \leq mt + s = lt + ht + s = w_\varphi(f_s^t)$.

If $f = f_r^i \neq f_s^t = g$ and $i = t$, then let, without loss of generality, $r < s$. In this case $w_\varphi(f_r^i) = mi + r < mi + s = w_\varphi(f_s^t)$. □

If $n_b = 1$ then by applying similar reasoning as in the proof of previous theorem to the i-sided faces, $i \leq c$, we have

Theorem 2.2. Let $G = (V, E, F)$ be a 2-connected plane graph with n_i i-sided faces, $i \geq 3$. Let $a = \min\{i | n_i \neq 0\}$, $b = \max\{i | n_i \neq 0\}$, $n_b = 1$ and $c = \max\{i | n_i \neq 0, \ i < b\}$. Then

$$efs(G) \geq \left\lceil \frac{2a + |F| - 1}{2c+1} \right\rceil. \quad (2)$$

The lower bound in Theorem 2.2 is tight. It can be seen from the following two theorems which determine the exact values of the entire face irregularity strength for certain families of plane graphs.

Theorem 2.3. Let $L_n \simeq P_n \square P_2$, $n \geq 3$, be a ladder. Then

$$efs(L_n) = \left\lceil \frac{n+7}{9} \right\rceil. \quad (3)$$

Proof. Let $L_n \simeq P_n \square P_2$, $n \geq 3$, be a ladder with $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. The ladder L_n contains $n-1$ 4-sided faces and the external $2n$-sided face. Denote by f_i^4 the 4-sided face surrounded by vertices $u_i, u_{i+1}, v_i, v_{i+1}$ and edges $u_i u_{i+1}, v_i v_{i+1}, u_i v_i, u_{i+1} v_{i+1}$, for $i = 1, 2, \ldots, n-1$, and denote by f^{ext} the external $2n$-sided face. From (2) we have $efs(L_n) \geq \lceil (n+7)/9 \rceil$.

Put $k = \lceil (n+7)/9 \rceil$. To show that k is an upper bound for entire face irregularity strength of L_n we describe an entire k-labeling $\varphi_1 : V \cup E \cup F \to \{1,2,\dots,k\}$ as follows:

$$\varphi_1(u_i) = \left\lfloor \frac{i+4}{9} \right\rfloor + 1, \text{ for } 1 \le i \le n$$

$$\varphi_1(v_i) = \left\lfloor \frac{i+2}{9} \right\rfloor + 1, \text{ for } 1 \le i \le n$$

$$\varphi_1(u_i u_{i+1}) = \left\lfloor \frac{i+7}{9} \right\rfloor + 1, \text{ for } 1 \le i \le n-1$$

$$\varphi_1(v_i v_{i+1}) = \left\lfloor \frac{i+6}{9} \right\rfloor + 1, \text{ for } 1 \le i \le n-1$$

$$\varphi_1(u_i v_i) = \left\lfloor \frac{i}{9} \right\rfloor + 1, \text{ for } 1 \le i \le n$$

$$\varphi_1(f_i^4) = \left\lceil \frac{i}{9} \right\rceil, \text{ for } 1 \le i \le n-1$$

$$\varphi_1(f^{ext}) = k.$$

It is easy to see that the labeling φ_1 is an entire k-labeling, the weights of 4-sided faces are all distinct and constitute the set $\{9,10,\dots,n+7\}$. Since for $n \ge 3$, $\sum_{i=1}^{n}(\varphi_1(u_i)+\varphi_1(v_i)) + \sum_{i=1}^{n-1}(\varphi_1(u_iu_{i+1})+\varphi_1(v_iv_{i+1})) \ge 4n-2$ and $\varphi_1(f_{n-1}^4) \le \varphi_1(f^{ext})$ then $w_{\varphi_1}(f_{n-1}^4) = n+7 < w_{\varphi_1}(f^{ext})$. This concludes the proof. □

Another variation of a ladder graph is specified as follows. A graph B_n, $n \ge 3$, is a plane graph obtained by completing the ladder $L_n \simeq P_n \square P_2$ by vertices z_i, for $1 \le i \le n$, and by edges u_iz_i, for $1 \le i \le n$, and $u_{i+1}z_i, z_iz_{i+1}$, for $1 \le i \le n-1$. The graph B_n contains $2n-2$ 3-sided faces, $n-1$ 4-sided faces and the external $(2n+2)$-sided face.

Theorem 2.4. For B_n, $n \ge 3$, we have

$$efs(B_n) = \left\lceil \frac{n+1}{3} \right\rceil. \qquad (4)$$

Proof. Let B_n, $n \ge 3$, be the plane graph with 3-sided faces, 4-sided faces and the external $(2n+2)$-sided face. Denote by g_i^3 the 3-sided face surrounded by vertices u_i, u_{i+1}, z_i and edges $u_iz_i, u_iu_{i+1}, u_{i+1}z_i$, for $i = 1,2,\dots,n-1$, and denote by h_i^3 3-sided face surrounded by vertices u_{i+1}, z_i, z_{i+1} and edges $u_{i+1}z_i, z_iz_{i+1}, u_{i+1}z_{i+1}$, for $i = 1,2,\dots,n-1$. Denote the 4-sided faces and the external face in the same way as in the proof of Theorem 2.3.

Let $k = \lceil \frac{n+1}{3} \rceil$. It follows from (2) that k is a lower bound for $efs(B_n)$. To show that k is an upper bound for $efs(B_n)$ it suffices to prove the existence of an optimal entire labeling $\varphi_2 : V(B_n) \cup E(B_n) \cup F(B_n) \to \{1,2,\dots,\lceil (n+1)/3 \rceil\}$. For $n \ge 3$ we construct the function φ_2 in the following way:

$$\varphi_2(z_i) = \varphi_2(u_i) = \begin{cases} 1, & \text{for } i = 1,2 \\ \lfloor \frac{i}{3} \rfloor + 1, & \text{for } 3 \le i \le n \end{cases}$$

$$\varphi_2(v_i) = \begin{cases} i+2, & \text{for } 1 \le i \le k-3 \\ k, & \text{for } k-2 \le i \le n \end{cases}$$

$$\varphi_2(z_iz_{i+1}) = \varphi_2(u_{i+1}z_i) = \left\lceil \frac{i}{3} \right\rceil, \text{ for } 1 \le i \le n-1$$

$$\varphi_2(u_iu_{i+1}) = \left\lceil \frac{i}{3} \right\rceil, \text{ for } 1 \le i \le n-1$$

$$\varphi_2(u_iz_i) = \left\lceil \frac{i}{3} \right\rceil, \text{ for } 1 \le i \le n$$

$$\varphi_2(v_iv_{i+1}) = \begin{cases} i+2, & \text{for } 1 \le i \le k-3 \\ k, & \text{for } k-2 \le i \le n-1 \end{cases}$$

$$\varphi_2(u_iv_i) = \begin{cases} i+2, & \text{for } 1 \le i \le k-3 \\ k, & \text{for } k-2 \le i \le n \end{cases}$$

$$\varphi_2(f_i^4) = \begin{cases} i, & \text{for } 1 \le i \le k-3 \\ k, & \text{for } k-2 \le i \le n-1 \end{cases}$$

$$\varphi_1(f^{ext}) = k$$

$$\varphi_2(h_i^3) = \begin{cases} 2, & \text{for } 1 \le i \le 7 \\ \lceil \frac{i-1}{3} \rceil, & \text{for } 8 \le i \le n-1 \end{cases}$$

$$\varphi_2(g_i^3) = \begin{cases} 1, & \text{for } i = 1,4 \\ 2, & \text{for } i = 2,3 \end{cases}$$

$$\varphi_2(g_i^3) = \begin{cases} \lceil \frac{i}{3} \rceil, & \text{for } i \ge 5,\ i \equiv 0,2 \ (mod\ 3) \\ \lceil \frac{i}{3} \rceil - 2, & \text{for } i \ge 7,\ i \equiv 1 \ (mod\ 3). \end{cases}$$

Observe that the face weights receive values

$$w_{\varphi_2}(f_i^4) = \begin{cases} 7(i+2), & \text{for } 1 \le i \le k-2 \\ 6k+2+i, & \text{for } k-1 \le i \le n-1 \end{cases}$$

$$w_{\varphi_2}(g_i^3) = \begin{cases} 2i+5, & \text{for } 1 \le i \le 7 \\ \frac{7i+10}{3}, & \text{for } i \ge 8, i \equiv 2 \pmod 3 \\ \frac{7i}{3}+3, & \text{for } i \ge 9, i \equiv 0 \pmod 3 \\ \frac{7i+8}{3}, & \text{for } i \ge 10, i \equiv 1 \pmod 3 \end{cases}$$

$$w_{\varphi_2}(h_i^3) = \begin{cases} 2i+6, & \text{for } 1 \le i \le 7 \\ \frac{7i+13}{3}, & \text{for } i \ge 8, i \equiv 2 \pmod 3 \\ \frac{7i}{3}+4, & \text{for } i \ge 9, i \equiv 0 \pmod 3 \\ \frac{7i+11}{3}, & \text{for } i \ge 10, i \equiv 1 \pmod 3. \end{cases}$$

It is a routine matter to verify that all vertex, edge and face labels are at most k and the face weights are different for all pairs of distinct faces. In fact, our labeling φ_2 has been chosen in such a way that the weights of 3-sided and 4-sided faces form the sequence of different integers from 7 up to $6k+n+1$.

Since for $n \ge 3$, $\varphi_2(u_{n-1}) \le \varphi_2(z_n)$, $\varphi_2(u_{n-1}u_n) = \varphi_2(z_{n-1}z_n)$, $\varphi_2(u_{n-1}v_{n-1}) < \varphi_2(u_nz_n) + \varphi_2(v_{n-2}v_{n-1})$ and $\varphi_2(f_{n-1}^4) \le \varphi_2(f^{ext})$ then $w_{\varphi_2}(f_{n-1}^4) < w_{\varphi_2}(f^{ext})$.

Thus, the labeling φ_2 is desired face irregular entire k-labeling. □

If we consider the maximum degree in a 2-connected plane graph, we obtain the following theorem.

Theorem 2.5. Let $G=(V,E,F)$ be a 2-connected plane graph with maximum degree Δ. Let x be a vertex of degree Δ and let the smallest face and the biggest face incident with x be an a-sided face and a b-sided face, respectively. Then

$$efs(G) \geq \left\lceil \frac{2a+\Delta-1}{2b} \right\rceil. \tag{5}$$

Proof. Suppose that φ is an optimal face irregular entire labeling of a 2-connected plane graph G. Let $f_1, f_2, \dots, f_\Delta$ be the faces incident with a fixed vertex x of maximum degree Δ in G and let a-sided face be the smallest and b-sided face be the biggest of them.

The face weights $w_\varphi(f_1), w_\varphi(f_2), \dots, w_\varphi(f_\Delta)$ are all distinct and each of them contains the value $\varphi(x)$. The largest among these face weights must be at least $\varphi(x)+2a+\Delta-1$. This weight can be the sum of at most $2b$ labels (without $\varphi(x)$). So at least one label is at least $\lceil(2a+\Delta-1)/(2b)\rceil$. $\square$.

The lower bound (5) is tight for wheels.

Theorem 2.6. Let W_n be a wheel on $n+1$ vertices, $n \geq 3$. Then

$$efs(W_n) = \left\lceil \frac{n+5}{6} \right\rceil. \tag{6}$$

Proof. A *wheel* W_n, $n \geq 3$, is a plane graph obtained by joining all vertices of cycle C_n to a further vertex v, called the *center*. Thus W_n contains $n+1$ vertices, say, $v, v_1, v_2, \dots, v_n$ and $2n$ edges, say, vv_i, $1 \leq i \leq n$, v_iv_{i+1}, $1 \leq i \leq n-1$, and v_nv_1. Denote by f_i^3 the 3-sided face surrounded by vertices v, v_i, v_{i+1} and edges $vv_i, v_iv_{i+1}, vv_{i+1}$, for $i=1,2,\dots,n-1$. Denote by f_n^3 the face surrounded by vertices v, v_1, v_n and edges vv_n, v_1v_n, vv_1 and the external n-sided face denote by f^{ext}.

In view of the lower bound (5) it suffices to prove the existence of an entire labeling $\varphi_3 : V(W_n) \cup E(W_n) \cup F(W_n) \to \{1,2,\dots,\lceil\frac{n+5}{6}\rceil\}$ such that $w_{\varphi_3}(f) \neq w_{\varphi_3}(g)$ for every $f, g \in F(W_n)$ with $f \neq g$.

For $n=3$ we define a desired entire labeling as follows: $\varphi_3(vv_i) = \varphi_3(f_i^3) = 1$, for $1 \leq i \leq 3$, $\varphi_3(v_i) = \varphi_3(v_iv_{i+1}) = 1$, for $1 \leq i \leq 2$, $\varphi_3(v) = 1$ and $\varphi_3(v_3) = \varphi_3(v_3v_1) = \varphi_3(f^{ext}) = 2$. We can see that $w_{\varphi_3}(f^{ext}) = 10$ and $w_{\varphi_3}(f_i^3) = 2n+i$, for $1 \leq i \leq 3$.

For $n \geq 4$, we describe the entire labeling in the following way:

$$\varphi_3(v) = 1$$

$$\varphi_3(v_i) = \begin{cases} 1, & \text{for } i=1,2 \\ \lceil\frac{i+1}{3}\rceil, & \text{for } 3 \leq i \leq \lceil\frac{n}{2}\rceil+1 \\ \lceil\frac{n-i+1}{3}\rceil+1, & \text{for } \lceil\frac{n}{2}\rceil+2 \leq i \leq n \end{cases}$$

$$\varphi_3(v_nv_1) = 1$$

$$\varphi_3(vv_i) = \begin{cases} 1, & \text{for } i=1,2,n \\ \lceil\frac{i+1}{3}\rceil, & \text{for } 3 \leq i \leq \lfloor\frac{n}{2}\rfloor+1 \\ \lceil\frac{n-i+2}{3}\rceil, & \text{for } \lfloor\frac{n}{2}\rfloor+2 \leq i \leq n-1 \end{cases}$$

$$\varphi_3(f_i^3) = \begin{cases} \lceil\frac{i}{3}\rceil, & \text{for } 1 \leq i \leq \lfloor\frac{n}{2}\rfloor \\ \lceil\frac{n+5}{6}\rceil, & \text{for } i = \lfloor\frac{n}{2}\rfloor+1 \\ \lceil\frac{n-i+2}{3}\rceil, & \text{for } \lfloor\frac{n}{2}\rfloor+2 \leq i \leq n \end{cases}$$

$$\varphi_3(f^{ext}) = \left\lceil \frac{n+5}{6} \right\rceil.$$

For n even we define

$$\varphi_3(v_iv_{i+1}) = \begin{cases} \lceil\frac{i}{3}\rceil, & \text{for } 1 \leq i \leq \frac{n}{2}+1 \\ \lceil\frac{n-i}{3}\rceil+1, & \text{for } \frac{n}{2}+2 \leq i \leq n-1. \end{cases}$$

For n odd we define

$$\varphi_3(v_iv_{i+1}) = \begin{cases} \lceil\frac{i}{3}\rceil, & \text{for } 1 \leq i \leq \frac{n-1}{2} \\ \lceil\frac{n-i}{3}\rceil+1, & \text{for } \frac{n+1}{2} \leq i \leq n-1. \end{cases}$$

The weights of 3-sided faces attain values

$$w_{\varphi_3}(f_i^3) = \begin{cases} 2i+5, & \text{for } 1 \leq i \leq \lfloor\frac{n}{2}\rfloor \\ 2n-2i+8, & \text{for } \lfloor\frac{n}{2}\rfloor+2 \leq i \leq n \end{cases}$$

that is, successively attain the values $7, 8, \dots, n+5$. The weight of the last 3-sided face $w_{\varphi_3}(f^3_{\lfloor\frac{n}{2}\rfloor+1}) = \varphi_3(v) + \varphi_3(v_{\lfloor\frac{n}{2}\rfloor+1}) + \varphi_3(v_{\lfloor\frac{n}{2}\rfloor+2}) + \varphi_3(v_{\lfloor\frac{n}{2}\rfloor+1}v_{\lfloor\frac{n}{2}\rfloor+2}) + \varphi_3(vv_{\lfloor\frac{n}{2}\rfloor+1}) + \varphi_3(vv_{\lfloor\frac{n}{2}\rfloor+2}) + \varphi_3(f^3_{\lfloor\frac{n}{2}\rfloor+1})$

for n even gives $w_{\varphi_3}(f^3_{\lfloor\frac{n}{2}\rfloor+1}) = 1 + \lceil\frac{n+4}{6}\rceil + \lceil\frac{n-2}{6}\rceil + 1 + \lceil\frac{n+2}{6}\rceil + \lceil\frac{n+4}{6}\rceil + \lceil\frac{n}{6}\rceil + \lceil\frac{n+5}{6}\rceil > n+5$

and for n odd gives $w_{\varphi_3}(f^3_{\lfloor\frac{n}{2}\rfloor+1}) = 1 + \lceil\frac{n+3}{6}\rceil + \lceil\frac{n-1}{6}\rceil + 1 + \lceil\frac{n-1}{6}\rceil + 1 + \lceil\frac{n+3}{6}\rceil + \lceil\frac{n+1}{6}\rceil + \lceil\frac{n+5}{6}\rceil > n+5$.

For $n=4$, $w_{\varphi_3}(f^3_{\lfloor\frac{n}{2}\rfloor+1}) = 11 < w_{\varphi_3}(f^{ext}) = 12$.

Since for $n \geq 5$, $\varphi_3(f^3_{\lfloor\frac{n}{2}\rfloor+1}) = \varphi_3(f^{ext})$ and $\varphi_3(v) + \varphi_3(vv_{\lfloor\frac{n}{2}\rfloor+1}) + \varphi_3(vv_{\lfloor\frac{n}{2}\rfloor+2}) < \varphi_3(v_{\lfloor\frac{n}{2}\rfloor-1}v_{\lfloor\frac{n}{2}\rfloor}) + \varphi_3(v_{\lfloor\frac{n}{2}\rfloor}v_{\lfloor\frac{n}{2}\rfloor+1}) + \varphi_3(v_{\lfloor\frac{n}{2}\rfloor+2}v_{\lfloor\frac{n}{2}\rfloor+3}) + \varphi_3(v_{\lfloor\frac{n}{2}\rfloor+3}v_{\lfloor\frac{n}{2}\rfloor+4})$ then the weight of the external n-sided face is greater than $w_{\varphi_3}(f^3_{\lfloor\frac{n}{2}\rfloor+1})$. So the labeling φ_3 has the required properties of a face irregular entire labeling. $\square$

3 Conclusion

In this paper we introduced a new graph parameter, the entire face irregularity strength, $efs(G)$, as a modification of the well-known irregularity strength, total edge irregularity strength and total vertex irregularity strength. We proved that for every 2-connected plane graph $G=(V,E,F)$ with n_i i-sided faces, $i \geq 3$, $\left\lceil\frac{2a+n_3+n_4+\cdots+n_b}{2b+1}\right\rceil \leq efs(G) \leq m$, where $a = \min\{i|n_i \neq 0\}$, $b = \max\{i|n_i \neq 0\}$ and $m = \max\{n_i|3 \leq i \leq b\}$.

This lower bound of the entire face irregularity strength can not be improved in general. Because if $n_b = 1$ and $c = \max\{i | n_i \neq 0,\ i < b\}$ then we obtain that $efs(G) \geq \left\lceil \frac{2a+|F|-1}{2c+1} \right\rceil$ and the sharpness of this lower bound is reached for graph of the ladder $L_n \simeq P_n \Box P_2$.

We suppose that the upper bound of the entire face irregularity strength can be improved.

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References

[1] M. Anholcer, M. Kalkowski and J. Przybylo, Discrete Math. **309**, 6316–6317 (2009).
[2] M. Bača, S. Jendroľ, M. Miller and J. Ryan, Discrete Math. **307**, 1378–1388 (2007).
[3] T. Bohman and D. Kravitz, J. Graph Theory **45**, 241–254 (2004).
[4] S. Brandt, J. Miškuf and D. Rautenbach, J. Graph Theory **57**, 333–343 (2008).
[5] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, Congr. Numer. **64**, 187–192 (1988).
[6] R.J. Faudree, M.S. Jacobson, J. Lehel and R.H. Schlep, Discrete Math. **76**, 223–240 (1988).
[7] A. Frieze, R.J. Gould, M. Karonski and F. Pfender, J. Graph Theory **41**, 120–137 (2002).
[8] K.M.M. Haque, Theory of Computing Systems **50**, 537–544 (2012).
[9] J. Ivančo and S. Jendroľ, Discussiones Math. Graph Theory **26**, 449–456 (2006).
[10] S. Jendroľ, J.Miškuf and R. Soták, Electron. Notes Discrete Math. **28**, 281–285 (2007).
[11] S. Jendroľ, J.Miškuf and R. Soták, Discrete Math. **310**, 400–407 (2010).
[12] S. Jendroľ, M. Tkáč and Z. Tuza, Discrete Math. **150**, 179–186 (1996).
[13] J. Miškuf and S. Jendroľ, Tatra Mt. Math. Publ. **36**, 147–151 (2007).
[14] T. Nierhoff, SIAM J. Discrete Math. **13**, 313–323 (2000).
[15] Nurdin, E.T. Baskoro, A.N.M. Salman and N.N. Goas, J. Combin. Math. Combin. Comput. **71**, 129–136 (2009).
[16] Nurdin, E.T. Baskoro, A.N.M. Salman and N.N. Goas, Discrete Math. **310**, 3043–3048 (2010).
[17] Nurdin, A.N.M. Salman and E.T. Baskoro, J. Combin. Math. Combin. Comput. **65**, 163–175 (2008).
[18] J. Przybylo, SIAM J. Discrete Math. **23**, 511–516 (2009).
[19] W. Wang and X. Zhu, J. Combin Theory Ser. B **101**, 490–501 (2011).

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