Application of Fractional Calculus to a Class of Multivalent $\beta$-Uniformly Convex Functions

S. M. Khairnar and Meena More
Department of Mathematics, Maharashtra Academy of Engineering
Alandi, Pune - 412105, Maharashtra State, India
Email Addresses: smkhairnar2007@gmail.com; meenamores@gmail.com
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In this paper we introduce a class of multivalent functions which is $\beta$-uniformly convex in the unit disc. Characterization property exhibited and relation with other fractional calculus operators are given. Connections with the popular classes like $\beta$-uniformly convex and parabolic convex functions are pointed out. Results on modified Hadamard product, extreme points, growth and distortion theorems, class preserving integral operators, region of $p$-valency and radius of $\beta$-uniform convexity are also derived.

Keywords: Multivalent, convex, $\beta$-uniformly convex, fractional calculus operator, region of $p$-valency, radius of $\beta$-uniform convexity.

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1 Introduction and Preliminaries

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \ (n, p \in \mathbb{N}),$$

(1.1)

which are analytic in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and let $S(p)$ denote the class of functions defined by (1.1) which are analytic and multivalent in $U$. Consider the subclass $T(p)$ of $S(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \ (a_k \geq 0, n, p \in \mathbb{N}).$$

(1.2)

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A function \( f(z) \in S(p) \) is said to be multivalently starlike of order \( s, 0 \leq s < p \) in \( U \), if
\[
\text{Re} \left\{ z \frac{f'(z)}{f(z)} \right\} > s \tag{1.3}
\]
and multivalently convex of order \( s, 0 \leq s < p \) in \( U \), if
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > s. \tag{1.4}
\]
A function \( f(z) \in S(p) \) is said to be uniformly convex in \( U \), if \( f(z) \) is convex in \( U \) and has the property that every circular arc \( \gamma \), contained in \( U \) with center \( \xi \) in \( U \), arc \( f(\gamma) \) is convex with respect to \( f(\xi) \).

This definition of uniformly convex functions was given by A. W. Goodman [4] in 1991. The class of uniformly convex functions is denoted by \( \text{UCV} \). We have the characterization:
\[
f \in \text{UCV}, \text{ if and only if } \quad \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|. \tag{1.5}
\]
We can further generalize the class \( \text{UCV} \) by introducing a parameter \( \alpha, -p \leq \alpha < p \).
\[
f \in \text{UCV}(\alpha), \text{ if and only if } \quad \text{Re} \left\{ 1 + z \frac{zf''(z)}{f'(z)} - \alpha \right\} \geq \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|. \tag{1.6}
\]
Further, let \( 0 \leq \beta < \infty \). Then the function \( f \in S(p) \) is said to be \( \beta \)-uniformly convex in \( U \), if the image of every circular arc \( \gamma \) contained in \( U \), with center \( \xi \) in \( U \), where \( |\xi| \leq \beta \), is convex. For fixed \( \beta \), the class of all \( \beta \)-uniformly convex functions is denoted by \( \beta - \text{UCV} \).

Notice that, \( 0 - \text{UCV} = \text{CV} \), set of all convex functions and \( 1 - \text{UCV} = \text{UCV} \) as defined in (1.6) as before. We again note that \( f \in \beta - \text{UCV}(\alpha) \), if and only if
\[
\text{Re} \left\{ 1 + z \frac{zf''(z)}{f'(z)} - \alpha \right\} \geq \beta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|. \tag{1.7}
\]
The class \( \beta - \text{UCV} \) was introduced by S. Kanas et al. [5], where its geometric properties and connections with convex domains were considered. S. Kanas and H. M. Srivastava [6] studied this class in detail. Later on, A. Gangadharan et al. [3] used linear operators to find the connections between the class \( \beta - \text{UCV} \) and the different subclasses of the class of analytic and univalent functions defined in the unit disc.

Let the function \( f(z) \) and \( g(z) \) defined by
\[
f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \tag{1.8}
\]
and
\[ g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k \] (1.9)
belong to \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) and \( K(\mu, \gamma, \eta, a, b, c, \xi, \beta) \), respectively. Then the modified Hadamard product of \( f \) and \( g \) is defined by
\[ (f * g)(z) = z^p - \sum_{k=p+1}^{\infty} a_k b_k z^k. \] (1.10)
The incomplete beta function \( \phi_p(a, c; z) \) is defined by
\[ \phi_p(a, c; z) = z^p + \sum_{k=p+1}^{\infty} \frac{(a)_k}{(c)_k} z^k, \] (1.11)
for \( a \in \mathbb{R} \) and \( c \in \mathbb{R} \setminus \{0\} \) where \( \mathbb{Z}_0 = \{0, -1, -2, \ldots\} \), \( z \in U \). \( (a)_k \) is the Pochhammer symbol defined by
\[ (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = \begin{cases} \frac{1}{a(a + 1) \cdots (a + k - 1)}, & k \in \mathbb{N} \end{cases} \]
Next, we consider the Carlson-Shaffer operator [1] defined by
\[ L_p(a, c) f(z) = \phi_p(a, c; z) * f(z), \quad \text{for } f \in S(p) = z^p + \sum_{k=p+1}^{\infty} \frac{(a)_k}{(c)_k} a_k z^k. \] (1.12)
The Gaussian hypergeometric function denoted by \( _2F_1(a, b; c; z) \) and is defined by
\[ _2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad z \in U \] (1.13)
and \( a + b < c \).
Now, using the convolution theorem we can define the Hohlov operator \( F_p(a, b; c) : T(p) \to T(p) \) by the following relation:
\[ F_p(a, b; c)(f(z)) = z^p _2F_1(a, b; c; z) * f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} (k - p)! a_k z^k, \] (1.14)
a, b \in \mathbb{R} \) and \( c \in \mathbb{R} \setminus \{0\} \), where \( \mathbb{Z}_0 = \{0, -1, -2, \ldots\} \), \( z \in U \). Notice that, Hohlov operator reduces to Carlson-Shaffer operator if \( b = 1 \). Also for \( a = m + 1, b = c = 1 \), we get the famous Ruscheweyh derivative operator of order \( m \). We can write
\[ F_p(a, b; c) f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} a_k z^k. \] (1.15)
\textbf{Definition 1.1.} Let \( \mu > 0 \) and \( \gamma, \eta \in \mathbb{R} \). Then the generalized fractional integral operator \( I_{0,x}^{\mu,\gamma,\eta} \) of a function \( f(z) \) is defined by

\[
I_{0,x}^{\mu,\gamma,\eta} f(z) = \frac{z^{-\mu-\gamma}}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} f(t) \, \frac{dF_1}{d(t)} (\mu + \gamma, -\eta; \mu; 1 - \frac{t}{z}) \, dt, \tag{1.16}
\]

where \( f(z) \) is analytic in a simply-connected region of the \( z \)-plane containing the origin, with order

\[
f(z) = 0(|z|^{r}), \quad z \to 0, \tag{1.17}
\]

where \( r > \max\{0, \mu - \eta\} - 1 \) and the multiplicity of \((z-t)^{-\mu-1}\) is removed by requiring \( \log(z-t) \) to be real, when \( (z-t) > 0 \) and is well defined in the unit disc.

\textbf{Definition 1.2.} Let \( 0 \leq \mu < 1 \) and \( \gamma, \eta \in \mathbb{R} \). Then the generalized fractional derivative operator \( J_{0,x}^{\mu,\gamma,\eta} \) of a function \( f(z) \) is defined by

\[
J_{0,x}^{\mu,\gamma,\eta} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \left\{ z^{-\mu-\gamma} \int_0^z (z-t)^{1-\mu} f(t) \, \frac{dF_1}{d(t)} (\gamma-\mu-1-\eta; 1-\mu; 1 - \frac{t}{z}) \, dt \right\}, \tag{1.18}
\]

where the function is analytic in the simply-connected region of \( z \)-plane containing the origin, with the order as given in (1.17) and multiplicity of \((z-t)^{1-\mu}\) is removed by requiring \( \log(z-t) \) to be real when \( (z-t) > 0 \). Notice that, we have the following relationships with the fractional integral and derivative operators of order \( \mu \).

\[
I_{0,x}^{\mu,\gamma,\eta} f(z) = D_{0,x}^{\gamma,\eta} f(z) \quad (\mu > 0),
\]

\[
J_{0,x}^{\mu,\gamma,\eta} f(z) = D_{0,x}^{\mu} f(z) \quad (0 \leq \mu < 1).
\]

Consider the fractional operator \( U_{0,x}^{\mu,\gamma,\eta} \) defined in terms of \( J_{0,x}^{\mu,\gamma,\eta} \) as follows:

\[
U_{0,x}^{\mu,\gamma,\eta} f(z) = \begin{cases} 
\frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p+\eta-\mu)} z^{\gamma} J_{0,x}^{\mu,\gamma,\eta} f(z), & 0 \leq \mu < 1 \\
\frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p+\eta-\mu)} z^{\gamma} I_{0,x}^{\mu,\gamma,\eta} f(z), & -\infty < \mu < 0
\end{cases}. \tag{1.19}
\]

Let

\[
L f(z) = M_{0,x}^{\mu,\gamma,\eta,a,b,c} f(z)
\]

\[
= f_p(a, b, c ; z) \ast U_{0,x}^{\mu,\gamma,\eta} f(z)
\]

\[
= z^p + \sum_{k=p}^{\infty} \left( \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p} \eta_{k-p}}{(c)_{k-p}(1+p)_{k-p} \eta_{k-p}} \right) a_k z^k \tag{1.20}
\]

for \( a, b \in \mathbb{R}, c \in \mathbb{R} \setminus \{0, 0, \alpha, \beta, \gamma, \eta\} = \{0, -1, -2, \ldots\}, -\infty < \mu < 1, -\infty < \gamma < 1, \eta \in \mathbb{R}^+, -p \leq \alpha < p, \beta \geq 0 \) and \( f \in S(p) \).
For convenience, we will write \( Lf \) as follows:

\[
Lf(z) = z^p + \sum_{k=p+n}^{\infty} g(k) a_k z^k,
\]

(1.21)

where

\[
g(k) = \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p} }{(c)_{k-p}(1+p+\eta-\mu)_{k-p}(1+p-\gamma)_{k-p}}.
\]

(1.22)

Let \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) denote the class of function \( f \in S(p) \) satisfying

\[
Re \left\{ \frac{1}{1 + z(Lf)'' - \alpha} \right\} \geq \beta \left| 1 + \frac{z(Lf)''}{(Lf)'} - p \right|,
\]

(1.23)

where \((a, b \in \mathbb{R}, c \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}, -\infty < \mu < 1, -\infty < \gamma < 1, \eta \in \mathbb{R}^+, \text{and} -p \leq \alpha < p, \beta \geq 0, z \in U)\).

It is very interesting to notice that the class \( K(\mu, \gamma, \eta, a, b, c) \) reduces to the class of convex, \( \beta \)-uniformly convex parabolic convex functions for suitable choice of the parameters \( a, b, c, \mu, \gamma, \eta, \alpha \) and \( \beta \). For instance,

1. If \( a = c, b = 1, \mu = \gamma = 0 \) the class reduces to \( \beta-UCV(\alpha) \).
2. If \( a = c, b = 1, \mu = \gamma = 0, \alpha = 2\rho - 1, (0 \leq \rho < 1) \) the class reduces to parabolic convex of order \( \rho \).

Other interesting classes studied by different authors can be derived from \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \).

2 Some Results for the Class \( K(\mu, \gamma, \eta, a, b, c) \)

Theorem 2.1. A function \( f \in T(p) \) is in the class \( K(\mu, \gamma, \eta, a, b, c) \) if and only if

\[
\sum_{k=p+n}^{\infty} k[k(1+\beta) - (\alpha + p\beta)]g(k)a_k \leq p(p-\alpha).
\]

(2.1)

The result is sharp for the function

\[
f(z) = z^p - \frac{p(p-\alpha)}{k[k(1+\beta) - (\alpha + p\beta)]g(k)} z^{p+n}, \ n \in \mathbb{N}.
\]

(2.2)

Proof. Assume that \( f \in K(\mu, \gamma, \eta, a, b, c) \) and \( z \) is real. Then we have from (1.23)

\[
\frac{p^2 - \sum_{k=p+n}^{\infty} k^2 g(k)a_k z^k}{p - \sum_{k=p+n}^{\infty} k g(k)a_k z^k} - \alpha \geq \beta \left| \sum_{k=p+n}^{\infty} (k-p)g(k)a_k z^{k-p} \right|.
\]

Allowing \( z \to 1 \) along the real axis, we obtain the desired inequality (2.1).
Conversely, let us assume that (2.1) holds, then we show that
\[
\beta \left| 1 + \frac{z(Lf)^{''}}{(Lf)^{''}} - p \right| - \text{Re} \left\{ 1 + \frac{z(Lf)^{''}}{(Lf)^{''}} - p \right\} \leq p - \alpha.
\]
Notice that
\[
\beta \left| 1 + z \left( \frac{(Lf)^{''}}{(Lf)^{''}} \right) - p \right| - \text{Re} \left\{ 1 + z \left( \frac{(Lf)^{''}}{(Lf)^{''}} \right) - p \right\} \leq \frac{(1 + \beta) \sum_{k=p+n}^{\infty} (k - p) g(k) a_k}{p - \sum_{k=p+n}^{\infty} k g(k) a_k}.
\]
This expression is bounded above by \((p - \alpha)\) if
\[
\sum_{k=p+n}^{\infty} k [k(1 + \beta) - (\alpha + p\beta)] g(k) a_k \leq p(p - \alpha).
\]

**Corollary 2.1.** Let the function \(f(z)\) defined by (1.2) be in the class \(K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)\). Then
\[
a_k \leq \frac{p(p - \alpha)}{k [k(1 + \beta) - (\alpha + p\beta)] g(k)}, \quad (k \geq p + n, n \in \mathbb{N})
\]
with equality for the function \(f(z)\) given by (2.2).

**Theorem 2.2.** Let the function \(f\) and \(g\) be in the class \(K(\mu, \gamma, \eta, a, b, c)\). Then for \(\lambda \in [0, 1]\), the function
\[
h(z) = (1 - \lambda)f(z) + \lambda g(z) = z^p - \sum_{k=p+n}^{\infty} d_k z^k
\]
is in the class \(K(\mu, \gamma, \eta, a, b, c)\).

**Proof.** Since \(f\) and \(g\) are in the class \(K(\mu, \gamma, \eta, a, b, c)\), they satisfy the inequality (2.1). Thus, the function \(h(z)\) defined by
\[
h(z) = (1 - \lambda)f(z) + \lambda g(z) = z^p - \sum_{k=p+n}^{\infty} [(1 - \lambda)a_k + \lambda b_k] z^k
\]
is also in the class \(K(\mu, \gamma, \eta, a, b, c)\). This immediately follows by setting \(d_k = (1 - \lambda)a_k + \lambda b_k \geq 0\). Therefore, \(K(\mu, \gamma, \eta, a, b, c)\) is a convex set.

**Theorem 2.3.** Let \(f(z)\) and \(g(z)\) defined by (1.8) and (1.9) be in the class \(K(\mu, \gamma, \eta, a, b, c)\). Then the function \(h(z)\) defined by
\[
h(z) = z^p - \sum_{k=p+n}^{\infty} (a_k^2 + b_k^2) z^k
\]
is in the class \(K(\mu, \gamma, \eta, a, b, c, \theta, \beta)\), where
\[
\theta = p - \frac{2p(1 + \beta)(p - \alpha)^2}{(1 + p)(1 + p + \beta - \alpha)^2 g(p + 1) - 2p(p - \alpha)^2}.
\]
Proof. In view of Theorem 2.1 it is sufficient to show that

\[
\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\theta + p\beta)]}{p(p - \theta)} g(k)(a_k^2 + b_k^2) \leq 1.
\]  

(2.3)

Notice that \( f(z) \) and \( g(z) \) belong to \( \mathcal{K}(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \), thus

\[
\sum_{k=p+n}^{\infty} \left( \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} \right)^2 \leq \frac{1}{2} \left( \sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k) a_k^2}{p(p - \alpha)} \right)^2 \leq 1,
\]

(2.4)

\[
\sum_{k=p+n}^{\infty} \left( \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} \right)^2 \leq \frac{1}{2} \left( \sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k) b_k^2}{p(p - \alpha)} \right)^2 \leq 1.
\]

(2.5)

Adding (2.4) and (2.5), we get

\[
\sum_{k=p+n}^{\infty} \frac{1}{2} \left( \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} \right)^2 (a_k^2 + b_k^2) \leq 1.
\]

(2.6)

Thus, (2.3) will hold if

\[
\frac{k(1+\beta) - (\theta + p\beta)}{p(p - \theta)} \leq \frac{1}{2} \frac{k[k(1+\beta) - (\alpha + p\beta)]^2 g(k)}{p(p - \alpha)^2}.
\]

That is, if

\[
\theta \leq p - \frac{2p(1 + \beta)(k - p)(p - \alpha)^2}{k[k(1+\beta) - (\alpha + p\beta)]^2 g(k) - 2p(p - \alpha)^2}.
\]

(2.7)

Notice that, \( \theta \) can be further improved by using the fact that \( g(k) \) is a non-increasing function of \( k \), for \( k \geq p + n, n \in \mathbb{N} \). Thus, \( g(p + n) \leq g(p + 1) \) for \( n \in \mathbb{N} \) and

\[
g(p + 1) = \frac{ab(1 + p)(1 + p + \eta - \gamma)}{c(1 + p + \eta - \mu)(1 + p - \gamma)}.
\]

(2.8)

Therefore,

\[
\theta = p - \frac{2p(1 + \beta)(p - \alpha)^2}{(1 + p)(1 + p + \beta - \alpha)^2 g(p + 1) - 2p(p - \alpha)^2},
\]

where \( g(p + 1) \) is given by (2.8). \( \square \)

Next, we give another inclusion property of the class.

**Theorem 2.4.** Let \( f_j(z) \) defined by

\[
f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k, \quad j = 1, 2, \ldots, \ell
\]
belong to the class $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$. Then the function

$$h(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z)$$

is also in the class $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$.

**Proof.** Since $f_j(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$, in view of Theorem 2.1, we have

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} a_{k,j} \leq 1.$$ [(2.9)]

Now,

$$h(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z) = z^p - \frac{1}{\ell} \sum_{j=1}^{\ell} \sum_{k=p+n}^{\infty} a_{k,j} z^k = z^p - \sum_{k=p+n}^{\infty} e_k z^k,$$

where

$$e_k = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j}.$$

Notice that

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j} \leq 1$$

using (2.9). Thus, $h(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$. \qed

### 3 Connections with Other Fractional Calculus Operators

**Theorem 3.1.** Let

$$\frac{ab(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)} \leq 1$$ [(3.1)]

for $a, b \in \mathbb{R}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0$, $\mathbb{Z}_0 = \{0, -1, -2, \ldots\}$, $-\infty < \mu < 1$, $-\infty < \gamma < 1$, $\eta \in \mathbb{R}^+$, $-p \leq \alpha < p, \beta \geq 0$. Also let the function $f(z)$ defined by (1.2) satisfy

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} a_k \leq \frac{c(1+p+\eta-\mu)(1+p-\gamma)}{ab(1+p)(1+p+\eta-\gamma)}.$$ [(3.2)]

Then $L f(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$ where $g(k)$ is given by (1.22).

**Proof.** We have,

$$L f(z) = z^p - \sum_{k=p+n}^{\infty} g(k) a_k z^k,$$ [(3.3)]

where

$$g(k) = \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p+\eta-\mu)_{k-p}(1+p-\gamma)_{k-p}}.$$
Under the hypothesis of the theorem, we observe that the function \( g(k) \) is a non-increasing function, that is, \( g(p + n) \leq g(p + 1) \), \( n \in \mathbb{N} \). Thus,

\[
0 < g(p + n) \leq g(p + 1) = \frac{ab(1 + p)(1 + p + \eta - \gamma)}{c(1 + p + \eta - \mu)(1 + p - \gamma)}.
\] (3.4)

In view of (3.2) and (3.4), we now have

\[
\sum_{k=p+n}^{\infty} \frac{k[k(1 + \beta) - (\alpha + p\beta)]g^2(k)}{p(p - \alpha)} a_k \leq g(p + 1),
\]

\[
\sum_{k=p+n}^{\infty} \frac{k[k(1 + \beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} \leq 1.
\]

Therefore, by Theorem 2.1, we conclude that

\[
\mathcal{L} f(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta).
\]

\[\square\]

**Remark 3.1.** The equality in (3.2) is attained for the function

\[
f(z) = z^p - \frac{cp(1 + p - \alpha)(1 + p + \eta - \mu)(1 + p - \gamma)}{ab(1 + p + \beta - \alpha)(1 + p)^2(1 + p + \eta - \gamma)} z^{p+1}.
\] (3.5)

**Corollary 3.1.** Let \( \mu, \gamma, \eta \) be such that \( \mu \geq 0, \gamma < 1+p \), and

\[
\max\{\mu, \gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma - (2+p))}{\gamma}.
\] (3.6)

Also let the function \( f(z) \) by (1.2) satisfy

\[
\sum_{k=p+n}^{\infty} \frac{k[k(1 + \beta) - (\alpha + p\beta)]g^2(k)}{p(p - \alpha)} a_k \leq \frac{(1 + p + \eta - \mu)(1 + p - \gamma)}{(1 + p)(1 + p + \eta - \gamma)}
\] (3.7)

for \( -p \leq \alpha < p, \beta \geq 0 \). Then

\[
\mathcal{L} f(z) = J_{0, z}^{\mu, \gamma, \eta} f(z) \in \beta - \text{UCV}(\alpha).
\]

**Proof.** The corollary follows from Theorem 3.1 by setting \( a = c, b = 1 \). \[\square\]

**Corollary 3.2.** Let \( \mu, \gamma, \eta \in \mathbb{R} \) such that \( \mu \geq 0, \gamma < 1+p \), and

\[
\max\{\mu, \gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma - (2+p))}{\gamma}.
\]

Also let the function \( f(z) \) defined by (1.2) satisfy

\[
\sum_{k=p+n}^{\infty} \frac{k[k(1 + \beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} a_k \leq \frac{(1 + p - \mu)}{(1 + p)}
\]

for \( -p \leq \alpha < p, \beta \geq 0 \). Then

\[
\mathcal{L} f(z) = D_{0, z}^{\mu} f(z) \in \beta - \text{UCV}(\alpha).
\]
Proof. The corollary follows from Theorem 3.1 by setting \( a = c, b = 1, \mu = \gamma \).

**Corollary 3.3.** Let \( \mu, \gamma, \eta \in \mathbb{R} \) such that \( \mu \geq 0, \gamma < 1 + p, \) and

\[
\max\{\mu, \gamma\} - (1 + p) < \eta \leq \frac{\mu(\gamma - (2 + p))}{\gamma}.
\]

Also, let the function \( f(z) \) defined by (1.2) satisfy

\[
\sum_{k=p+n}^{\infty} \frac{k[k(1 + \beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} a_k \leq \frac{c}{ab}.
\]

Then \( Lf(z) = F_p(a, b; c)f(z) \in \beta - UCV(\alpha) \).

**Proof.** Corollary follows from Theorem 3.1 by setting \( \mu = \gamma = 0 \).

**Corollary 3.4.** Let the hypothesis of Corollary 3.3 be true and

\[
\sum_{k=p+n}^{\infty} \frac{k[k(1 + \beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} a_k \leq \frac{c}{a}.
\]

then

\[
Lf(z) = L_p(a, c)f(z) \in \beta - UCV(\alpha).
\]

**Proof.** The corollary follows from Theorem 3.1 by setting \( \mu = \gamma = 0, b = 1 \).

## 4 Results on Modified Hadamard Product

**Theorem 4.1.** Let the function \( f(z) \) and \( g(z) \) defined by

\[
f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k
\]

and

\[
g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k
\]

belong to \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) and \( K(\mu, \gamma, \eta, a, b, c, \xi, \beta) \), respectively. Also assume that

\[
ab(1 + p)(1 + p + \eta - \gamma) \leq c(1 + p + \eta - \mu)(1 + p - \gamma) \leq 1.
\]

Then \( (f * g)(z) \in K(\mu, \gamma, \eta, a, b, c, \delta, \beta) \), where

\[
\delta = p - \frac{p(1 + \beta)(p - \alpha)(p - \xi)}{k(1 + p + \beta - \alpha)(1 + p + \beta - \xi)g(p + 1) - p(p - \alpha)(p - \xi)}.
\]

\[
(4.3)
\]
Application of Fractional Calculus

and the result is sharp for

\[
\begin{align*}
    f(z) &= z^p - \frac{p(p-\alpha)}{(p+1)(1+\beta-\alpha)g(p+1)}z^{p+1}, \\
    g(z) &= z^p - \frac{p(p-\xi)}{(p+1)(1+\beta-\xi)g(p+1)}z^{p+1}.
\end{align*}
\]

Proof. To prove the theorem it is sufficient to show that

\[
\sum_{k=p+1}^{\infty} \frac{k[k(1+\beta) - (\delta + p\beta)]}{p(p-\delta)} g(k) a_k b_k \leq 1,
\]

(4.4) where \(g(p+1)\) is defined by (3.4).

Since \(f(z) \in K(\mu, \gamma, \eta, a, b, c, a, \xi)\) and \(g(z) \in K(\mu, \gamma, \eta, a, b, c, \xi)\), we have

\[
\sum_{k=p+1}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]}{p(p-\alpha)} a_k \leq 1,
\]

(4.5)

\[
\sum_{k=p+1}^{\infty} \frac{k[k(1+\beta) - (\xi + p\beta)]}{p(p-\xi)} b_k \leq 1.
\]

(4.6)

Applying Cauchy-Schwarz inequality to (4.5) and (4.6), we get

\[
\sum_{k=p+1}^{\infty} \frac{k\sqrt{k(1+\beta) - (\alpha + p\beta)[k(1+\beta) - (\xi + p\beta)]g(k)}}{p\sqrt{(p-\alpha)(p-\xi)}} \sqrt{a_k b_k} \leq 1.
\]

(4.7)

In view of (4.4) it suffices to show that

\[
\sum_{k=p+1}^{\infty} \frac{k\sqrt{k(1+\beta) - (\delta + p\beta)]}{p(p-\delta)} g(k) a_k b_k}
\leq \sum_{k=p+1}^{\infty} \frac{k\sqrt{k(1+\beta) - (\alpha + p\beta)[k(1+\beta) - (\xi + p\beta)]g(k)}}{p\sqrt{(p-\alpha)(p-\xi)}} \sqrt{a_k b_k}.
\]

Or equivalently, for \(k \geq p + 1\).

\[
\sqrt{a_k b_k} \leq \frac{\sqrt{k(1+\beta) - (\alpha + p\beta)]}[k(1+\beta) - (\xi + p\beta)]}{\sqrt{(p-\alpha)(p-\xi)}} \sqrt{k(1+\beta) - (\delta + p\beta)]} \frac{(p-\delta)}{[k(1+\beta) - (\delta + p\beta)]}.
\]

(4.8)

In view of (4.7) and (4.8) it is enough to show that

\[
\frac{p\sqrt{(p-\alpha)(p-\xi)}}{k\sqrt{k(1+\beta) - (\alpha + p\beta)]}[k(1+\beta) - (\xi + p\beta)]g(k)}
\leq \frac{\sqrt{k(1+\beta) - (\alpha + p\beta)]}[k(1+\beta) - (\xi + p\beta)]}{\sqrt{(p-\alpha)(p-\xi)}} \sqrt{k(1+\beta) - (\delta + p\beta)]}.
\]
which simplices to

\[
\delta \leq p - \frac{p(k-p)(1+\beta)(p-\alpha)(p-\xi)}{k(k(1+\beta) - (1+\alpha+p\beta))}[k(1+\beta) - (\xi + p\beta)]g(k) - p(p-\alpha)(p-\xi)
\]

with \( g(k) \) given by (1.22). Using the fact that \( g(k) \) is a decreasing function of \( k \) \((k \geq p+1)\), we can choose \( \delta \) such that

\[
\delta = p - \frac{p(1+\beta)(p-\alpha)(p-\xi)}{k(1+p+\beta - \alpha)(1+p+\beta - \xi)g(p+1) - p(p-\alpha)(p-\xi)}
\]

where

\[
g(p+1) = \frac{ab(1+p)(1+p+\eta - \gamma)}{c(1+p+\eta - \mu)(1+p - \gamma)}.
\]

**Theorem 4.2.** Let the function \( f(z) \) and \( g(z) \) defined by (4.1) and (4.2) be in the class \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \). Then \((f \ast g)(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) where

\[
\delta = p - \frac{p(1+\beta)(p-\alpha)^2}{k(1+p+\beta - \alpha)^2g(p+1) - p(p-\alpha)^2}
\]

for \( g(p+1) \) given by (2.8).

**Proof.** Substituting \( \alpha = \beta \) in Theorem 4.1, the result follows.

**Corollary 4.1.** Let the function \( f(z) \) defined by (1.2) be in the class \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) and let \( g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k \) for \(|b_k| \leq 1\). Then \((f \ast g)(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \).

5 Extreme Points of the Class \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \)

**Theorem 5.1.** Let \( f(z)_p = z^p \) and

\[
f_k(z) = z^k - \frac{p(p-\alpha)}{k[k(1+\beta) - (1+\alpha+p\beta)]}g(k)z^k, \quad (k \geq p+1).
\]

Then \( f(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) if and only if \( f(z) \) can be expressed in the form

\[
f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z),
\]

where \( \lambda_k \geq 0 \) and \( \sum_{k=p}^{\infty} \lambda_k = 1 \).

**Proof.** Let \( f(z) \) be expressible in the form

\[
f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z) = z^k - \sum_{k=p+1}^{\infty} \frac{p(p-\alpha)}{k[k(1+\beta) - (1+\alpha+p\beta)]}g(k)z^k.
\]
Also we see that

**Theorem 6.1.** Let the function

\[ f \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta). \]

Conversely, suppose that \( f(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \). Setting

\[ \lambda_k = \frac{k[k(1 + \beta) - (\alpha + p\beta)]g(k)}{p(p - \alpha)} \text{ and } \lambda_p = 1 - \sum_{k=p+1}^{\infty} \lambda_k \]

we see that \( f(z) \) can be expressed in the form (5.1).

**Corollary 5.1.** The extreme points of the class \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) are \( f_p(z) = z^p \) and

\[ f_k(z) = z^p - \frac{p(p - \alpha)}{k[k(1 + \beta) - (\alpha + p\beta)]g(k)}z^k, \quad k \geq p + 1. \]

### 6 Growth and Distortion Theorems

**Theorem 6.1.** Let the function \( f(z) \) defined by (1.2) be in the class \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \). Then

\[ ||Lf(z)|| - |z|^p \leq \frac{cp(1 + p - \gamma)(1 + p + \eta - \mu)}{ab(1 + p)(1 + p + \eta - \gamma)(1 + p + \beta - \alpha)}|z|^{p+1}, \quad (6.1) \]

and

\[ ||Lf(z)'|| - p|z|^{p-1} \leq \frac{cp(1 + p - \gamma)(1 + p + \eta - \mu)}{ab(1 + p + \eta - \gamma)(1 + p + \beta - \alpha)}|z|^p. \quad (6.2) \]

**Remark 6.1.** The result (6.1) and (6.2) are sharp for the extremal function \( f(z) \) given by

\[ f(z) = z^p - \frac{cp(1 + p - \gamma)(1 + p + \eta - \mu)}{ab(1 + p)(1 + p + \eta - \gamma)(1 + p + \beta - \alpha)}z^{p+1}. \quad (6.3) \]

**Corollary 6.1.** Let \( Lf(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \) then the disc \( |z| < 1 \) is mapped onto a domain that contains the disc

\[ |w| < 1 - \frac{cp(1 + p - \gamma)(1 + p + \eta - \mu)}{ab(1 + p)(1 + p + \eta - \gamma)(1 + p + \beta - \alpha)}. \]

Also \( (Lf(z))' \) maps the disc \( |z| < 1 \) onto a domain that contains the disc

\[ |w| < p - \frac{cp(1 + p - \gamma)(1 + p + \eta - \mu)}{ab(1 + p + \eta - \gamma)(1 + p + \beta - \alpha)}. \]

**Remark 6.2.** We can obtain the Growth and Distortion Theorems for \( J_{\delta, \gamma}^{\alpha, \beta} f(z), D_{\alpha, \beta}^{\gamma} f(z), F_{\alpha, \beta} f(z) \) and \( L_{\alpha, \beta} f(z) \) by accordingly initializing the parameters.
7 Class Preserving Integral Operators

The integral operator \( F(z) \) defined by

\[
F(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p} \, dt
\]

is class preserving. The Komatu integral operator \([5]\) is defined by

\[
H(z) = P_{c,p}^d f(z) = \left( \frac{c + p}{c + k} \right)^d \int_0^z \left( \frac{\log \frac{z}{t}}{7} \right)^{d-1} f(t) \, dt, \quad d > 0, \ c > -p, \ z \in U
\]

and the integral operator

\[
I(z) = Q_{c,p}^d f(z) = \left( \frac{d + c + p - 1}{c + k} \right)^d \int_0^z \left( 1 - \frac{t}{z} \right)^{d-1} f(t) \, dt,
\]

\((d > 0, c > -p, z \in U)\), is also class preserving. We note that

\[
H(z) = z^p - \sum_{k=p+n}^{\infty} \frac{c + p}{c + k}^d a_k z^k
\]

and

\[
I(z) = z^p - \sum_{k=p+n}^{\infty} \frac{c + p}{c + k} \frac{k(z \text{ } + \text{ } c + k)^d}{(c + k)(c + k + d)(c + p)^d} a_k z^k.
\]

It can be easily proved that these are class preserving integral operators.

**Theorem 7.1.** Let \( d > 0, c > -p \) and \( f(z) \) belong to the class \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \). Then the function \( H(z) \) defined by (7.2) is \( p \)-valent in the disc \( |z| < R_1 \), where

\[
R_1 = \inf_{k} \left\{ \frac{\left[k(1 + \beta) - (\alpha + p \beta)]g(k)(c + k)^d\right]^{1/(k-p)}}{(p - \alpha)(c + p)^d} \right\}.
\]

The result is sharp for the function \( f(z) \) given by

\[
f(z) = z^p - \frac{(p - \alpha)(c + 1)^d}{\left[k(1 + \beta) - (\alpha + p \beta)]g(k)(c + k)^d\right]} z^{p+n}, \ n \in \mathbb{N}.
\]

**Proof.** We show that

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq p \quad \text{in} \quad |z| < R_1,
\]

where \( R_1 \) is given by (7.6). In view of (7.4), we have

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| = \left| - \sum_{k=p+n}^{\infty} k \left( \frac{c + p}{c + k} \right)^d a_k z^{k-p} \right| \leq \sum_{k=p+n}^{\infty} k \left( \frac{c + p}{c + k} \right)^d a_k |z|^{k-p}.
\]
Theorem 7.2. Let \( d > 0, c > -p \) and \( f(z) \) belong to the class \( K(\mu, \gamma, \eta, a, b, c, \alpha, \beta) \). Then the function \( f(z) \) is \( p \)-valent in the disk \( |z| < R_2 \), where

\[
R_2 = \inf_k \left\{ \frac{|k(1 + \beta) - (\alpha + p\beta)|g(k)(c + k)^d}{(p - \alpha)\Gamma(c + k)\Gamma(c + d + k)} \right\}^{1/(k-p)}.
\]

The result is sharp for the function given by

\[
f(z) = z^p - \left\{ \frac{(p - \alpha)\Gamma(c + k)\Gamma(d + c + k)}{|k(1 + \beta) - (\alpha + p\beta)|\Gamma(c + d + k)\Gamma(c + p)g(k)} \right\}^{1/(k-p)} z^{p+n}, \quad n \in \mathbb{N}.
\]

Proof. In view of the arguments similar to Theorem 7.1 and relation (7.5), we get

\[
|z| \geq \left\{ \frac{|k(1 + \beta) - (\alpha + p\beta)|g(k)(c + k)^d}{(p - \alpha)\Gamma(c + k)\Gamma(c + d + k)} \right\}^{1/(k-p)} \quad \text{for } k \geq p + n, \quad n \in \mathbb{N}.
\]

\[
The result follows by setting \( |z| = R_1 \). \]

\[
Theorem 8.1. \text{Let the function } f(z) \text{ defined by (1.2) be in the class } K(\mu, \gamma, \eta, a, b, c, \alpha, \beta). \text{ Then } f(z) \text{ is } \beta\text{-uniformly convex in } |z| < R_3 \text{ of order } 0 \leq \delta < p, 0 \leq \alpha + \delta < p \text{ where}
\]

\[
|z| < R_3 = \inf_k \left\{ \frac{|k(1 + \beta) - (\alpha + p\beta)|g(k)(p - \alpha)\Gamma(k - p) - (p - \delta - \alpha)\Gamma(k - p)}{\beta(k - p) - (p - \delta - \alpha)} \right\}.
\]

The result is sharp for

\[
f(z) = z^p - \sum_{k=p+n}^{\infty} \frac{p(p - \alpha)}{k(k(1 + \beta) - (\alpha + p\beta))} z^k \quad \text{for some } k.
\]
Proof. To prove the result it is sufficient to show that
$$\beta \left| 1 + \frac{z f''(z)}{f'(z)} - p \right| + \alpha \leq p - \delta. \quad (8.1)$$
Simplifying by fairly straightforward calculations and using Theorem 2.1, we get
$$|z|^k - p \leq \frac{k(1 + \beta) - (\alpha + p\beta)\theta(k)(p - \delta - \alpha)}{(p - \alpha)[\beta(k - p) - (p - \delta - \alpha)]}. \quad (8.2)$$
Setting $|z| = R_3$ in (8.2) we get the desired result. \hfill \Box

References


Shamkant Madhav Khairnar earned his M. Phil. in Complex Analysis and Ph.D. in Univalent and Multivalent function theory. He has 20 years of teaching and 15 years of research experience. He has published more than 40 research papers in peer reviewed International journals. He is a recognized M.Phil. and Ph.D. guide by three renowned universities in India and Life member of ten national and international journals and societies. He is presently supervising four Ph.D. students and two M.Phil students. He has visited eight International countries and delivered plenary and invited talks at various International Conferences. He has organized several conferences, workshops and paper presentation contests for faculty and students. He is working on five R&D projects funded by Department of Science and Technology (DST), Department of Atomic Energy (DAE), Government of India, UGC, New Delhi and BCUD, University of Pune, India. He is Principal Editor of two and Associate Editor of two International Journals. He has worked on several bodies of the universities like, the Board of Studies, Research Recognition Committee, Staff Selection Committee and several others. He has received many national and international recognition related with his academic and research excellence.

Meena More is presently pursuing Ph.D. degree in Univalent and Multivalent function theory. She has five years of teaching and three years of research experience. She published eighteen research papers in peer reviewed International journals. She is working on four research projects as Co. Principal Investigator, funded by University of Pune, Department of Science and Technology and Department of Atomic Energy, Government of India. She is a Life member of two international journals and two technical societies. She has participated and presented research papers in ten national/international conferences.