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Performance of New Liu-Type Logistic Estimators in Combating Multicollinearity

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Abstract: Multicollinearity problem in logistic regression causes an inflation in the variance of the Maximum Likelihood (ML) estimator. To overcome this serious problem, some biased estimators such as: ridge estimator, Liu estimator and Liu-type estimator, were suggested as a way of obtaining smaller Mean Squared Error (MSE) than ML estimator. This paper discusses these different biased estimators in the logistic regression and applies some ridge estimators that were not applied to the Liu-type estimation before. A Monte Carlo simulation study was conducted to assess the performances of ridge and Liu-type estimators in the sense of MSE and Bias criteria. It was concluded that the new estimators perform well in the Liu-type estimation.

Keywords: Liu-type Estimator, Logistic Regression, Maximum Likelihood Estimator, MSE, Multicollinearity, Ridge Estimator

1 Introduction

The use of logistic regression modeling has exploded during the past decade. Binary logistic regression became a widely used statistical method as it studies the relationship between a categorical outcome variable, which is dichotomous (taking on two values such as success/ failure), and a set of regressors. From its original acceptance in epidemiologic research, the logistic regression is now commonly employed in many fields not nearly limited to biomedical research and health policy [1].
The specific form of the logistic regression model is defined as:

$$E(y) = \pi(x) = \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}} = \frac{1}{1+e^{-(\beta_0+\beta_1 x)}} \tag{1}$$

The logarithm of the odds ratio $\frac{\pi(x)}{1-\pi(x)}$ is called the log odds or the logit transformation, which is defined as follows:

$$l = \ln\left(\frac{\pi(x)}{1-\pi(x)}\right) = \beta_0 + \beta_1 x \tag{2}$$

It is an important function having many of the desired properties of the linear regression model, as it is linear in its parameters and connects the random component (response variable) with the linear predictor $(\eta = \beta_o + \beta_1 x + \beta_2 x + \ldots\ldots\ldots\ldots + \beta_p x)$.
ML estimation is the most common method of estimating the coefficients of the logistic regression model. Multicollinearity is the statistical phenomena that occurs when two or more explanatory variables are correlated among themselves. It was known that the presence of multicollinearity problem affects the ML estimator, increasing its variance, resulting in unstable estimates of the coefficients. As a way of dealing with multicollinearity, biased estimation methods, mainly, ridge, Liu and Liu-type estimators were proposed.
Schaefer et al.(1984)[2] extended the ridge estimator, that first proposed by Hoerl and Kennard(1970) [3], to be applicable to the logistic model. Following them, Mansson and Shukur (2011) [4], Kibria et al. (2012) [5] suggested many methods to estimate the ridge parameter of the logistic ridge estimator as a way to reduce its MSE.

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Urgan and Tez (2008)[6] and Mansson et al. (2012)[7] applied the Liu estimator (LL), that first introduced by Liu (1993) [8], on the logistic regression. They showed theoritically that the Logistic Liu estimator has smaller MSE than Logistic Ridge (LR) estimator for some k and d in the presence of multicollinearity.
To combat multicollinearity in logistic regression model, Haung (2012)[9] proposed a logistic Liu type (LLT) estimator, but Inan and Erdogan (2013) [10] applied the Liu type estimator that first produced by Liu (2003)[11] in the linear model, to be applicable to the logistic model as well.
Lukman et al. (2019a)[12] suggested a modified ridge-type estimator in the linear regression model, which was extended to the logistic version by Lukman et al. (2020)[13].
The purpose of this article is to apply a number of LR regression parameters to LLT estimators and investigate them in the presence of different levels of correlation between the explanatory variables.
In the next section, the different methods of biased estimation, LR estimator, LL estimator, and LLT estimators are presented and a number of variants logistic ridge parameters that exists in the literature are defined. In Sec.(3), the design of a Monte Carlo simulation with main factors such as the number of regressors, the sample size and the degree of correlation, was presented. In Sec.(4), the results concerning the various parameters in term of MSE and Bias criteria were provided. The conclusions of the article are presented in Sec.(5).

2 Methods of Biased Estimation

2.1 Logistic Ridge Estimator

It is known that one of the most common disadvantages of the ML estimation is that its variance becomes so large in the presence of multicollinearity. The ML estimator of β, that obtained by using iteratively reweighted Least Squares algorithm, can be expressed as follows:

$$\hat{\beta}_{ML} = (X^{'}\hat{W}X)^{-1}X^{'}\hat{W}\hat{z}, \tag{3}$$

where

$$\hat{W} = diag(\hat{\pi}_i(1-\hat{\pi}_i)),$$

where $\hat{W}$ is the weight matrix, and

$$\hat{z}_i = \log(\hat{\pi}_i) + \frac{y_i - \hat{\pi}_i}{\hat{\pi}_i(1-\hat{\pi}_i)}$$

$\hat{z}_i$ is the i^{th} element of the vector $\hat{z}$. $\hat{\pi}_i$ is the i^{th} element of the proportion of observations with an outcome of 1, The hats in the equations indicate the iterative process. As presented by Schaefer (1979) [14], that under certain regular conditions, increasing of the sample size n, makes $\hat{\beta}_{ML}$ asymptotically approaches β; following $N(0,(X^{'}WX)^{-1})$ and thus the asymptotic MSE of unbiased $\hat{\beta}_{ML}$ equals

$$MSE(\hat{\beta}_{ML}) = E[(\hat{\beta}_{ML}-\beta)^{'}(\hat{\beta}_{ML}-\beta)] = tr(X^{'}\hat{W}X)^{-1} = \sum_j \frac{1}{\hat{\lambda}_j}, \tag{4}$$

where $\hat{\lambda}_j \geq 0$ is the j^{th} eigenvalue of the semi-definite matrix $(X^{'}\hat{W}X)$.[15].
As a consequence of near multicollinearity between the explanatory variables of the logistic regression, similar to linear regression, some of the eigenvalues of the matrix $(X^{'}\hat{W}X)$ become very small when it is ill-conditioned, then the MSE of ML estimator becomes so large, producing unstable parameter estimates.[16].
As a way of solving this problem, Schaefer et al. (1984)[2] considered the ridge estimation, that first introduced by Hoerl and Kennard (1970)[3] for the linear model, they developed a logistic version of the ridge estimator. The LR estimator was defined as follows:

$$\hat{\beta}_{LR} = [(X^{'}\hat{W}X)+kI]^{-1}(X^{'}\hat{W}X)\hat{\beta}_{ML}, \tag{5}$$

where k is the unknown positive ridge parameter.
This estimator has smaller total MSE than the ML estimator. The bias of $\hat{\beta}_{LR}$ can be obtained as

$$\begin{aligned} Bais(\hat{\beta}_{LR}) &= E(\hat{\beta}_{LR})-\beta \\ &= [(X^{'}\hat{W}X+kI)^{-1}X^{'}\hat{W}X-I]\beta \\ &= (X^{'}\hat{W}X+kI)^{-1}[X^{'}\hat{W}X-(X^{'}\hat{W}X+kI)]\beta \\ &= -k(X^{'}\hat{W}X+kI)^{-1}\beta \end{aligned} \tag{6}$$

Then the MSE of the LR estimator can be defined as

$$MSE(\hat{\beta}_{LR}) = E(\hat{\beta}_{LR} - \beta)'(\hat{\beta}_{LR} - \beta),$$
$$= \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j+k)^2} + k^2\beta'[(X'\hat{W}X) + kI]^{-2}\beta \quad (7)$$
$$= \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p} \frac{\hat{\alpha}_j^2}{(\lambda_j+k)^2},$$

where $\hat{\alpha}_j$ is the j^{th} element of $P'\hat{\beta}_{ML}$, such that $\hat{\alpha} = P'\hat{\beta}_{ML}$.
P is the eigenvector and Λ is the diagonal matrix of eigenvalues of $(X'\hat{W}X)$ (equals to $diag(\lambda_j)$) such that $(X'\hat{W}X) = (P'\Lambda P)$. The first term of the MSE is the total variance of $\hat{\beta}_{LR}$ and the second term is the squared bias.
Schaefer et al. (1984)[2] showed that in the LR regression, if the multicollinearity is quite strong, then MSE$(\hat{\beta}_{LR})$ is smaller than $MSE(\hat{\beta}_{ML})$ for large n and small ridge parameter k. They conducted an empirical study, first, to determine the degree of multicollinearity and the range of the sample size for which the LR estimator was proposed to outperform the ML estimator, and second, to determine whether the small positive value (k) could be estimated from the data. It was concluded that the LR estimator has a smaller MSE then ML estimator.
The very important objective of the ridge regression study is finding an appropriate value of k satisfying that: the increase of the second term in the MSE (the bias) is exceeded by the decrease of the first term (the variance). In this regard, many methods have been proposed for the linear ridge regression model, and these methods have been generalized to be applicable for the LR regression model.

Some existing ridge parameters from the literature such as Hoerl and Kennard (1970)[3], Schaefer et al.(1984)[2], Kibria et al. (2012)[5] and Wu and Asar (2016)[17] are applied in this paper and listed respectively as follows:

$$K1 = \frac{\hat{\sigma}^2}{\hat{\alpha}_{max}^2}, \quad (8)$$

$$K2 = \frac{1}{\hat{\alpha}_{max}^2}, \quad (9)$$

$$K3 = max\left(\frac{1}{\sqrt{q_i}}\right) \quad (10)$$

.

$$K4 = \frac{p}{\sum_{i=1}^{p}[\alpha_i^2/[1+(1+\lambda_i\alpha_i^2)^{1/2}]]} \quad (11)$$

2.2 Logistic Liu Estimator

Liu (1993)[8] proposed another estimator, called Liu estimator , with the benefit of being a linear function of the parameter d. It was to avoid the major disadvantage of the ridge regression, that the estimated parameters are non-linear of the ridge parameter k, which takes on values between zero and infinity, conducting to complicated equations when selecting k.
This Liu estimator that was extended to the logistic regression model by Urgan and Tez (2008)[6] and Kibria et al. (2012) [5], who showed that the LL estimator has a smaller MSE than the ML estimator in the existence of multicollinearity. The LL estimator is obtained as

$$\hat{\beta}_d = (X'\hat{W}X + I)^{-1}(X'\hat{W}X + dI)\hat{\beta}_{ML} = Z\hat{\beta}_{ML}. \quad (12)$$

The MSE of the LL estimator is given by

$$MSE(\hat{\beta}_d) = E(\hat{\beta}_d - \beta)'(\hat{\beta}_d - \beta)$$
$$= \sum_{j=1}^{p} \frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2} + (d-1)^2\sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j+1)^2}, \quad (13)$$

where α_j^2 is the square of α_j which is the j^{th} element of $P'\beta$ and P is the eigenvector such that $X'\hat{W}X = P'\Lambda P$, where Λ represents $dig(\lambda_j)$.

2.3 Logistic Liu-Type Estimator

The Liu-type estimator was introduced by Liu (2003)[11] to overcome the disadvantages of the ridge regression, as a combination of ridge and Liu estimators for the linear models. It has been adjusted to the binary logistic model as a way of solving the multicollinearity problem and decreasing the variance in the case of dichotomous response variable, so that the estimates become stable having smaller MSE.

Haung (2012)[9] proposed a LLT estimator which can be denoted by (LLT1), to overcome the multicollinearity problem in the logistic regression model, by combining the LR estimator Eq.(5) and the LL estimator Eq. (12), hoping that the combination of these two estimators may inherit the advantages of both estimators. The proposed biased estimator was defined as

$$\hat{\beta}_{k,d} = (X^{'}\hat{W}X + kI)^{-1}(X^{'}\hat{W}X + kdI)\hat{\beta}_{ML}, \tag{14}$$

where $k > 0$ and $0 < d < 1$. The MSE of this estimator is given by

$$\begin{aligned} MSE(\hat{\beta}_{k,d}) &= E(\hat{\beta}_{k,d} - \beta)^{'}(\hat{\beta}_{k,d} - \beta) \\ &= \sum_{j=1}^{p} \frac{(\lambda_j + kd)^2}{\lambda_j(\lambda_j + k)^2} + \sum_{j=1}^{p} \frac{k^2(d-1)^2\alpha_j^2}{(\lambda_j + k)^2}, \end{aligned} \tag{15}$$

where the first term and the second term represent the variance of the estimator and the squared bias respectively. The aim of this biased estimator is to choose appropriate values of k and d for which the reduction in the variance term exceeds the increase of the squared bias. The optimal value of d can be obtained by differentiating the Eq.(15) with respect to d, to have the formula

$$d_{opt} = \frac{\sum_{j=1}^{p}((k\alpha_j^2 - 1)/(\lambda_j + k)^2)}{\sum_{j=1}^{p}(k(\lambda_j\alpha_j^2 + 1)/\lambda_j(\lambda_j + k)^2)} \tag{16}$$

Inan and Erdogan (2013)[10] defined the LLT estimator, which can be denoted by (LLT2), as a generalization of the Liu-type estimator that first introduced by Liu (2003)[11] for the linear model. They introduced a LLT estimator that was expected to generate a smaller MSE than the LR estimator which was proposed by Schaefer (1984)[2]. The LLT2 was defined as

$$\hat{\beta}_{k,d} = (X^{'}\hat{W}X + kI)^{-1}(X^{'}\hat{W}X - dI)\hat{\beta}_{ML}, \tag{17}$$

where $k > 0$ and $-\infty < d < \infty$.

Asar (2017)[18] considered LLT2 estimator that introduced by Inan and Erdogan (2013)[10] to overcome the multicollinearity problem in binary logistic regression model. The author introduced some new methods of estimating the shrinkage parameter k to be used in LLT2, expecting that these new estimators have smaller MSE than ML estimator in the presence of multicollinearity in binary logistic regression model.

The MSE of LLT2 can be defined as

$$\begin{aligned} MSE(\hat{\beta}_{k,d}) &= E(\hat{\beta}_{k,d} - \beta)^{'}(\hat{\beta}_{k,d} - \beta) \\ &= \sum_{j=1}^{p} \frac{(\lambda_j - d)^2}{\lambda_j(\lambda_j + k)^2} + \sum_{j=1}^{p} \frac{(k+d)^2\alpha_j^2}{(\lambda_j + k)^2} \end{aligned} \tag{18}$$

That is a quadratic function of d for a fixed k. However, there is no conclusive rule to choose k, but the general idea is to control the increase of the MSE resulted from the bias term by using the bias correction parameter d. Therefore, the optimum value of d that minimizes this function, can be found easily to be

$$d_{opt} = \frac{\sum_{j=1}^{p}((1 - k\alpha_j^2)/(\lambda_j + k)^2)}{\sum_{j=1}^{p}((\lambda_j\alpha_j^2 + 1)/\lambda_j(\lambda_j + k)^2)} \tag{19}$$

3 The design of a Monte Carlo Simulation study

A comparative simulation study was conducted to evaluate the performances of the different estimators, ML, LR, LLT1 and LLT2 estimators applying the four shrinkage parameters k. it is known that the design of a good simulation depends on two main factors:

1) The criteria used to judge the performance of the estimators

To investigate the performance of the estimators and to check whether the reduction in variance of LR, LLT estimators exceeds the increase in bias, the MSE was calculated, following most of the articles (Alkhamisi et al., (2006)[19]; Khalaf and Shukur, (2005)[20]; Kibria, (2003)[21]; Muniz and Kibria, 2009)[22]) who compared the performance of the estimators based on the smaller MSE criterion. Also the Bias was taken in consideration as a criterion of judgement.

The following equations were used:

$$MSE(\hat{\beta}_i) = \frac{1}{R}\sum_{i=1}^{R}(\hat{\beta}_i - \beta)'(\hat{\beta}_i - \beta), \tag{20}$$

where $(\hat{\beta}_i)$ represents the vector of estimated parameters of (β) obtained from either ML estimator, LR estimators, and LLT estimators. R is the number of replicates in the Monte Carlo simulation which is set to be 1000.

2) The factors affecting the performance of the investigated estimators

- **The strength of correlation among the predictors** (ρ^2)

The degree of correlation between the explanatory variables, which assumed to have a negative effect on MSE, is one of most important factors affecting the performance of the estimators. To vary the strength of the correlation, following Gibbons (1981)[23], the explanatory variables are generated using the following equation:

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{ip}, \quad i = 1,2,...n, \quad j = 1,2,....,p. \tag{21}$$

where ρ is specified so that the correlation between any two explanatory variables is given by ρ^2 and z_{ij} are independent standard normal pseudo random numbers. The dependent variable of the logistic regression is generated from the Bernoulli(π_i) distribution.

In this simulation, there are five degrees of correlation $\rho = 0.7, 0.8, 0.9, 0.95$ and 0.98. representing high, strong and severe multicollinearity respectively.

- **The number of explanatory variables** (p)

The number of the regressors has also an obvious effect on assessing the performance of the estimators, as it negatively affects the MSE. The main objective of varying the number of regressors is to investigate which parameter is the best for a specific number of predictors. There are three cases of this factor that are used in the simulation $P = 2, 4$ and 8.

- **The sample size** (n)

The sample size is an important factor that must be taken into consideration. It is supposed to positively affect the MSE, during comparing different estimation methods because increasing the sample size reduces the variance of the estimated parameters. The sample sizes that used in this study are $10, 20, 40, 100, 200$ and 500.

- **The parameter's values**

The values of β are chosen so that $\beta'\beta = 1$, following Newhouse and Oman (1971)[24], which is a common restriction in many simulation studies (See Kibria, (2003)[21] and Alkhamisi and Shukur, (2008)[25]). The intercept β_0 is set to be zero, following Månsson and Shukur (2011)[4] and Al Somahi et al.(2015)[26] and Månsson et al.(2010)[27].

4 Results and Discussion

The MSE and Bias of the estimators for the different values of n, ρ and p are simulated,and the results were presented in tables from (1) to (6) according to the number of regressors.

The results of the simulation study showed that the increase in the degree of correlation between the regressors causes an inflation of MSE and an oscillating change in the values of the Bias. Generally, increasing the sample size has a positive effect on both criteria of judgement, since it causes a gradually decrease in the MSE and Bias for all different estimators. In addition, within all different estimators, ML estimator has always the maximum MSE and Bias values, so it can be said that the LR, LLT1 and LLT2 outperform the ML in the presence of different degrees of correlation and with changing the sample sizes. Now the different biased estimators have been compared using the different k parameters with changing the number of regressors:

In case of two regressors $(p = 2)$**:**

According to MSE criterion presented in table (1), figures (1, 2) and (3):

Increasing the degree of correlation, causes an increase in MSE, but since the results of both 0.7 and 0.8 degrees of correlation have the same trend and to save space, their results will be presented together indicating high multicollinearity and the same manner for 0.95 and 0.98 degrees of correlation indicating severe multicollinearity. The results will be presented through three cases at the different levels of correlation:

1) In the existence of high multicollinearity ($\rho = 0.7$ and 0.8):

Considering LR and LLT2, it is notable that the shrinkage parameter $K3$ has the minimum MSE within all other parameters, even with increasing the sample sizes from 10 to 500. However, LLT1 using $K3$ outperforms them to some extent. It also has the minimum MSE among other parameters when $n = 10, 20$, but in the larger sample sizes, it was found that LLT1 using $K1$ is superior to all other estimators ever.
However, LLT2 using $K4$ outperforms the LR in the case of $n = 20, 200$ and 500, despite their equivalence in the other sample sizes. Also, LLT2 using $K4$ has smaller MSE values than its counterparts among all the sample sizes. In general, when the sample size increases, the behavior of the different parameters in both LLT1 and LLT2 becomes equivalent.

2) In the existence of strong multicollinearity ($\rho = 0.9$):
Using the parameters $K3$ and $K4$ in LLT1 performs better than LR and LLT2. Also the estimator LLT2 using the parameters $K1$ and $K2$ performs better than its counterparts of LR, however, LLT1 using them is more effective.

3) In the existence of severe multicollinearity ($\rho = 0.95$ and 0.98):
It is obvious that $K3$ parameter has the minimum MSE among other parameters in estimating the LR except in the largest sample size 500, in such case, $K4$ has the smallest MSE. Among LLT2 estimators, the shrinkage parameter $K3$ outperforms the other parameters, although their equivalent behavior with the large sample sizes $200, 500$. That is the same for the LLT1 using $K3$ parameter, however it has the superiority to the other estimators.
Using the ridge parameter $K4$ in LLT2 shows its superiority to LR in case of small sample size $n = 10$ and 20 with 0.95 degree of correlation, and its near behavior to LR with 0.98 degree of correlation. It is noted that LLT1 using $K4$ has smaller values of MSE than LR and LLT2.
According to the Bias criterion, as shown in table (2), it is noted that the parameter $K1$ has the minimum bias among the different shrinkage parameters. However, LLT1 using $K1$ has a bias which is greater than its counterpart of LR and LLT2. That causes more control in its MSE leading to superiority, which retracts with increasing the degree of correlation to be limited with $n = 200$ and 500 in the severe multicollinearity.

In case of four regressors ($p = 4$)

According to MSE criterion presented in table (3), figures (4, 5) and (6).

1) In the existence of high multicollinearity ($\rho = 0.7$ and 0.8):

It was found that LLT2 using the parameter $K3$ outperforms its counterpart of LR in case of small and medium sample sizes. However, $K4$ shows more efficiency in the large samples. Also, LLT1 using $K4$ behaves better than LR and LLT2. In general, it can be said that LLT1 using $K3$ has the superiority to all other biased estimators.

2) In the existence of strong multicollinearity ($\rho = 0.9$):

Using the parameter $K3$ in LLT1 has the superiority within all other estimators. In large sample sizes, it is noted that the shrinkage parameter $K4$ seems to be equivalent to $K3$.

3) In the existence of severe multicollinearity ($\rho = 0.95$ and 0.98):

It is shown that, for the three different biased estimators, the shrinkage parameter $K3$ performs better than other parameters in case of sample sizes from 10 to 100, but when $n = 200$ and 500, $K4$ has a better performance in reducing MSE. Based on that, and comparing LR, LLT1 and LLT2, it is noted that the last one has the superiority to other estimators.

According to Bias criterion as shown in table (4), The shrinkage parameter $K1$ has always the minimum value of Bias, despite its high MSE of the LR, LLT1 and LLT2, which is interpreted by the increase of their variances especially in the small sample sizes in low correlation, extending to medium and large sample sizes with increasing the degree of correlation. However, LLT2 outperforms LLT1 using the different shrinkage parameters, having smaller bias. Also, it is noted that the parameter $K3$ has always smaller bias than $K4$ among all biased estimators.

In the case of eight regressors ($p = 8$), figures (7, 8) and (9)

According to the MSE criterion as presented in table (5), for the different biased estimators, it is notable that the shrinkage parameters $K3$ and $K4$ have smaller MSE values then other parameters for different degrees of correlation and for all sample sizes except in case of $n = 10$, where $K1$ is superior to them.

1) In the existence of high multicollinearity ($\rho = 0.7$ and 0.8):

It is found that the two types of LLT estimator using $K3$ outperform the LR estimator. However, $K3$ has the smallest MSE for the medium and large samples, having the superiority of LLT1 to other biased estimators. However in small samples, using the parameter $K4$ in both types of LLT estimator has more efficiency than $K3$.

2) In the existence of strong multicollinearity ($\rho = 0.9$):

It is noted that $K3$ outperforms other parameters in estimating LLT1 and LLT2 except when $n = 10, 500$, $K4$ is the best. However, the superiority of the LLT1 among all biased estimators using $K3$ and $K4$.

3) In the existence of severe multicollinearity ($\rho = 0.95$ and 0.98):

It is obvious that, LR using the shrinkage parameter $K4$ outperforms $K3$ with all different sample sizes, achieving the superiority to all biased estimators. However, at both types of LLT estimator, the parameter $K4$ behaves better than $K3$ when $n = 10$ and 500.

According to the Bias criterion as shown in table (6), for all biased estimators, with changing the sample sizes and for degrees of correlation of $\rho = 0.7$ and 0.8, it is found that $K1$ has the smallest bias except when $n = 10$, where $K3$ is better. It is also noted that considering LLT2 at the large sample sizes, the parameters $K3$ and $K4$ are equivalent to $K1$, however their superiority when $n = 500$. In the presence of high multicollinearity $(0.9, 0.95$ and $0.98)$, the parameter $K1$ is the best except when $n = 10$, where $K2$ is superior.

5 Conclusions

In the sense of MSE criterion, with different degrees of multicollinearity, it is found that LLT1 is relatively superior to the other estimators especially when using the shrinkage parameter $K3$. Although that, in case of small number of regressors, the $K1$ parameter often outperforms $K3$, with decreasing the degree of correlation and increasing the sample sizes. Also, it is noted that, the $K4$ parameter often behaves better than other estimators in case of large number of regressors and increasing the degrees of correlation especially when $n = 10$ and 500. In the sense of Bias criterion, among the different shrinkage parameters of each estimator, the parameters with smaller Bias are $K1, K3$ and $K4$ respectively. Since the ML is asymptotically unbiased estimator with increasing the sample size which reflects the increase of its Bias in small samples, it is noted that, the proportion of reduction among the different estimators in small sample sizes is greater than large samples.

Generally, it can be said that the shrinkage parameter $K3$ is recommended in the different types of biased estimation, especially in LLT1, in case of large number of explanatory variables and high degrees of correlation. The behavior of each of the parameters $K3$ and $K4$ for both types of LLT estimator converges with increasing the degree of correlation especially in case of small samples, with relative outperformance of LLT1 over LLT2. Finally, It can be concluded that the shrinkage parameters $K3$ and $K4$ are recommended in the different types of biased estimation, especially in LLT1.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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Appendix A

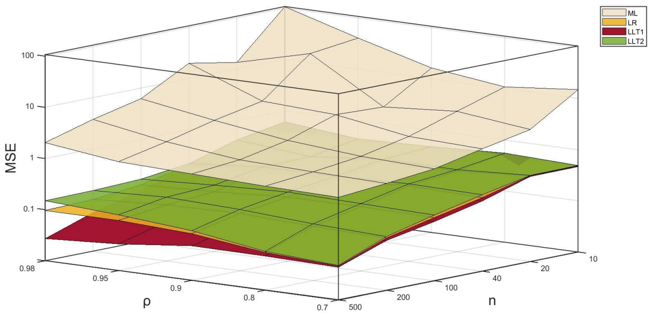

Fig. 1: MSE of ML estimator, LR, LLT1 and LLT2 estimators using K3 when p = 2

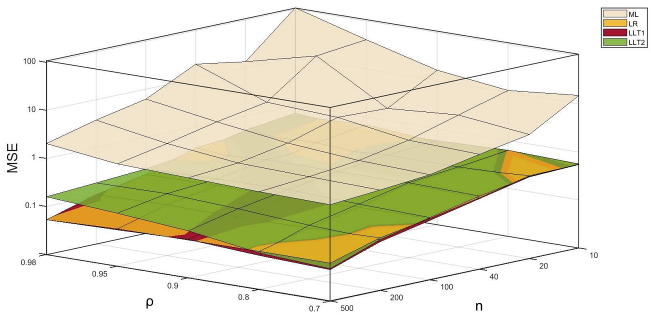

Fig. 2: MSE of ML estimator, LR, LLT1 and LLT2 estimators using K4 when p = 2

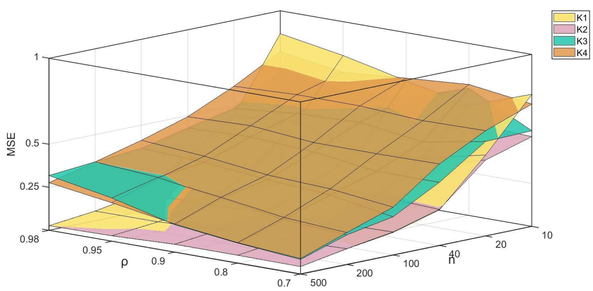

Fig. 3: MSE of LLT1 estimator using K1, K2, K3 and K4 when p = 2

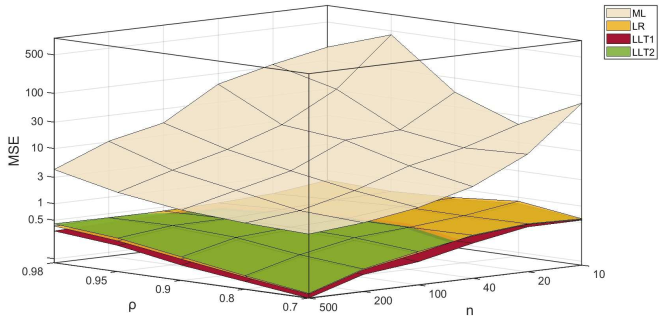

Fig. 4: MSE of ML estimator, LR, LLT1 and LLT2 estimators using K3 when p = 4

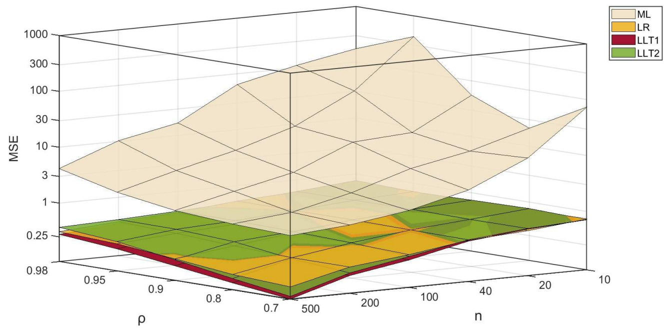

Fig. 5: MSE of ML estimator, LR, LLT1 and LLT2 estimators using K4 when p = 4

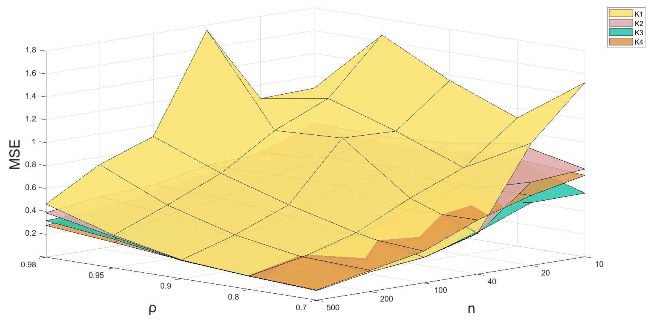

Fig. 6: MSE of LLT1 estimator using K1, K2, K3 and K4 when p = 4

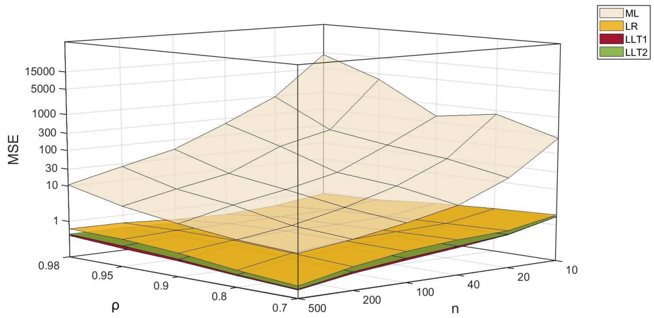

Fig. 7: MSE of ML estimator, LR, LLT1 and LLT2 estimators using K3 when p = 8

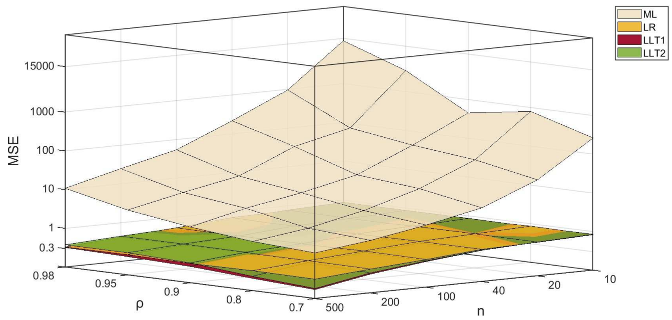

Fig. 8: MSE of ML estimator, LR, LLT1 and LLT2 estimators using K4 when p = 8

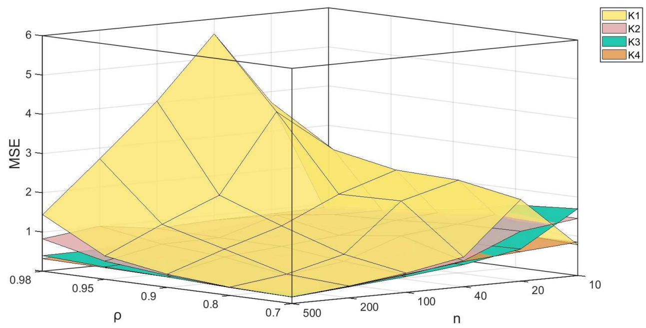

Fig. 9: MSE of LLT1 estimator using K1, K2, K3 and K4 when p = 8

Table 1: MSE values for the different estimators when p = 2

n	ML	LR				LLT1				LLT2			
		K1	K2	K3	K4	K1	K2	K3	K4	K1	K2	K3	K4
Roo = 0.7													
10	14.97	2.099	0.679	0.482	0.550	0.768	0.522	0.467	0.547	0.796	0.552	0.492	0.561
20	3.824	0.928	0.642	0.479	0.532	0.471	0.442	0.461	0.501	0.491	0.482	0.486	0.511
40	2.465	0.663	0.432	0.274	0.307	0.207	0.210	0.24	0.294	0.317	0.311	0.312	0.333
100	1.542	0.285	0.233	0.172	0.189	0.131	0.134	0.143	0.172	0.193	0.192	0.193	0.198
200	1.230	0.137	0.124	0.105	0.124	0.092	0.092	0.094	0.103	0.113	0.113	0.113	0.113
500	1.013	0.051	0.049	0.046	0.062	0.043	0.043	0.043	0.045	0.047	0.047	0.047	0.047
Roo = 0.8													
10	10.87	1.481	0.715	0.552	0.621	0.640	0.540	0.538	0.613	0.677	0.569	0.553	0.620
20	5.581	1.374	0.598	0.377	0.438	0.389	0.337	0.344	0.431	0.470	0.415	0.404	0.464
40	2.777	0.712	0.465	0.299	0.341	0.222	0.227	0.254	0.327	0.332	0.326	0.328	0.362
100	1.748	0.345	0.264	0.181	0.185	0.125	0.127	0.135	0.174	0.209	0.208	0.208	0.216
200	1.334	0.170	0.148	0.115	0.114	0.080	0.081	0.085	0.105	0.129	0.129	0.128	0.130
500	1.079	0.064	0.061	0.054	0.062	0.045	0.046	0.046	0.051	0.057	0.057	0.057	0.057
Roo = 0.9													
10	16.44	1.942	0.640	0.470	0.526	0.719	0.488	0.455	0.550	0.752	0.52	0.477	0.563
20	4.380	1.234	0.610	0.392	0.380	0.369	0.322	0.366	0.444	0.444	0.418	0.418	0.48
40	4.753	1.161	0.462	0.256	0.323	0.239	0.217	0.218	0.301	0.327	0.305	0.297	0.353
100	2.304	0.562	0.331	0.18	0.169	0.119	0.118	0.126	0.185	0.227	0.225	0.225	0.248
200	1.478	0.257	0.200	0.133	0.107	0.076	0.077	0.081	0.114	0.159	0.158	0.158	0.164
500	1.203	0.123	0.108	0.082	0.061	0.046	0.046	0.047	0.062	0.094	0.094	0.094	0.095
Roo = 0.95													
10	40.63	1.875	0.555	0.432	0.685	0.802	0.491	0.426	0.525	0.815	0.505	0.436	0.531
20	31.09	1.960	0.412	0.292	0.483	0.473	0.341	0.285	0.377	0.490	0.360	0.303	0.390
40	5.630	1.359	0.443	0.252	0.284	0.242	0.219	0.226	0.317	0.318	0.294	0.289	0.355
100	3.651	0.967	0.344	0.147	0.172	0.108	0.103	0.102	0.159	0.205	0.201	0.198	0.237
200	2.144	0.500	0.277	0.125	0.114	0.058	0.057	0.059	0.101	0.178	0.177	0.177	0.194
500	1.346	0.204	0.158	0.096	0.060	0.032	0.032	0.033	0.057	0.123	0.123	0.123	0.126
Roo = 0.98													
10	107.16	0.997	0.675	0.602	0.606	0.866	0.649	0.600	0.683	0.871	0.653	0.603	0.684
20	13.464	1.184	0.523	0.403	0.441	0.443	0.382	0.389	0.497	0.481	0.416	0.413	0.508
40	19.907	1.657	0.313	0.204	0.360	0.266	0.221	0.194	0.276	0.290	0.246	0.219	0.297
100	6.218	1.248	0.302	0.129	0.169	0.104	0.099	0.098	0.163	0.174	0.169	0.167	0.219
200	3.776	0.896	0.297	0.103	0.106	0.057	0.056	0.057	0.107	0.155	0.154	0.154	0.186
500	2.039	0.466	0.249	0.096	0.053	0.027	0.027	0.027	0.053	0.149	0.149	0.149	0.159

Table 2: Bias values for the different estimators when p = 2

n	ML	LR				LLT1				LLT2			
		K1	K2	K3	K4	K1	K2	K3	K4	K1	K2	K3	K4
Roo = 0.7													
10	2.536	0.183	0.354	0.535	0.693	0.338	0.421	0.560	0.679	0.315	0.391	0.526	0.647
20	1.450	0.132	0.262	0.449	0.660	0.450	0.480	0.522	0.557	0.347	0.382	0.437	0.498
40	1.240	0.023	0.108	0.260	0.480	0.264	0.300	0.358	0.411	0.146	0.171	0.220	0.291
100	1.088	0.011	0.054	0.143	0.362	0.231	0.238	0.251	0.264	0.091	0.097	0.110	0.146
200	1.028	0.006	0.031	0.085	0.288	0.168	0.168	0.168	0.163	0.058	0.059	0.062	0.076
500	0.975	0.002	0.011	0.034	0.194	0.081	0.081	0.080	0.074	0.023	0.024	0.024	0.028
Roo = 0.8													
10	2.157	0.190	0.362	0.568	0.734	0.332	0.451	0.600	0.705	0.302	0.415	0.565	0.678
20	1.637	0.071	0.210	0.396	0.601	0.247	0.329	0.454	0.567	0.196	0.264	0.375	0.493
40	1.271	0.043	0.125	0.274	0.512	0.252	0.296	0.370	0.445	0.153	0.186	0.248	0.339
100	1.144	0.014	0.059	0.142	0.363	0.222	0.232	0.253	0.294	0.094	0.099	0.114	0.159
200	1.060	0.005	0.029	0.081	0.274	0.180	0.183	0.189	0.199	0.052	0.054	0.058	0.079
500	1.001	0.002	0.012	0.036	0.194	0.110	0.110	0.110	0.102	0.024	0.024	0.025	0.031
Roo = 0.9													
10	2.553	0.123	0.304	0.522	0.679	0.202	0.355	0.546	0.684	0.189	0.332	0.516	0.656
20	1.517	0.048	0.191	0.415	0.561	0.242	0.348	0.480	0.556	0.176	0.261	0.386	0.481
40	1.588	0.041	0.132	0.268	0.500	0.171	0.216	0.317	0.468	0.131	0.166	0.247	0.382
100	1.229	0.020	0.075	0.163	0.350	0.181	0.196	0.237	0.341	0.101	0.110	0.136	0.219
200	1.077	0.009	0.039	0.094	0.269	0.176	0.181	0.194	0.244	0.063	0.065	0.072	0.108
500	1.029	0.004	0.019	0.051	0.196	0.146	0.147	0.151	0.170	0.034	0.034	0.036	0.050
Roo = 0.95													
10	3.435	0.102	0.277	0.505	0.786	0.129	0.294	0.513	0.675	0.126	0.286	0.501	0.663
20	3.362	0.077	0.175	0.360	0.633	0.112	0.189	0.368	0.557	0.108	0.182	0.356	0.541
40	1.661	0.037	0.131	0.292	0.475	0.135	0.193	0.328	0.484	0.110	0.157	0.272	0.418
100	1.423	0.023	0.078	0.146	0.356	0.126	0.138	0.180	0.336	0.091	0.100	0.130	0.252
200	1.212	0.009	0.041	0.088	0.283	0.107	0.114	0.137	0.260	0.056	0.060	0.072	0.146
500	1.052	0.004	0.020	0.050	0.120	0.110	0.113	0.120	0.187	0.033	0.033	0.036	0.064
Roo = 0.98													
10	6.136	0.275	0.448	0.641	0.730	0.289	0.458	0.645	0.783	0.286	0.454	0.641	0.779
20	2.359	0.096	0.234	0.447	0.608	0.147	0.277	0.470	0.634	0.135	0.256	0.440	0.605
40	2.740	0.051	0.117	0.247	0.539	0.083	0.130	0.256	0.467	0.079	0.123	0.242	0.445
100	1.736	0.030	0.077	0.142	0.354	0.094	0.107	0.160	0.346	0.078	0.089	0.134	0.293
200	1.427	0.017	0.051	0.095	0.275	0.080	0.088	0.117	0.276	0.059	0.064	0.086	0.208
500	1.187	0.007	0.029	0.058	0.190	0.081	0.083	0.091	0.192	0.042	0.043	0.047	0.104

Table 3: MSE values for the different estimators when p = 4

n	ML	LR				LLT1				LLT2			
		K1	K2	K3	K4	K1	K2	K3	K4	K1	K2	K3	K4
Roo = 0.7													
10	74.14	0.930	2.860	0.588	0.734	0.784	1.543	0.569	0.714	0.766	1.522	0.556	0.712
20	11.47	1.07	2.316	0.629	0.574	0.807	1.152	0.612	0.628	0.729	1.069	0.549	0.612
40	3.907	0.804	1.151	0.514	0.494	0.528	0.557	0.498	0.503	0.355	0.359	0.367	0.474
100	2.142	0.442	0.546	0.318	0.347	0.352	0.360	0.340	0.321	0.222	0.224	0.222	0.278
200	1.498	0.250	0.276	0.209	0.265	0.219	0.220	0.217	0.208	0.172	0.173	0.172	0.186
500	1.230	0.110	0.115	0.10	0.149	0.103	0.103	0.103	0.101	0.085	0.085	0.085	0.090
Roo = 0.8													
10	20.96	1.232	1.856	0.846	0.729	0.931	1.162	0.766	0.766	0.894	1.119	0.740	0.762
20	10.38	1.073	2.239	0.608	0.622	0.683	0.896	0.567	0.625	0.567	0.760	0.483	0.606
40	6.430	0.959	1.960	0.493	0.514	0.592	0.726	0.495	0.512	0.443	0.571	0.368	0.470
100	3.007	0.584	0.843	0.376	0.379	0.458	0.484	0.423	0.385	0.317	0.345	0.285	0.321
200	1.718	0.322	0.378	0.248	0.265	0.278	0.282	0.272	0.252	0.204	0.207	0.198	0.210
500	1.35	0.163	0.176	0.141	0.169	0.150	0.150	0.149	0.143	0.118	0.118	0.117	0.119
Roo = 0.9													
10	55.56	1.027	2.141	0.680	0.675	0.821	1.376	0.634	0.738	0.801	1.352	0.620	0.735
20	15.19	1.087	2.601	0.620	0.622	0.730	1.063	0.584	0.642	0.663	0.985	0.533	0.629
40	12.99	0.944	2.799	0.489	0.529	0.669	1.088	0.491	0.504	0.585	1.001	0.417	0.471
100	4.75	0.746	1.407	0.404	0.40	0.540	0.632	0.453	0.416	0.398	0.495	0.311	0.338
200	2.77	0.548	0.786	0.344	0.293	0.416	0.437	0.387	0.347	0.284	0.307	0.256	0.264
500	1.60	0.262	0.304	0.206	0.219	0.229	0.231	0.226	0.210	0.163	0.165	0.160	0.164
Roo = 0.95													
10	422.3	0.811	2.015	0.610	0.715	0.778	1.657	0.603	0.705	0.776	1.654	0.601	0.705
20	57.34	0.863	2.320	0.562	0.616	0.677	1.196	0.536	0.635	0.658	1.176	0.520	0.631
40	17.07	0.932	2.862	0.501	0.529	0.649	1.041	0.496	0.529	0.587	0.973	0.440	0.505
100	8.021	0.875	2.136	0.433	0.384	0.565	0.786	0.448	0.441	0.456	0.675	0.347	0.384
200	4.440	0.736	1.338	0.402	0.333	0.514	0.594	0.440	0.394	0.390	0.474	0.316	0.310
500	2.31	0.444	0.596	0.305	0.276	0.361	0.373	0.342	0.302	0.273	0.287	0.251	0.223
Roo = 0.98													
10	174.3	0.877	1.544	0.673	0.729	0.793	1.103	0.659	0.763	0.788	1.097	0.655	0.762
20	111.5	0.715	2.256	0.506	0.610	0.608	1.093	0.496	0.619	0.598	1.081	0.488	0.616
40	64.33	0.737	3.539	0.450	0.530	0.592	1.771	0.444	0.502	0.576	1.755	0.429	0.494
100	16.82	0.857	2.853	0.425	0.423	0.538	0.964	0.420	0.446	0.480	0.903	0.366	0.413
200	10.29	0.859	2.443	0.414	0.367	0.539	0.821	0.423	0.401	0.453	0.733	0.343	0.342
500	4.156	0.697	1.255	0.391	0.308	0.499	0.572	0.430	0.370	0.388	0.464	0.318	0.277

Table 4: Bias values for the different estimators when p = 4

n	ML	LR				LLT1				LLT2			
		K1	K2	K3	K4	K1	K2	K3	K4	K1	K2	K3	K4
Roo = 0.7													
10	4.536	0.280	0.397	0.544	0.835	0.344	0.426	0.561	0.821	0.335	0.416	0.550	0.815
20	2.306	0.181	0.336	0.473	0.729	0.393	0.462	0.543	0.722	0.338	0.401	0.481	0.679
40	1.497	0.059	0.150	0.293	0.668	0.370	0.408	0.461	0.560	0.214	0.248	0.302	0.464
100	1.215	0.018	0.070	0.161	0.557	0.302	0.312	0.330	0.378	0.117	0.126	0.142	0.236
200	1.085	0.011	0.043	0.108	0.481	0.247	0.248	0.250	0.253	0.087	0.089	0.093	0.134
500	1.049	0.003	0.016	0.044	0.354	0.146	0.147	0.147	0.141	0.034	0.034	0.035	0.051
Roo = 0.8													
10	3.054	0.397	0.482	0.593	0.835	0.550	0.597	0.663	0.827	0.512	0.559	0.628	0.810
20	2.253	0.140	0.281	0.435	0.764	0.361	0.434	0.526	0.715	0.293	0.361	0.451	0.667
40	1.832	0.084	0.228	0.359	0.687	0.421	0.434	0.466	0.613	0.308	0.323	0.358	0.529
100	1.381	0.040	0.128	0.236	0.586	0.335	0.348	0.369	0.456	0.179	0.188	0.205	0.314
200	1.131	0.015	0.062	0.143	0.482	0.277	0.281	0.287	0.311	0.104	0.107	0.113	0.164
500	1.074	0.006	0.028	0.074	0.383	0.191	0.192	0.194	0.199	0.053	0.054	0.055	0.074
Roo = 0.9													
10	4.945	0.352	0.459	0.572	0.805	0.423	0.506	0.600	0.833	0.410	0.492	0.586	0.824
20	2.629	0.232	0.383	0.506	0.768	0.470	0.514	0.571	0.744	0.417	0.462	0.521	0.710
40	2.480	0.149	0.318	0.423	0.670	0.376	0.430	0.477	0.644	0.326	0.376	0.422	0.594
100	1.632	0.065	0.193	0.302	0.604	0.364	0.380	0.400	0.513	0.248	0.260	0.279	0.397
200	1.336	0.034	0.125	0.236	0.518	0.363	0.368	0.376	0.423	0.192	0.197	0.205	0.268
500	1.121	0.012	0.052	0.120	0.447	0.255	0.257	0.262	0.278	0.087	0.089	0.093	0.125
Roo = 0.95													
10	12.455	0.448	0.529	0.597	0.831	0.465	0.536	0.601	0.823	0.464	0.535	0.599	0.822
20	4.837	0.376	0.467	0.541	0.766	0.469	0.511	0.563	0.771	0.456	0.498	0.550	0.761
40	2.819	0.217	0.378	0.465	0.705	0.450	0.482	0.511	0.671	0.406	0.438	0.467	0.633
100	1.998	0.123	0.284	0.390	0.600	0.431	0.451	0.466	0.571	0.346	0.366	0.383	0.494
200	1.579	0.080	0.218	0.335	0.559	0.432	0.439	0.443	0.495	0.300	0.307	0.313	0.375
500	1.259	0.034	0.120	0.230	0.511	0.348	0.353	0.360	0.380	0.183	0.186	0.191	0.220
Roo = 0.98													
10	8.580	0.509	0.562	0.639	0.839	0.547	0.581	0.648	0.856	0.542	0.577	0.644	0.853
20	6.751	0.397	0.474	0.535	0.763	0.462	0.496	0.545	0.764	0.456	0.490	0.538	0.758
40	5.174	0.360	0.470	0.513	0.709	0.474	0.507	0.527	0.681	0.463	0.495	0.515	0.670
100	2.781	0.235	0.391	0.464	0.632	0.500	0.506	0.509	0.610	0.453	0.461	0.465	0.569
200	2.261	0.157	0.338	0.432	0.591	0.482	0.495	0.497	0.551	0.408	0.422	0.428	0.486
500	1.527	0.081	0.227	0.348	0.545	0.439	0.448	0.454	0.473	0.307	0.315	0.323	0.351

Table 5: MSE values for the different estimators when p = 8

n	ML	LR				LLT1				LLT2			
		K1	K2	K3	K4	K1	K2	K3	K4	K1	K2	K3	K4
Roo = 0.7													
10	260.1	0.794	1.615	1.927	0.828	0.779	1.401	1.699	0.850	0.780	1.453	1.702	0.850
20	30.28	3.693	1.906	1.063	0.736	2.077	1.269	0.817	0.748	2.085	1.282	0.834	0.750
40	8.04	2.306	1.547	0.919	0.682	0.755	0.660	0.561	0.643	0.814	0.721	0.620	0.651
100	4.058	1.290	0.996	0.700	0.522	0.491	0.448	0.402	0.462	0.529	0.494	0.457	0.483
200	2.283	0.616	0.537	0.429	0.415	0.306	0.297	0.283	0.306	0.327	0.321	0.314	0.330
500	1.521	0.251	0.237	0.214	0.318	0.170	0.169	0.167	0.171	0.177	0.176	0.176	0.183
Roo = 0.8													
10	789.4	0.872	1.539	1.656	0.864	0.858	1.481	1.589	0.834	0.859	1.482	1.590	0.834
20	62.21	4.225	1.747	0.947	0.765	2.366	1.247	0.785	0.760	2.376	1.259	0.796	0.762
40	15.76	3.874	1.750	0.847	0.683	1.980	1.091	0.601	0.608	1.982	1.106	0.632	0.619
100	6.007	2.052	1.288	0.735	0.542	0.765	0.587	0.431	0.464	0.783	0.619	0.483	0.496
200	2.942	0.883	0.715	0.528	0.427	0.404	0.379	0.346	0.350	0.418	0.398	0.374	0.382
500	1.874	0.379	0.343	0.288	0.329	0.215	0.211	0.204	0.209	0.225	0.223	0.220	0.236
Roo = 0.9													
10	450.3	0.809	1.397	1.348	0.795	0.778	1.258	1.219	0.856	0.780	1.260	1.220	0.856
20	93.16	4.616	1.576	0.849	0.763	2.418	1.138	0.719	0.729	2.425	1.146	0.728	0.731
40	22.33	5.015	1.858	0.758	0.664	1.946	0.897	0.489	0.641	1.961	0.929	0.528	0.649
100	10.78	3.509	1.644	0.749	0.515	1.281	0.759	0.459	0.497	1.294	0.791	0.510	0.523
200	5.995	2.005	1.164	0.645	0.427	0.834	0.634	0.431	0.393	0.840	0.648	0.465	0.440
500	2.782	0.804	0.632	0.442	0.317	0.338	0.310	0.267	0.265	0.353	0.330	0.301	0.327
Roo = 0.95													
10	3486.5	0.769	1.078	0.987	0.833	0.763	1.059	0.971	0.859	0.763	1.059	0.971	0.859
20	161.7	4.262	1.413	0.745	0.704	2.727	1.074	0.649	0.743	2.733	1.080	0.655	0.744
40	75.16	5.836	1.671	0.704	0.636	3.840	1.206	0.595	0.665	3.842	1.214	0.605	0.668
100	18.21	4.999	1.773	0.672	0.507	1.857	0.775	0.402	0.499	1.871	0.808	0.447	0.519
200	9.63	3.066	1.479	0.668	0.426	1.260	0.756	0.405	0.408	1.269	0.779	0.454	0.449
500	4.738	1.617	1.035	0.573	0.334	0.599	0.465	0.316	0.297	0.616	0.496	0.375	0.375
Roo = 0.98													
10	12828	0.667	0.933	0.886	0.815	0.666	0.925	0.880	0.865	0.666	0.925	0.880	0.865
20	967.6	4.537	0.965	0.567	0.742	3.727	0.887	0.553	0.764	3.728	0.888	0.554	0.764
40	216.4	8.274	1.147	0.468	0.687	5.620	0.871	0.424	0.631	5.623	0.876	0.429	0.632
100	54.24	8.486	1.552	0.500	0.526	4.046	0.805	0.374	0.506	4.051	0.820	0.392	0.515
200	24.73	5.257	1.710	0.604	0.427	2.722	1.005	0.416	0.426	2.725	1.020	0.444	0.448
500	10.79	3.304	1.457	0.622	0.329	1.443	0.826	0.399	0.320	1.447	0.841	0.437	0.375

Table 6: Bias values for the different estimators when p = 8

n	ML	LR				LLT1				LLT2			
		K1	K2	K3	K4	K1	K2	K3	K4	K1	K2	K3	K4
Roo = 0.7													
10	9.053	0.726	0.530	0.503	0.903	0.733	0.550	0.525	0.912	0.730	0.547	0.523	0.911
20	3.697	0.286	0.378	0.468	0.847	0.445	0.484	0.540	0.835	0.434	0.469	0.521	0.824
40	2.042	0.103	0.196	0.326	0.812	0.395	0.435	0.497	0.720	0.338	0.372	0.427	0.682
100	1.553	0.041	0.104	0.195	0.706	0.338	0.364	0.402	0.563	0.273	0.289	0.313	0.484
200	1.261	0.015	0.056	0.124	0.628	0.319	0.326	0.339	0.399	0.241	0.240	0.240	0.295
500	1.117	0.006	0.025	0.061	0.547	0.217	0.219	0.222	0.237	0.157	0.155	0.153	0.147
Roo = 0.8													
10	13.70	0.656	0.556	0.549	0.922	0.660	0.562	0.556	0.906	0.659	0.562	0.555	0.906
20	5.145	0.302	0.411	0.506	0.865	0.404	0.473	0.547	0.854	0.397	0.464	0.536	0.848
40	2.748	0.140	0.257	0.350	0.815	0.338	0.378	0.431	0.741	0.331	0.361	0.402	0.712
100	1.837	0.040	0.138	0.254	0.723	0.307	0.355	0.413	0.610	0.276	0.309	0.347	0.533
200	1.384	0.024	0.083	0.168	0.640	0.332	0.347	0.371	0.468	0.280	0.284	0.290	0.360
500	1.209	0.008	0.036	0.087	0.560	0.252	0.256	0.264	0.310	0.191	0.190	0.188	0.190
Roo = 0.9													
10	12.83	0.660	0.546	0.550	0.887	0.667	0.560	0.564	0.919	0.665	0.558	0.562	0.918
20	6.376	0.361	0.452	0.514	0.865	0.462	0.502	0.543	0.841	0.456	0.495	0.535	0.836
40	3.266	0.134	0.260	0.371	0.806	0.328	0.378	0.443	0.771	0.314	0.357	0.416	0.749
100	2.347	0.068	0.209	0.334	0.708	0.321	0.390	0.448	0.663	0.302	0.356	0.400	0.612
200	1.821	0.036	0.131	0.239	0.643	0.233	0.283	0.354	0.570	0.219	0.258	0.308	0.484
500	1.369	0.013	0.059	0.129	0.553	0.236	0.252	0.281	0.425	0.196	0.203	0.214	0.288
Roo = 0.95													
10	24.64	0.618	0.532	0.546	0.908	0.619	0.534	0.547	0.922	0.619	0.533	0.547	0.922
20	8.471	0.327	0.436	0.514	0.834	0.380	0.467	0.533	0.852	0.377	0.463	0.528	0.850
40	5.749	0.277	0.417	0.492	0.790	0.369	0.471	0.521	0.802	0.365	0.465	0.512	0.795
100	2.999	0.070	0.218	0.344	0.705	0.226	0.341	0.421	0.680	0.215	0.319	0.390	0.646
200	2.233	0.045	0.149	0.263	0.645	0.197	0.272	0.355	0.602	0.187	0.250	0.315	0.540
500	1.664	0.020	0.088	0.179	0.570	0.211	0.246	0.303	0.495	0.186	0.210	0.244	0.384
Roo = 0.98													
10	50.02	0.663	0.514	0.525	0.899	0.664	0.514	0.525	0.926	0.663	0.514	0.525	0.926
20	18.09	0.399	0.482	0.550	0.856	0.416	0.488	0.553	0.868	0.415	0.488	0.552	0.868
40	9.655	0.247	0.383	0.448	0.822	0.294	0.403	0.457	0.786	0.293	0.401	0.454	0.784
100	4.991	0.161	0.344	0.424	0.718	0.289	0.415	0.456	0.699	0.285	0.405	0.444	0.687
200	3.412	0.090	0.242	0.356	0.647	0.191	0.325	0.409	0.635	0.188	0.313	0.388	0.608
500	2.340	0.043	0.149	0.264	0.568	0.166	0.247	0.340	0.542	0.160	0.232	0.308	0.482