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On Multiobjective Mathematical Programming Problems with Equilibrium Constraints

Kunwar V. K. Singh and S. K. Mishra[*]

Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India

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Abstract: In this paper, we consider a multiobjective mathematical programming problem with equilibrium constraints. We define the generalized Guignard constraint qualification for a multiobjective mathematical programming problem with equilibrium constraints. We derive the Karush-Kuhn-Tucker type necessary optimality conditions for multiobjective mathematical programming problem with equilibrium constraints and also derive sufficient optimality conditions for multiobjective mathematical programming problem with equilibrium constraints under assumptions of pseudoconvexity and quasiconvexity. Further, we formulate Wolfe type dual as well as Mond-Weir type dual models and establish weak and strong duality theorems under pseudoconvexity and quasiconvexity assumptions.

Keywords: Mathematical programming problems with equilibrium constraints, Optimality conditions, Constraint qualifications, Efficient solutions, Duality.

1 Introduction

Consider a multiobjective mathematical programming problem with equilibrium constraints as follows:

$$
\begin{array}{lll}
(MMPEC) & \min & (f_1(z),\ldots,f_l(z)) \\
\text{subject to} & & g(z) \leqq 0,\ \ h(z) = 0, \\
& & G(z) \geqq 0,\ H(z) \geqq 0, \\
& & G(z)^T H(z) = 0,
\end{array}
$$

where $f : R^n \to R^l$, $g : R^n \to R^p$, $h : R^n \to R^q$, $G : R^n \to R^m$ and $H : R^n \to R^m$ are continuously differentiable on R^n and $G(z)^T$ indicates the transpose of the $G(z)$.

The concept of mathematical programming problems with equilibrium constraints (MPEC) has been coined by Harker and Pang [1] in 1988. MPEC form a relatively new and interesting subclass of nonlinear programming problems. MPEC arises frequently in various real world problems e.g., in chemical process engineering [2], hydroeconomic river basin model [3], capacity enhancement in traffic, dynamic pricing in telecommunication networks [4], multilevel games [5], chemical equilibria, environmental economics problems [6] and several other problems [7,8].

Luo *et al.* [7] presented a comprehensive study of MPEC. Fukushima and Pang [9] studied some feasibility conditions in MPEC. Outrata [10] established necessary optimality conditions for a class of MPEC, provided the complementarity problem is strongly regular at the solution. Scheel and Scholtes [11] studied MPEC and introduced several stationary point concepts. Ye [12] established necessary and sufficient optimality conditions for MPEC and obtained new constraint qualifications for MPEC. It is easy to see that the standard Mangasarian-Formovitz constraint qualification is violated at every feasible point of a MPEC [13]. Flegel and Kanzow [14] introduced Abadie-type and Slater-type constraint qualifications for MPEC. Flegel and Kanzow [15] proved that first order optimality conditions for MPEC may be obtained under assumption of Guignard constraint qualification.

To the best of our knowledge, there are only a few papers on the (MMPEC). Bao *et al.* [16] and Mordukhovich [17] studied multiobjective optimization problems with equilibrium constraints described by parametric generalized equations. Recently, Pandey and Mishra [18] studied a multiobjective semi-infinite mathematical programming problems with equilibrium constraints and defined the concept of Mourdukhovich stationary point for the nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints in terms of Clarke subdifferentials. Zhang *et al.* [19] defined constraint

[*] Corresponding author e-mail: bhu.skmishra@gmail.com

qualifications for multiobjective problems with equilibrium constraints and established relationships among them, also gave various stationarity conditions in the proper Pareto sense of multiobjective problem with equilibrium constraints.

The outline of this paper is as follows: in Sect. 2, we give some basic definitions. In Sect. 3, we derive Karush-Kuhn-Tucker type conditions for the (MMPEC) using generalized Guignard constraint qualification. In Sect. 4, we derive sufficient optimality conditions for the (MMPEC). In Sect. 5, we formulate Wolfe type dual and Mond-Weir type dual and establish duality results for the (MMPEC). In Sect. 6, we give an application of the (MMPEC).

2 Preliminaries

In this section, we give some basic definitions and results, which will be used in the sequel. For any two vectors x and y in R^n, we shall use the following conventions:

$$\begin{aligned} x \leqq y &\iff x_i \leqq y_i\ ,\ \forall\ i = 1, \ldots, n, \\ x \leq y &\iff x \leqq y \ \text{ and } \ x \neq y, \\ x < y &\iff x_i < y_i\ ,\ \forall\ i = 1, \ldots, n. \end{aligned}$$

The following definitions are taken from Cambini and Martein [20].

Definition 1. *A differentiable function f defined on an open convex set $S \subseteq R^n$ is said to be pseudoconvex at z^* over S, iff following implication holds:*

$$z,\ z^* \in S,\ \ \langle \nabla f(z^*), z - z^* \rangle \geq 0 \Rightarrow f(z) \geq f(z^*).$$

Definition 2. *A differentiable function f defined on an open convex set $S \subseteq R^n$ is said to be strictly pseudoconvex at z^* over S, iff following implication holds:*

$$z,\ z^* \in S \text{ and } z \neq z^*,\ \langle \nabla f(z^*), z - z^* \rangle \geq 0 \Rightarrow f(z) > f(z^*).$$

Definition 3. *A differentiable function f defined on an open convex set $S \subseteq R^n$ is said to be quasiconvex at z^* over S, iff following implication holds:*

$$z,\ z^* \in S,\ \ f(z) \leq f(z^*) \Rightarrow \langle \nabla f(z^*), z - z^* \rangle \leq 0.$$

The set

$$\begin{aligned} P := \{z \in R^n \mid g(z) \leqq 0,\ h(z) = 0,& \\ G(z) \geqq 0,\ H(z) \geqq 0,\ G(z)^T H(z) = 0\}& \end{aligned}$$

is the set of feasible solutions of the (MMPEC).

The following definitions are taken from Maeda [21].

Definition 4. *A vector $z^* \in P$ is said to be a weak efficient solution to the (MMPEC), if there is no $z \in P$, such that*

$$f(z) < f(z^*),$$

Definition 5. *A vector $z^* \in P$ is said to be an efficient solution to the (MMPEC), if there is no $z \in P$, such that*

$$f(z) \leq f(z^*)\ .$$

Given a feasible point $z^* \in P$, we consider the following index sets:

$$\begin{aligned} &I_g := \{i : g_i(z^*) = 0\}, \\ &\alpha := \alpha(z^*) = \{i : G_i(z^*) = 0\ ,\ H_i(z^*) > 0\}, \\ &\beta := \beta(z^*) = \{i : G_i(z^*) = 0\ ,\ H_i(z^*) = 0\}, \\ &\gamma := \gamma(z^*) = \{i : G_i(z^*) > 0\ ,\ H_i(z^*) = 0\}. \end{aligned}$$

The set β is known as the degenerate set. If β is empty, then the vector z^* is said to satisfy the strict complementarity condition. Also, consider the following function

$$\theta(z) := G(z)^T H(z) \tag{1}$$

and the gradient is given by

$$\nabla\theta(z) := \sum_{i=1}^{m} [H_i(z)\nabla G_i(z) + G_i(z)\nabla H_i(z)]. \tag{2}$$

3 Necessary Optimality conditions for MMPEC

In 1994, Maeda [21] derived some relations among various constraint qualifications for multiobjective nonlinear programming problems and Mishra *et al.* [22] extended the concept of Maeda [21] for multiobjective optimization problems with vanishing constraints. In this section, we discuss the generalized Guignard constraint qualification for the (MMPEC) under which the Karush-Kuhn-Tucker type necessary optimality conditions for a feasible solution to be an efficient solution will be given.

Let Q be a non-empty subset of R^n. The tangent cone to Q at $z^* \in clQ$ is the set $T(Q; z^*)$ defined by,

$$\begin{aligned} T(Q, z^*) := \{d \in R^n \mid \exists \{z^n\} \subseteq Q, \{t_n\} \searrow 0 : z^n \to z^*& \\ \text{and } \frac{z^n - z^*}{t_n} \to d\},& \end{aligned}$$

where clQ denotes the closure of Q.

The following sets of Q^k and Q for $k = 1, \ldots, l$ will be used in the sequel.

$$\begin{aligned} Q^k := \{z \in R^n \mid f_i(z) &\leqq f_i(z^*),\ \forall\, i = 1, \ldots, l \ \text{ and } i \neq k, \\ g_i(z) &\leqq 0,\ \forall\, i = 1, \ldots, p, \\ h_i(z) &= 0,\ \forall\, i = 1, \ldots, q, \\ G_i(z) &\geqq 0,\ \forall\, i = 1, \ldots, m, \\ H_i(z) &\geqq 0,\ \forall i = 1, \ldots, m, \\ G(z)^T\ H(z) = 0\ \},& \end{aligned} \tag{3}$$

$$\begin{aligned} Q := \{z \in R^n \mid f_i(z) &\leqq f_i(z^*), \forall\, i = 1,\ldots,l,\\ g_i(z) &\leqq 0,\ \forall\, i = 1,\ldots,p,\\ h_i(z) &= 0, \forall i = 1,\ldots,q,\\ G_i(z) &\geqq 0, \forall i = 1,\ldots,m,\\ H_i(z) &\geqq 0, \forall i = 1,\ldots,m,\\ G(z)^T\ H(z) = 0\}. \end{aligned} \tag{4}$$

For scalar objective programming problems, $Q^1 = P$.

We now extend the Definition 3.1 of Maeda [21] for the (MMPEC).

$$\begin{aligned} T^{Lin}(Q,z^*) = \{d \in R^n \mid \nabla f_i(z^*)^T d \leqq 0\ \forall\, i = 1,\ldots,l,\\ \nabla g_i(z^*)^T d \leqq 0,\ \forall\, i \in I_g,\\ \nabla h_i(z^*)^T d = 0,\ \forall\, i = 1,\ldots,q,\\ \nabla G_i(z^*)^T d = 0,\ \forall\, i \in \alpha,\\ \nabla H_i(z^*)^T d = 0,\ \forall\, i \in \gamma,\\ \nabla G_i(z^*)^T d \geqq 0,\ \forall\, i \in \beta,\\ \nabla H_i(z^*)^T d \geqq 0,\ \forall\, i \in \beta\}. \end{aligned} \tag{5}$$

It is known that the inclusion

$$T(Q,z^*) \subseteq T^{Lin}(Q,z^*)$$

holds and the standard Abadie constraint qualification (ACQ) for a general nonlinear programming problems is given by

$$T(Q,z^*) = T^{Lin}(Q,z^*).$$

The above condition is likely to satisfied by the standard nonlinear programming problems, but it is not appropriate for the (MMPEC). Since, in general tangent cone $T(Q,z^*)$ is not convex, but $T^{Lin}(Q,z^*)$ is polyhedral, so convex. Therefore, we modify above definition as follows:

$$\begin{aligned} T^{Lin}_{MMPEC}(Q,z^*) = \{d \in R^n \mid \nabla f_i(z^*)^T d \leqq 0,\ \forall\, i = 1,\ldots,l,\\ \nabla g_i(z^*)^T d \leqq 0,\ \forall\, i \in I_g,\\ \nabla h_i(z^*)^T d = 0,\ \forall\, i = 1,\ldots,q,\\ \nabla G_i(z^*)^T d = 0,\ \forall\, i \in \alpha,\\ \nabla H_i(z^*)^T d = 0,\ \forall\, i \in \gamma,\\ \nabla G_i(z^*)^T d \geqq 0,\ \forall\, i \in \beta,\\ \nabla H_i(z^*)^T d \geqq 0,\ \forall\, i \in \beta,\\ (\nabla(G_i(z^*)^T d).\nabla(H_i(z^*)^T d) = 0,\ \forall\, i \in \beta\}. \end{aligned} \tag{6}$$

Clearly $\quad T^{Lin}_{MMPEC}(Q,z^*) \subseteq T^{Lin}(Q,z^*)$.

Definition 6. *Let $z^* \in P$ be a feasible solution of the (MMPEC). Then, the generalized Guignard constraint qualification (GGCQ) holds at z^*, iff*

$$T^{Lin}_{MMPEC}(Q,z^*) \subseteq \bigcap_{k=1}^{l} cl\ coT(Q^k,z^*),$$

where $cl\ coT(Q^k,z^*)$ denotes the closure of the convex hull of $T(Q^k,z^*)$.

We define constraint qualifications which are sufficient condition for the GGCQ.

Definition 7. *Let $z^* \in P$ be a feasible solution of the (MMPEC). The Abadie constraint qualification holds at z^*, iff*

$$T^{Lin}_{MMPEC}(Q,z^*) \subseteq T(Q,z^*),$$

and the generalized Abadie constraints qualification (GACQ) holds at z^*, iff

$$T^{Lin}_{MMPEC}(Q,z^*) \subseteq \bigcap_{k=1}^{l} T(Q^k,z^*).$$

The following result gives the KKT type necessary optimality conditions for efficiency, when the GGCQ holds at an efficient solution of the (MMPEC).

Theorem 1. *Let $z^* \in P$ be an efficient solution of the (MMPEC). If the GGCQ holds at z^*, then there exist Lagrange multipliers τ_i $(i = 1,\ldots,l)$, λ_i^g $(i = 1,\ldots,p)$, λ_i^h $(i = 1,\ldots,q)$, λ_i^G $(i = 1,\ldots,m)$ and λ_i^H $(i = 1,\ldots,m)$, such that $\tau_i > 0,\ \forall\ i = 1,\ldots,l,$*

$$\begin{aligned} \sum_{i=1}^{l} \tau_i \nabla f_i(z^*) + \sum_{i=1}^{p} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*)\\ - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)] = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} g_i(z^*) \leqq 0\ , \lambda_i^g \geqq 0,\ \lambda_i^g g_i(z^*) = 0,\ \forall\ i = 1,\ldots,p,\\ h_i(z^*) = 0,\ \forall\ i = 1,\ldots,q, \end{aligned} \tag{8}$$

λ_i^G *free* $i \in \alpha$, $\lambda_i^G \geqq 0,\ i \in \beta,\ \lambda_i^G = 0,\ i \in \gamma,$

λ_i^H *free* $i \in \gamma$, $\lambda_i^H \geqq 0,\ i \in \beta,\ \lambda_i^H = 0,\ i \in \alpha.$

Proof. Suppose that $z^* \in P$ is an efficient solution of the (MMPEC) such that GGCQ holds at z^*. Then, by Theorem 3.2 of [21], there exist Lagrange multipliers $\tau_i (i = 1,\ldots,l)$, $\lambda_i^g (i = 1,\ldots,p)$, $\rho_i^+, \rho_i^- \in R$ $(i = 1,\ldots,q)$, $\lambda_i \in R$ $(i = 1,\ldots,m)$, $\mu_i \in R$ $(i = 1,\ldots,m)$, $\delta \in R$, such that the following conditions hold:

$$\begin{aligned} \sum_{i=1}^{l} \tau_i\ \nabla f_i(z^*) + \sum_{i=1}^{p} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \rho_i^+ \nabla h_i(z^*) - \sum_{i=1}^{q} \rho_i^- \nabla h_i(z^*)\\ - \sum_{i=1}^{m} \lambda_i \nabla G_i(z^*) - \sum_{i=1}^{m} \mu_i \nabla H_i(z^*) + \delta \nabla \theta(z^*) = 0, \end{aligned}$$

and $\quad \tau_i > 0,\ \forall\ i = 1,\ldots l,$

$$\begin{aligned} g_i(z^*) \leqq 0,\ \lambda_i^g \geqq 0,\ \lambda_i^g g_i(z^*) = 0,\ \forall\ i = 1,\ldots,p,\\ h_i(z^*) \leqq 0,\ \rho_i^+ \geqq 0,\ \rho_i^+ h_i(z^*) = 0,\ \forall\ i = 1,\ldots,q,\\ -h_i(z^*) \leqq 0,\ \rho_i^- \geqq 0,\ \rho_i^-(h_i(z^*)) = 0,\ \forall\ i = 1,\ldots,q,\\ -G_i(z^*) \leqq 0,\ \lambda_i \geqq 0,\ \lambda_i(-G_i(z^*)) = 0,\ \forall\ i = 1,\ldots,m,\\ -H_i(z^*) \leqq 0,\ \mu_i \geqq 0,\ \mu_i(-H_i(z^*)) = 0,\ \forall\ i = 1,\ldots,m,\\ \theta(z^*) \leqq 0,\ \delta \geqq 0,\ \delta\theta(z^*) = 0, \end{aligned}$$

where θ denotes the function defined in (1). Now using the gradient of composite function and setting

$$\rho_i^+ - \rho_i^- = \lambda_i^h, \ \forall \ i = 1,\dots,q,$$
$$\lambda_i - \delta H_i(z^*) = \lambda_i^G, \ \forall \ i = 1,\dots,m,$$
$$\mu_i - \delta G_i(z^*) = \lambda_i^H, \ \forall \ i = 1,\dots,m,$$

we get the required KKT type necessary optimality conditions (7) and (8).

4 Sufficient Optimality Conditions for MMPEC

In a standard nonlinear programming problem there is only one stationary condition, that is, KKT condition, but in MPEC there are several stationary point conditions given in literature, such as W-stationary point, A-stationary point, C-stationary point, M-stationary point, S-stationary point [12]. S-stationary point is the strongest condition among them and M-stationary point condition is the second strongest stationary condition for MPEC [12,14,15].

The following definition of M-stationary point for the (MMPEC) is an extension of Ye [12].

Definition 8.*(M-stationary point) A feasible point z^* of the (MMPEC) is called Mordukhovich stationary point, if there exist $\tau_i > 0$ $(i = 1,\dots,l)$ and $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$, such that the following conditions hold:*

$$\sum_{i=1}^{l} \tau_i \nabla f_i(z^*) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)] = 0, \quad (9)$$

$$\lambda_{I_g}^g \geqq 0, \ \lambda_\gamma^G = 0, \ \lambda_\alpha^H = 0,$$
$$\forall i \in \beta, \ \textit{either} \ \lambda_i^G > 0, \ \lambda_i^H > 0 \ \textit{or} \ \lambda_i^G \lambda_i^H = 0.$$

Before going to next result, we define some index sets as follow:

$$\beta^+ := \{i \in \beta : \lambda_i^G > 0, \lambda_i^H > 0\},$$
$$\beta_G^+ := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H > 0\},$$
$$\beta_G^- := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H < 0\},$$
$$\beta_H^+ := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G > 0\},$$
$$\beta_H^- := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G < 0\},$$
$$\alpha^+ := \{i \in \alpha : \lambda_i^G > 0\},$$
$$\alpha^- := \{i \in \alpha : \lambda_i^G < 0\},$$
$$\gamma^+ := \{i \in \gamma : \lambda_i^H > 0\},$$
$$\gamma^- := \{i \in \gamma : \lambda_i^H < 0\},$$
$$J^+ := \{i : \lambda_i^h > 0\}, \quad J^- := \{i : \lambda_i^h < 0\}.$$

Theorem 2. *Let $z^* \in P$ be a feasible point of the (MMPEC) and M-stationary condition holds at z^*. Suppose that each f_i $(i = 1,\dots,l)$ is pseudoconvex at z^*, g_i $(i \in I_g)$, h_i $(i \in J^+)$, $-h_i$ $(i \in J^-)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \gamma^- \cup \beta_G^-)$, $-H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are quasiconvex at z^*. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then z^* is a weak efficient solution for the (MMPEC).*

Proof. Suppose to the contrary that $z^* \in P$ is not a weak efficient solution for the (MMPEC). Then there exists a feasible point $z \in P$ such that

$$f_i(z) < f_i(z^*), \ \forall \ i = 1,\dots,l.$$

Since, f_i is pseudoconvex, $\forall \ i = 1,\dots,l$, it follows that

$$\langle \nabla f_i(z^*), \ z - z^* \rangle < 0, \ \forall \ z \in P.$$

Since $\tau_i > 0$, $\forall \ i = 1,\dots,l$, we have

$$\left\langle \sum_{i=1}^{l} \tau_i \nabla f_i(z^*), z - z^* \right\rangle < 0. \quad (10)$$

Since, z^* is M-stationary point, then from (9) and (10), we get

$$\left\langle \sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) \quad (11)$$
$$+ \lambda_i^H \nabla H_i(z^*)], z - z^* \right\rangle > 0. \quad (12)$$

Since, for each $i \in I_g(z^*)$, $g_i(z^*) \leqq 0 = g_i(z^*)$. Hence, by quasiconvexity of g_i, we get

$$\langle \nabla g_i(z^*), z - z^* \rangle \leqq 0, \ \forall z \in P, \ \forall i \in I_g. \quad (13)$$

For any feasible point z of the (MMPEC) and each $i \in J^-$, $h_i(z^*) \leqq 0 = h_i(z^*)$. Hence, by definition of quasiconvexity of h_i, we get

$$\langle \nabla h_i(z^*), z - z^* \rangle \leqq 0, \ \forall z \in P, \ \forall i \in J^-. \quad (14)$$

Similarly, we have

$$\langle \nabla h_i(z^*), z - z^* \rangle \geqq 0, \ \forall z \in P, \ \forall i \in J^+. \quad (15)$$

Since, $-G_i(z^*) \leqq 0 = -G_i(z^*)$, $\forall \ i \in \alpha^+ \cup \beta_H^+$, we get

$$\langle \nabla G_i(z^*), z - z^* \rangle \geqq 0, \ \forall z \in P, \ \forall i \in \alpha^+ \cup \beta_H^+. \quad (16)$$

Similarly

$$\langle \nabla H_i(z^*), z - z^* \rangle \geqq 0, \ \forall z \in P, \ \forall i \in \gamma^+ \cup \beta_G^+. \quad (17)$$

Since $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then from equations (13) to (17), we get

$$\left\langle \sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*), z - z^* \right\rangle \leqq 0,$$
$$\left\langle \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*), z - z^* \right\rangle \leqq 0,$$
$$\left\langle \sum_{i\in\alpha\cup\beta} \lambda_i^G \nabla G_i(z^*), z - z^* \right\rangle \geqq 0,$$
$$\left\langle \sum_{i\in\beta\cup\gamma} \lambda_i^H \nabla H_i(z^*), z - z^* \right\rangle \geqq 0.$$

Therefore, from above equations, we get

$$\left\langle \sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],\ z - z^* \right\rangle \leqq 0,\ \forall z \in P,$$

this implies that

$$\sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)] \leqq 0,$$

which contradicts (11). Therefore, z^* is a weak efficient solution for the (MMPEC). □

The following example illustrates the above result.

Example 1. Consider the following MMPEC in R^2

$$\min\ f(z_1,z_2) := (z_1^3 + z_1,\ z_2),$$

$$\text{subject to } G(z_1,z_2) := z_1 \geqq 0,\quad H(z_1,z_2) := z_2 \geqq 0,$$
$$G(z_1,z_2)^T H(z_1,z_2) := z_1 z_2 = 0,\quad \forall z_1, z_2 \in R.$$

Let $f_1(z_1,z_2) = z_1^3 + z_1$ and $f_2(z_1,z_2) = z_2$. The feasible region of MMPEC is $P_1 = \{(z_1,z_2) \in R^2$, such that either $z_1 = 0$ and $z_2 \geqq 0$ or $z_2 = 0$ and $z_1 \geqq 0\}$. If we take point $(z_1^*, z_2^*) = (0,0)$ in feasible region, then index sets $\alpha(0,0)$ and $\gamma(0,0)$ are empty, but $\beta(0,0) = \{1\}$. Also $\tau_1 \nabla f_1(0,0) + \tau_2 \nabla f_2(0,0) - \mu \nabla G(0,0) - \nu \nabla H(0,0) = 0$ and either $\mu\nu = 0$ or $\mu > 0$ and $\nu > 0$, then $\tau_1 - \mu = 0,\ \tau_2 - \nu = 0$. If we take $\tau_1 = 1,\ \tau_2 = 1$, then $\mu = 1$ and $\nu = 1$, such that MMPEC M-stationary conditions hold. Therefore, by Theorem 2, $(z_1^*, z_2^*) = (0,0)$ is a weak efficient solution of MMPEC.

Theorem 2 can be obtained under slightly weaker assumption on the objective function and the proof will follow on the lines of the proof of Theorem 2.

Theorem 3. *Let $z^* \in P$ be a feasible point of the (MMPEC) and M-stationary condition holds at z^*. Suppose that each f_i $(i = 1,\dots,l)$ is strictly pseudoconvex at z^*, g_i $(i \in I_g)$, h_i $(i \in J^+)$, $-h_i$ $(i \in J^-)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \gamma^- \cup \beta_G^-)$, $-H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are quasiconvex at z^*. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then z^* is an efficient solution for the (MMPEC).*

5 Duality

In this section, we formulate a Wolfe type dual model and a Mond-Weir type dual model for the (MMPEC) and establish usual duality theorems under convexity and generalized convexity assumptions.

We consider the following Wolfe type dual of the (MMPEC):

$$(WDMMPEC)\quad \max\ f(u) + \Big[\sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) - \sum_{i=1}^{m} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e,$$
$$\text{where } e = (1,1,\dots,1) \in R^l,$$

$$\text{subject to } \sum_{i=1}^{l} \tau_i \nabla f_i(u) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)] = 0,$$

$$\lambda_{I_g}^g \geqq 0,\ \lambda_\gamma^G = 0,\ \lambda_\alpha^H = 0,$$
$$\forall i \in \beta,\ \text{either } \lambda_i^G > 0,\ \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0,$$
$$\text{where } \lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m},$$
$$\tau_i \geqq 0,\ i = 1,2,\dots,l \ \text{ and } \sum_{i=1}^{l} \tau_i = 1.$$

Remark. The Wolfe type dual model (WDMPEC) given by Pandey and Mishra [23], is a special case of the (WDMMPEC).

Theorem 4. *(Weak Duality) Let z be a feasible for the (MMPEC) and (u,τ,λ) be feasible for the (WDMMPEC) where $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in R^{p+q+2m}$, $\tau \in R^l$. f_i $(i = 1,\dots,l)$, g_i $(i \in I_g)$, h_i $(i \in J^+)$, $-h_i$ $(i \in J^-)$, G_i $(i \in \alpha^- \cup \beta_H^-)$, $-G_i$ $(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \gamma^- \cup \beta_G^-)$, $-H_i$ $(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are convex functions. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$ and $\tau_i > 0$, $\forall\ i = 1,\dots,l$, then*

$$f(z) \nleq f(u) + \Big[\sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) - \sum_{i=1}^{m} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e. \tag{18}$$

Proof. Suppose to the contrary that equation (18) hold that is

$$f(z) \nleq f(u) + \Big[\sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) - \sum_{i=1}^{m} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e. \tag{19}$$

The above equation can be expressed as

$$f_i(z) \leqq f_i(u) + \sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) - \sum_{i=1}^{m} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)],\ \forall\ i = 1,2,\dots,l,$$

$$f_j(z) < f_j(u) + \sum_{i\in I_g}\lambda_i^g g_i(u) + \sum_{i=1}^{q}\lambda_i^h h_i(u) - \sum_{i=1}^{m}[\lambda_i^G G_i(u) + \lambda_i^H H_i(u)],\ \ \forall\, i \neq j.$$

Since z^* is feasible for the (MMPEC) and (u,τ,λ) is feasible for the (WDMMPEC), then $\lambda_{I_g}^g \geqq 0,\ \lambda_\gamma^G = 0,\ \lambda_\alpha^H = 0$, and index sets implies that

$$\sum_{i=1}^{l}\tau_i f_i(z) + \sum_{i\in I_g}\lambda_i^g g_i(z) + \sum_{i=1}^{q}\lambda_i^h h_i(z) - \sum_{i=1}^{m}[\lambda_i^G G_i(z) + \lambda_i^H H_i(z)] \leqq \sum_{i=1}^{l}\tau_i f_i(u) + \sum_{i\in I_g}\lambda_i^g g_i(u) + \sum_{i=1}^{q}\lambda_i^h h_i(u) - \sum_{i=1}^{m}[\lambda_i^G G_i(u) + \lambda_i^H H_i(u)],\ \ \forall\, i = 1,2,\dots,l,$$

and

$$\sum_{i=1}^{l}\tau_i f_j(z) + \sum_{i\in I_g}\lambda_i^g g_i(z) + \sum_{i=1}^{q}\lambda_i^h h_i(z) - \sum_{i=1}^{m}[\lambda_i^G G_i(z) + \lambda_i^H H_i(z)] < \sum_{i=1}^{l}\tau_i f_j(u) + \sum_{i\in I_g}\lambda_i^g g_i(u) + \sum_{i=1}^{q}\lambda_i^h h_i(u) - \sum_{i=1}^{m}[\lambda_i^G G_i(u) + \lambda_i^H H_i(u)],\ \ \forall\, i \neq j.$$

Since $\sum_{i=1}^{l}\tau_i = 1$ and $\tau_i > 0,\ \forall\, i = 1,\dots,l$, which implies that

$$\sum_{i=1}^{l}\tau_i f_j(z) + \sum_{i\in I_g}\lambda_i^g g_i(z) + \sum_{i=1}^{q}\lambda_i^h h_i(z) - \sum_{i=1}^{m}[\lambda_i^G G_i(z) + \lambda_i^H H_i(z)] < \sum_{i=1}^{l}\tau_i f_i(u) + \sum_{i\in I_g}\lambda_i^g g_i(u) + \sum_{i=1}^{q}\lambda_i^h h_i(u) - \sum_{i=1}^{m}[\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]. \quad (20)$$

Since $f_i (i = 1,\dots,l)$, $g_i\ (i \in I_g)$, $h_i\ (i \in J^+)$, $-h_i\ (i \in J^-)$, $G_i\ (i \in \alpha^- \cup \beta_H^-)$, $-G_i\ (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $H_i\ (i \in \gamma^- \cup \beta_G^-)$, $-H_i\ (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ are convex and from (20), we get

$$\left\langle \sum_{i=1}^{l}\tau_i \nabla f_i(u) + \sum_{i\in I_g}\lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q}\lambda_i^h \nabla h_i(u) - \sum_{i=1}^{m}[\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)],\ z-u \right\rangle < 0.$$

Then, we have

$$\sum_{i=1}^{l}\tau_i \nabla f_i(u) + \sum_{i\in I_g}\lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q}\lambda_i^h \nabla h_i(u) - \sum_{i=1}^{m}[\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)] < 0,$$

which contradicts the assumption of duality. □

Corollary 1. *Suppose that Theorem 4 holds for the (MMPEC) and (WDMMPEC). If (z^*,τ^*,λ^*) is feasible for the (WDMMPEC) and z^* is feasible for the (MMPEC). Then, z^* is an efficient solution for the (MMPEC) and (z^*,τ^*,λ^*) is an efficient solution for the (WDMMPEC).*

Proof. Suppose that z^* is not an efficient solution for the (MMPEC), then there exists a feasible solution z for the (MMPEC) such that

$$f_i(z) \leqq f_i(z^*) \ \text{ for some } \ i = 1,\dots,l,$$
$$f_j(z) < f_j(z^*) \ \text{ for some } \ i \neq j,$$

from the feasibility condition of the (MMPEC) and (WDMMPEC) i.e., for $g_i(z^*) = 0,\ \ \forall i \in I_g$, $h_i(z^*) = 0,\ \ G_i(z^*) = 0,\ \ \forall\ i \in \alpha\cup\beta$ and $H_i(z^*) = 0,\ \ \forall\ i \in \beta\cup\gamma$ and $\lambda_{I_g}^g \geqq 0,\ \lambda_\gamma^G = 0,\ \lambda_\alpha^H = 0$, then

$$\sum_{i\in I_g}\lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q}\lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m}[\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)] = 0,$$

and we can write

$$f_i(z) < f_i(z^*) + \sum_{i\in I_g}\lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q}\lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m}[\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)],$$
$$\forall\, i = 1,\dots,l,$$

$$f_j(z) \leqq f_j(z^*) + \sum_{i\in I_g}\lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q}\lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m}[\lambda_i^G \nabla Gi(z^*) + \lambda_i^H \nabla H_i(z^*)],\ i \neq j,$$

which is contradiction to weak duality Theorem 4.

The following lemma is extension of Lemma 2 of Egudo [24] to the (MMPEC).

Lemma 1. *A feasible solution $z^* \in P$ is an efficient solution for the (MMPEC) iff z^* solve*

$$\begin{aligned} P_k(\varepsilon) \quad & \textit{min} \ \ f_k(z), \\ \textit{subject to} \quad & f_j(z) \leqq f_j(z^*), \quad \forall\ j \neq k, \\ \textit{for each} \ \ k = 1,\dots,l, \quad & \\ & g(z) \leqq 0, h(z) = 0, \\ & G(z) \geqq 0,\ H(z) \geqq 0, \\ & G\,(z)\ ^T H(z) = 0. \end{aligned}$$

Theorem 5. *(Strong Duality) Let z^* be an efficient solution for the (MMPEC) and assume that z^* satisfies the GGCQ for $P_k(\varepsilon)$ for at least one $k = 1,\dots,l$; then there* *exist $\tau^* \in R^l$ and $\lambda^* \in R^{p+q+2m}$, such that (z^*,τ^*,λ^*) is feasible for the (WDMMPEC) and if also weak duality Theorem 4 holds between the (MMPEC) and (WDMMPEC), then (z^*,τ^*,λ^*) is an efficient solution for the (WDMMPEC).*

Proof. Since z^* is an efficient solution for the (MMPEC), then from Lemma 1, z^* solves $P_k(\varepsilon)$, $\forall\ k=1,...,l$. Then, by hypothesis there exist $k=1,...,l$ for which z^* satisfies GGCQ for $P_k(\varepsilon)$, then from Theorem 1 there exist $\tau_i > 0$, for $i \neq k$ and $\lambda \in R^{p+q+2m}$, such that

$$f_k(z^*) + \sum_{i\neq k} \tau_i \nabla f_i(z^*) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(z^*) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)] = 0$$

$$g_i(z^*) \leqq 0\ ,\lambda_i^g \geqq 0,\quad \lambda_i^g g_i(z^*) = 0\ \forall\ i=1,...,p,$$
$$h_i(z^*) = 0\ \forall\ i=1,...,q,$$

λ_i^G free $i\in\alpha$, $\lambda_i^G \geqq 0,\ i\in\beta,\ \lambda_i^G = 0,\ i\in\gamma,$

λ_i^H free $i\in\gamma$, $\lambda_i^H \geqq 0,\ i\in\beta,\ \lambda_i^H = 0,\ i\in\alpha.$

Dividing above equations by $1+\sum_{i\neq k}\tau_i$ and setting $\tau_k^* = \dfrac{1}{1+\sum_{i\neq k}\tau_i} > 0$ and $\tau_j^* = \dfrac{1}{1+\sum_{i\neq k}\tau_i} > 0,\ \forall\ j\neq k,$ then we get (z^*,τ^*,λ^*) is feasible solution for the (WDMMPEC) and from Corollary 1, we get result. □

We consider the following Mond-Weir type dual of the (MMPEC).

$$(MWDMMPEC) \qquad \max\ f(u),$$

subject to
$$\sum_{i=1}^{l} \tau_i \nabla f_i(u) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)] = 0, \tag{21}$$

$$\sum_{i\in I_g} \lambda_i^g g_i(u) \geqq 0,\quad \sum_{i=1}^{q} \lambda_i^h h_i(u) \geqq 0,$$

$$\sum_{i=1}^{m} \lambda_i^G G_i(u) \leqq 0,\quad \sum_{i=1}^{m} \lambda_i^H \nabla H_i(u) \leqq 0,$$

$$\lambda_{I_g}^g \geqq 0,\ \lambda_\gamma^G = 0,\ \lambda_\alpha^H = 0,$$

$\forall i\in\beta$ either $\lambda_i^G > 0,\ \lambda_i^H > 0$ or $\lambda_i^G\lambda_i^H = 0.$

where $\lambda = (\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in R^{p+q+2m}$,

$$\tau_i \geqq 0,\ i=1,2,...,l\ \text{ and } \sum_{i=1}^{l}\tau_i = 1.$$

Remark. The Mond-Weir type dual model (MWDMPEC) given in Pandey and Mishra [23] is a special case of the (MWDMMPEC).

Theorem 6. *(Weak Duality) Let z be a feasible point for the (MMPEC) and (u,τ,λ) be feasible point for the (MWDMMPEC), where $\lambda = (\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in R^{p+q+2m}$. Suppose that g_i $(i\in I_g)$, h_i $(i\in J^+)$, $-h_i$ $(i\in J^-)$, G_i $(i\in\alpha^-\cup\beta_H^-)$, $-G_i$ $(i\in\alpha^+\cup\beta_H^+\cup\beta^+)$, H_i $(i\in\gamma^-\cup\beta_G^-)$, $-H_i$ $(i\in\gamma^+\cup\beta_G^+\cup\beta^+)$ be quasiconvex at u. If $\alpha^-\cup\gamma^-\cup\beta_G^-\cup\beta_H^- = \phi$ and any one of the following holds:*

(a) $\tau_i > 0, \forall i\in\{1,...,l\}$ and f_i, $i=1,...,l$ are pseudoconvex at u;

(b) $\tau_i > 0, \forall i\in\{1,...,l\}$ and $\sum_{i=1}^{l}\tau_i f_i(.)$ is pseudoconvex at u;

(c) $\sum_{i=1}^{l}\tau_i f_i(.)$ is strictly pseudoconvex at u. Then,

$$f(z^*) \nleq f(u). \tag{22}$$

Proof. Since g_i $(i\in I_g)$, h_i $(i\in J^+)$, $-h_i$ $(i\in J^-)$, G_i $(i\in\alpha^-\cup\beta_H^-)$, $-G_i$ $(i\in\alpha^+\cup\beta_H^+\cup\beta^+)$, H_i $(i\in\gamma^-\cup\beta_G^-)$, $-H_i$ $(i\in\gamma^+\cup\beta_G^+\cup\beta^+)$ are quasiconvex at u. If $\alpha^-\cup\gamma^-\cup\beta_G^-\cup\beta_H^- = \phi$, then we get

$$\sum_{i\in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) - \sum_{i=1}^{m} [\lambda_i^G \nabla Gi(u) + \lambda_i^H \nabla H_i(u)] \leqq 0,$$

from (21), we get

$$\sum_{i=1}^{l} \tau_i \nabla f_i(u) \geqq 0,$$

which implies that

$$\left\langle \sum_{i=1}^{l} \tau_i \nabla f_i(u), z-u \right\rangle \geqq 0. \tag{23}$$

Suppose to the contrary that result (22) hold, i.e.

$$f(z) \leq f(u). \tag{24}$$

The above inequality can be expressed as

$$f_i(z) \leqq f_i(u),\quad i=1,...,l, \tag{25}$$
$$f_j(z) < f_j(u),\quad i\neq j, \tag{26}$$

for $\tau_i > 0$,

$$\tau_i f_i(z) \leqq \tau_i f_i(u),\quad i=1,...,l,$$
$$\tau_i f_j(z) < \tau_i f_j(u),\quad i\neq j.$$

Then,

$$\sum_{i=1}^{l} \tau_i f_i(z) \leqq \sum_{i=1}^{l} \tau_i f_i(u). \tag{27}$$

Since f_i, $i=1,...,l$ are pseudoconvex, we get

$$\left\langle \sum_{i=1}^{l} \tau_i \nabla f_i(u), z-u \right\rangle < 0, \tag{28}$$

which contradicts equation (23).

(b) Since $\sum_{i=1}^{l} \tau_i f_i(.)$ is pseudoconvex at u, then (27) implies (28) again we get contradiction.

(c) Since $\tau_i \geqq 0$, $i = 1,...,l$, then from (25) and (26), we get

$$\sum_{i=1}^{l} \tau_i f_i(z) \leqq \sum_{i=1}^{l} \tau_i f_i(u). \tag{29}$$

Since $\sum_{i=1}^{l} \tau_i f_i(.)$ is strictly pseudoconvex at u, again we get contradiction. □

Corollary 2. *Suppose that Theorem 6 holds for the (MMPEC) and (MWDMMPEC). If (z^*,τ^*,λ^*) is feasible for the (MWDMMPEC) and z^* is feasible for the (MMPEC). Then, z^* is an efficient solution for the (MMPEC) and (z^*,τ^*,λ^*) is an efficient solution for the (MWDMMPEC).*

Proof. Suppose that z^* is not an efficient solution for the (MMPEC), then there exists a feasible solution z for the (MMPEC), such that Theorem 6 holds. Since (z^*,τ^*,λ^*) is feasible for the (MWDMMPEC), then we get a contradiction. Therefore, z^* must be an efficient solution for the (MMPEC). Similarly assuming that (z^*,τ^*,λ^*) is not an efficient solution for the (MWDMMPEC), then we get contradiction.

Hence, (z^*,τ^*,λ^*) is an efficient solution for the (MWDMMPEC).

Theorem 7. *(Strong Duality) Let z^* be an efficient solution for the (MMPEC) and assume that z^* satisfies the GGCQ for $P_k(\varepsilon)$, for at least one $k = 1,...,l$; then there exist $\tau^* \in R^l$ and $\lambda^* \in R^{p+q+2m}$, such that (z^*,τ^*,λ^*) is feasible for the (MWDMMPEC) and if also weak duality Theorem 6 holds between the (MMPEC) and (MWDMMPEC), then (z^*,τ^*,λ^*) is an efficient solution for the (MWDMMPEC).*

Proof. Since z^* is an efficient solution for the (MMPEC), then from Lemma 1, z^* solves $P_k(\varepsilon)$, $\forall\ k = 1,...,l$. Then, by hypothesis there exist $k = 1,...,l$ for which z^* satisfies GGCQ for $P_k(\varepsilon)$, then from Theorem 1 there exist $\tau_i > 0$ for $i \neq k$ and $\lambda \in R^{p+q+2m}$, such that

$$f_k\ (z^*) + \sum_{i \neq k} \tau_i \nabla f_i(z^*) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*)$$
$$+ \sum_{i=1}^{q} \lambda_i^h \nabla h_i(z^*) - \sum_{i=1}^{m} [\lambda_i^G \nabla G_i(z^*) + \lambda_i^H \nabla H_i(z^*)] = 0,$$
$$g_i(z^*) \leqq 0\ , \lambda_i^g \geqq 0,\quad \lambda_i^g g_i(z^*) = 0,\ \forall\ i = 1,...,p,$$
$$h_i(z^*) = 0,\ \forall i = 1,...,q,$$

λ_i^G free $i \in \alpha$, $\lambda_i^G \geqq 0$, $i \in \beta$, $\lambda_i^G = 0$, $i \in \gamma$,
λ_i^H free $i \in \gamma$, $\lambda_i^H \geqq 0$, $i \in \beta$, $\lambda_i^H = 0$, $i \in \alpha$.

Dividing above equations by $1+\sum_{i \neq k} \tau_i$ and setting $\tau_k^* = \dfrac{1}{1+\sum_{i \neq k} \tau_i} > 0$ and $\tau_j^* = \dfrac{1}{1+\sum_{i \neq k} \tau_i} > 0$, $\forall\ j \neq k$, we get (z^*,τ^*,λ^*) is a feasible solution for the (MWDMMPEC) and from Corollary 2, we get result. □

6 Applications

A special case of the multiobjective mathematical programming problem with equilibrium constraints (MMPEC) studied in the paper, is the following multiobjective optimization problem with equilibrium constraints:

$$(MOPEC) \qquad \min\ \ (f_1(z),...,f_n(z))$$
$$\text{subject to}\quad z \geqq 0,\ H(z) \geqq 0,$$
$$z^T H(z) = 0.$$

Siddiqui and Christensen [25] used (MOPEC) to model energy markets and climate policy along with related tradeoff. In (MOPEC), the objective functions are social welfare, greenhouse gas emission (GHG), producer profit, $G(z) = (G_1(z),...,G_m(z))$ as taxes, caps and other climate policy instruments, consumption, price of energy markets, and $H(z) = (H_1(z),...,H_m(z))$ as constraints for the energy and climate policies and markets.

7 Conclusions

We defined the GGCQ for the (MMPEC). We have derived the KKT type necessary optimality condition using GGCQ and sufficient optimality conditions for the (MMPEC) using generalized convexity. We have formulated Wolfe type dual and Mond-Weir type dual models and established weak and strong duality conditions for the (MMPEC) using generalized convexity.

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Kunwar V. K. Singh obtained M.Sc. degree from Department of Mathematics, Institute of Science, Banaras Hindu University (BHU), India in 2011. Presently, he is a junior Research Fellow of council of Scientific and Industrial Research New Delhi, India, and working for Ph.D. at the Banaras Hindu University, varanasi, India. He is working on Mathematical Programming Problems with Equilibrium Constraints(MPEC) and Vanishing Constraints (MPVC).

S. K. Mishra is a professor of Department of Mathematics, Institute of Science at Banaras Hindu University (BHU), India. He obtained M.Sc. degree from Concordia University, Montreal, Canada in 1990 and Ph.D. from IIT (BHU), India in 1995. He has been awarded Young Scientists Award by DST, Government of India in 2001-02. He obtained D.Sc. degree in 2002 from Dr. BRA University, Agra, India. He has been Research Fellow in Department of Management Science, City University of Hong Kong in 2005-2007. His main research interest is the field of Generalized Convexity, Variational Inequalities, Vector Optimization, Nonsmooth optimization and Mathematical Programs with Equilibrium Constraints. S. K. Mishra has authored/ edited six books/text books published in Springer and CRC press and more than 170 research articles in SCI/peer-reviewed journals. He has guest edited special issues of the Journal of Global Optimization, Optimization Letters and Optimization. He is currently member of the Editorial Board of several reputed international/ national journal in the field of Optimization/Operations Research. He has supervised 12 Ph.D. students. He has organized several international workshops/conferences.