Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/080203

On Recurrence Relations for the GS-Distribution of Lower Generalized Order Statistics

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Received: 31 Jul. 2020, Revised: 9 Nov. 2020, Accepted: 11 Nov. 2020
Published online: 1 May 2021

Abstract: We establish recurrence relations for the single and product moments from the generalized S-distribution (GS-distribution) of lower generalized order statistics. Characterizations for the GS-distribution are derived. The GS-distribution is a family of distributions for continuous unimodal variables which includes distributions such as Uniform, Exponential, generalized exponential, Logistic and beta distributions, among others. Uniform and Exponential distributions are given as applications to illustrate the results.

Keywords: Generalized order statistics, GS-distribution, Ordinary order statistics, Record values, Moments, Characterizations.

1 Introduction

The density function of GS-distribution is given by

$$f(x) = \alpha F^{\lambda}(1-F^{\theta})^{\beta}, \qquad \alpha,\lambda,\theta,\beta > 0, \qquad a < x < b, \tag{1}$$

since β is positive integers, then by using binomial theorem, we get

$$(1-F^{\theta})^{\beta} = \sum_{\nu=0}^{\beta}\binom{\beta}{\nu}(-1)^{\nu}F^{\theta\nu} = \sum_{\nu=0}^{\beta}\eta_{\nu}F^{\theta\nu}, \tag{2}$$

where

$$\eta_{\nu} = \binom{\beta}{\nu}(-1)^{\nu}. \tag{3}$$

Thus

$$f(x) = \alpha\sum_{\nu=0}^{\beta}\eta_{\nu}F^{\theta\nu+\lambda}. \tag{4}$$

The random variables $X'(r,n,\overline{m},k)$, $1 \le r \le n$, are said to be lower generalized order statistics(lgos) if their joint density function is given by

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = k\left(\prod_{i=1}^{n-1}\gamma_i\right)\left(\prod_{i=1}^{n-1}(F(x_i))^{m_i}f(x_i)\right)(F(x_n))^{k-1}f(x_n), \tag{5}$$

$F^{-1}(1) > x_1 > x_2 > \ldots > x_n > F^{-1}(0)$, $n \in N$, $\overline{m} = (m_1,\ldots,m_{n-1}) \in \Re^{n-1}$, $k \ge 1$, $\gamma_i = k+n-i+M_i > 0$, and $M_i = \sum_{j=i}^{n-1} m_j$ for all $1 \le i \le n-1$.

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The pdf of X'_r is given by

$$f_{X'_r}(x) = C_{r-1} \sum_{i=1}^{r} a_i(r) \left(F(x)\right)^{\gamma_i - 1} f(x), \quad 1 \leq r \leq n, \tag{6}$$

and the joint pdf of X'_r and X'_s is

$$f_{X'_r, X'_s}(x,y) = C_{s-1} \left[\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \right] \left[\sum_{i=1}^{r} a_i(r) \left(F(x)\right)^{\gamma_i} \right] \frac{f(x) f(y)}{F(x) F(y)}, \tag{7}$$

where $x > y$, $1 \leq r < s \leq n$, $X'_i \equiv X'(i, n, \overline{m}, k)$, $1 \leq i \leq n$,

$$C_{\ell-1} = \prod_{i=1}^{\ell} \gamma_i, \quad a_i^{(r)}(s) = \prod_{j=r+1, j \neq i}^{s} \frac{1}{\gamma_j - \gamma_i} \quad \text{and} \quad a_i(s) = a_i^{(0)}(s).$$

The conditional pdf of X'_s given X'_r, $1 \leq r < s \leq n$, is given by

$$f_{X'_s | X'_r}(y|x) = \frac{C_{s-1}}{C_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{F(y)}{F(x)} \right)^{\gamma_i} \frac{f(y)}{F(y)}, \quad y < x. \tag{8}$$

Aboutahoun and Al-otaibi. [1] presented recurrence relations between the single and the product moments for order statistics from doubly truncated Makeham distribution and its characterizations. Al-Hussaini et al. [2] obtained recurrence relations for moment and conditional moment generating functions of gos based on member of a doubly truncated class of distributions and characterize members of the class based on recurrence relations for conditional moment generating functions of gos. Athar and Islam. [3] established recurrence relations between the single and the product moments for gos based on modified Burr XII-geometric distribution and its characterizations. Bieniek. [4] characterize distributions by using linearity of regression of gos. Burkschat et al. [5] defined concept of gos. El-Baset. [6] established recurrence relations between the single and the product moments for gos based on a general class of doubly truncated Marshall-Olkin extended distributions and its characterizations. El-Baset and Mohammad. [7] discussed recurrence relations for single moments of generalized order statistics from doubly truncated distributions. Kamps [9] and Khan and Kumar [10] defined concept of lgos. Khan and Kumar et al. [10] established explicit expressions and some recurrence relations for single and product moments of lgos from exponentiated Pareto distribution. Mohie El-Din and Kotb. [11,12] established recurrence relations between the single and the product moments for gos based on a general class of distribution and doubly truncated Burr Type XII Distribution and its characterizations. Mohie El-Din and Kotb. [13] derived recurrence relations for quotient moments of gos. Muino et al. [14] derived GS-distribution. Samuel. [15] introduced general forms of many well-known continuous probability distributions which are characterized by conditional expectation of some functions of gos. Saran and Pandey. [16] established give some recurrence relations satisfied by marginal and joint moment generating functions of gos from power function distribution.

2 Recurrence Relation for Single Moments

Theorem 1. Let X be a r.v. has a distribution function $F(x)$ defined on (a,b). For any measurable function $\psi(x)$, the recurrence relation for the single moments of lgos is

$$\begin{aligned} E\left[\psi'(X'(r,n,\overline{m},k))\right] &= \alpha \sum_{\nu=0}^{\beta} \eta_\nu \gamma_r \grave{C}_{r-2} E\left[\psi(X'(r-1,n-1,\widetilde{m},\grave{k}))\right] \\ &- \alpha \sum_{\nu=0}^{\beta} \eta_\nu \gamma_r \grave{C}_{r-1} E\left[\psi(X'(r,n-1,\widetilde{m},\grave{k}))\right] \\ &- \alpha \sum_{\nu=0}^{\beta} \eta_\nu (\theta\nu + \lambda - 1) \grave{C}_{r-1} E\left[\psi(X'(r,n-1,\widetilde{m},\grave{k}))\right], \end{aligned} \tag{9}$$

where $\grave{k} = k + m_{n-1} + \theta\nu + \lambda$, $\grave{C}_{r-2} = \frac{C_{r-2}}{C^{\star}_{r-2}}$, $C^{\star}_{r-1} = \prod_{i=1}^{r}(\gamma_i + \theta\nu + \lambda - 1)$, $' \equiv \frac{d}{dx}$ and $\widetilde{m} = (m_1, \ldots, m_{n-2}) \in \Re^{n-2}$.

Proof: From Eqs. (4) and (6), we have

$$E\left[\psi^{'}(X'(r,n,\overline{m},k))\right] = C_{r-1}\sum_{i=1}^{r}a_i(r)\int_a^b \psi'(x)\,(F(x))^{\gamma_i-1}f(x)dx$$
$$= \alpha C_{r-1}\sum_{\nu=0}^{\beta}\eta_\nu\sum_{i=1}^{r}a_i(r)\int_a^b \psi'(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-1}dx. \tag{10}$$

Integrating by parts, we get

$$\begin{aligned}E\left[\psi^{'}(X'(r,n,\overline{m},k))\right] &= -\alpha C_{r-1}\sum_{\nu=0}^{\beta}\eta_\nu\sum_{i=1}^{r}a_i(r)(\gamma_i+\theta\nu+\lambda-1)\\ &\times\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\ &= -\alpha C_{r-1}\sum_{\nu=0}^{\beta}\eta_\nu\sum_{i=1}^{r}a_i(r)\gamma_i\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\ &-\alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)C_{r-1}\sum_{i=1}^{r}a_i(r)\\ &\times\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx.\end{aligned} \tag{11}$$

Using $a_i(r-1)=(\gamma_r-\gamma_i)a_i(r)$ and $\gamma_r C_{r-2}=C_{r-1}$, we get

$$\begin{aligned}E\left[\psi^{'}(X'(r,n,\overline{m},k))\right] &= \alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_r C_{r-2}\sum_{i=1}^{r-1}a_i(r-1)\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\ &-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_r C_{r-1}\sum_{i=1}^{r}a_i(r)\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\ &-\alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)C_{r-1}\sum_{i=1}^{r}a_i(r)\\ &\times\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx,\end{aligned} \tag{12}$$

thus

$$\begin{aligned}E\left[\psi^{'}(X'(r,n,\overline{m},k))\right] &= \alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_r \grave{C}_{r-2}C^{\star}_{r-2}\sum_{i=1}^{r-1}a_i(r-1)\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\ &-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_r \grave{C}_{r-1}C^{\star}_{r-1}\sum_{i=1}^{r}a_i(r)\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\ &-\alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)\grave{C}_{r-1}C^{\star}_{r-1}\sum_{i=1}^{r}a_i(r)\\ &\times\int_a^b \psi(x)\,(F(x))^{\gamma_i+\theta\nu+\lambda-2}f(x)dx.\end{aligned} \tag{13}$$

After simplifying, Eq. (9) is proved.

Remark 1. Put $\psi(x)=x^{\ell}$ in Eq. (9) satisfy the ℓ-th moment,

$$\begin{aligned}\mu'^{(\ell-1)}(r,n,\overline{m},k) = \frac{\alpha}{\ell}\sum_{\nu=0}^{\beta}\eta_\nu[\gamma_r \grave{C}_{r-2}\mu'^{(\ell)}(r-1,n-1,\widetilde{m},\grave{k})\\ -\gamma_r \grave{C}_{r-1}\mu'^{(\ell)}(r,n-1,\widetilde{m},\grave{k})\\ -(\theta\nu+\lambda-1)\grave{C}_{r-1}\mu'^{(\ell)}(r,n-1,\widetilde{m},\grave{k})].\end{aligned} \tag{14}$$

Similarly, put $\psi(x)=\exp(tx)$ in Eq. (9) satisfy the moment generating function.

3 Recurrence Relation For Double Moments

Theorem 2. Let X be a r.v. has a distribution function $F(x)$ defined on (a,b). For $r \leq s < n$ and any measurable function $\psi(x,y)$, the recurrence relation for the product moments of lgos is

$$\begin{aligned} E\left[\psi^{'}(X'(r,s,n,\overline{m},k))\right] &= \alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_s\grave{C}_{s-2}E\left[\psi(X'(r,s-1,n-1,\widetilde{m},\grave{k}))\right] \\ &- \alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_s\grave{C}_{s-1}E\left[\psi(X'(r,s,n-1,\widetilde{m},\grave{k}))\right] \\ &- \alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)\grave{C}_{s-1}E\left[\psi(X'(r,s,n-1,\widetilde{m},\grave{k}))\right], \end{aligned} \tag{15}$$

where $' \equiv \frac{d}{dy}$.

Proof: From Eqs. (4) and (7), we have

$$\begin{aligned} E\left[\psi^{'}(X'(r,s,n,\overline{m},k))\right] &= C_{s-1}\int_a^b\int_a^x\psi'(x,y)\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}\right] \\ &\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{\gamma_i}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx \\ &= \alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{s-1}\int_a^b\int_a^x\psi'(x,y)\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right] \\ &\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)}{F(x)}dydx. \end{aligned} \tag{16}$$

Integrating by parts , we get

$$\begin{aligned} E\left[\psi^{'}(X'(r,s,n,\overline{m},k))\right] &= -\alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{s-1}\int_a^b\int_a^x\psi(x,y) \\ &\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)(\gamma_i+\theta\nu+\lambda-1)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right] \\ &\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx \\ &= -\alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{s-1}\int_a^b\int_a^x\psi(x,y) \\ &\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\gamma_i\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right] \\ &\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx \\ &- \alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)C_{s-1}\int_a^b\int_a^x\psi(x,y) \\ &\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right] \\ &\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx. \end{aligned} \tag{17}$$

Using $a_i^{(r)}(s-1)=(\gamma_s-\gamma_i)a_i^{(r)}(s)$ and $\gamma_s C_{s-2}=C_{s-1}$, we get

$$\begin{aligned}
E\left[\psi^{'}(X'(r,s,n,\overline{m},k))\right] &= \alpha\sum_{v=0}^{\beta}\eta_v\gamma_s C_{s-2}\int_a^b\int_a^x\psi(x,y) \\
&\times\left[\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta v+\lambda-1)}\right] \\
&\times\left[\sum_{i=1}^{r}a_i(r)\,(F(x))^{(\gamma_i+\theta v+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx \\
&-\alpha\sum_{v=0}^{\beta}\eta_v\gamma_s C_{s-1}\int_a^b\int_a^x\psi(x,y) \\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta v+\lambda-1)}\right] \\
&\times\left[\sum_{i=1}^{r}a_i(r)\,(F(x))^{(\gamma_i+\theta v+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx \\
&-\alpha\sum_{v=0}^{\beta}\eta_v(\theta v+\lambda-1)C_{s-1}\int_a^b\int_a^x\psi(x,y) \\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta v+\lambda-1)}\right] \\
&\times\left[\sum_{i=1}^{r}a_i(r)\,(F(x))^{(\gamma_i+\theta v+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx,
\end{aligned}\tag{18}$$

which reduce to

$$\begin{aligned}
E\left[\psi^{'}(X'(r,s,n,\overline{m},k))\right] &= \alpha\sum_{v=0}^{\beta}\eta_v\gamma_s \grave{C}_{r-2}C^{\star}_{r-2}\int_a^b\int_a^x\psi(x,y) \\
&\times\left[\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta v+\lambda-1)}\right] \\
&\times\left[\sum_{i=1}^{r}a_i(r)\,(F(x))^{(\gamma_i+\theta v+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx \\
&-\alpha\sum_{v=0}^{\beta}\eta_v\gamma_s \grave{C}_{r-1}C^{\star}_{r-1}\int_a^b\int_a^x\psi(x,y) \\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta v+\lambda-1)}\right] \\
&\times\left[\sum_{i=1}^{r}a_i(r)\,(F(x))^{(\gamma_i+\theta v+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx \\
&-\alpha\sum_{v=0}^{\beta}\eta_v(\theta v+\lambda-1)\grave{C}_{r-1}C^{\star}_{r-1}\int_a^b\int_a^x\psi(x,y) \\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta v+\lambda-1)}\right] \\
&\times\left[\sum_{i=1}^{r}a_i(r)\,(F(x))^{(\gamma_i+\theta v+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx.
\end{aligned}\tag{19}$$

After simplifying, Eq. (15) is proved.

Remark 2. Put $\psi(x,y)=x^{\ell}y^{\tau}$ in Eq. (15) satisfy the ℓ-th moment,

$$\mu'^{(\ell,\tau-1)}(r,s,n,\overline{m},k)=\frac{\alpha}{\tau}\sum_{\nu=0}^{\beta}\eta_{\nu}[\gamma_s C^{`}_{s-2}\mu'^{(\ell,\tau)}(r,s-1,n-1,\widetilde{m},\grave{k})$$
$$-\gamma_s C^{`}_{s-1}\mu'^{(\ell,\tau)}(r,s,n-1,\widetilde{m},\grave{k})$$
$$-(\theta\nu+\lambda-1)C^{`}_{s-1}\mu'^{(\ell,\tau)}(r,s,n-1,\widetilde{m},\grave{k})]. \tag{20}$$

Similarly, put $\psi(x,y)=\exp(tx+ty)$ in Eq. (15) satisfy the moment generating function.

4 Characterization of lgOSs

Theorem 1. For any arbitrary continuous distribution function and for any continuous function $\phi(y)$, then

$$E(\phi^{\iota}(X'_{s,n,\tilde{m},k})|X'_{r,n,\tilde{m},k}=x)=\phi^{\iota}(x)+\iota\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}\frac{\Psi_{i,\iota}(x)}{(F(x))^{\gamma_i}}, \tag{21}$$

where $\iota=1,2,\ldots$ and

$$\Psi_{i,\iota}(x)=\int_{-\infty}^{x}\phi'(y)\phi^{\iota-1}(y)(F(y))^{\gamma_i}dy. \tag{22}$$

Proof:let

$$E(\phi^{\iota}(X'_s)|X'_r=x)=E(\phi^{\iota}(X'_{s,n,\tilde{m},k})|X'_{r,n,\tilde{m},k}=x)$$
$$=\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_{-\infty}^{x}\phi^{\iota}(y)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}\frac{f(y)}{F(y)}dy$$
$$=\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_{-\infty}^{x}\phi^{\iota}(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i-1)}\frac{f(y)}{F(x)}dy. \tag{23}$$

Integrating by parts, we get

$$E(\phi^{\iota}(X'_s)|X'_r=x)=\phi^{\iota}(x)\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}$$
$$-\iota\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}\int_{-\infty}^{x}\phi'(y)\phi^{\iota-1}(y)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}dy. \tag{24}$$

From the pdf of Eq. (6) and Eq. (7), respectively, we get

$$\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i}=\frac{1}{C_{r-1}}, \tag{25}$$

and

$$\left(\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}\right)\left(\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i}\right)=\frac{1}{C_{s-1}}. \tag{26}$$

Thus

$$\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i}=\frac{C_{r-1}}{C_{s-1}}. \tag{27}$$

From Eq. (27), then Eq. (21) is proved.
We shall require the following definition (see Hwang. [8]):
Definition 2: (Hwang. [8]). Let $L_{\delta}(A)$ be the space of δ-integrable functions on a measurable set $A\subset R$. A sequence $(f_n)_{n\in N}$ of functions in $L_{\delta}(A)$ is called complete on $L_{\delta}(A)$, if for all functions $g\in L_{\delta}(A)$ the condition

$\int_A g(x)f_n(x)dx = 0$ for all $n \in N$ implies: $g(x) = 0$ a.e on A.

Theorem 3. Let X be a r.v. has a distribution function $F(y)$ defined on (a,b). For any measurable function $\psi(y)$, the conditional expectation of lgos is given by

$$
\begin{aligned}
& E\left[\psi^{'}(X'(s,n,\overline{m},k))|X'(r,n,\overline{m},k)=x\right] = \\
& \alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\gamma_s\grave{C}_{r-1,s-2}E\left[\psi(X'(s-1,n,\overline{m},\grave{k}))|X'(r,n,\overline{m},k)\right] \\
& -\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\gamma_s\grave{C}_{r-1,s-1}E\left[\psi(X'(s,n,\overline{m},\grave{k}))|X'(r,n,\overline{m},k)\right] \\
& -\alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)F(x)^{(\theta\nu+\lambda-1)}\grave{C}_{r-1,s-1}E\left[\psi(X'(s,n,\overline{m},\grave{k}))|X'(r,n,\overline{m},k)\right],
\end{aligned}
\tag{28}
$$

where $\grave{C}_{r-1,s-2} = \frac{\grave{C}_{s-2}}{\grave{C}_{r-1}}$ and $\grave{C}_{r-1,s-1} = \frac{\grave{C}_{s-1}}{\grave{C}_{r-1}}$.

Proof: From Eqs. (4) and (8), we have

$$
\begin{aligned}
& E\left[\psi^{'}(X'(s,n,\overline{m},k))|X'(r,n,\overline{m},k)=x\right] \\
& = \alpha\sum_{\nu=0}^{\beta}\eta_\nu\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_a^x\psi'(y)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}F(y)^{(\theta\nu+\lambda-1)}dy \\
& = \alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_a^x\psi'(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}dy.
\end{aligned}
\tag{29}
$$

Integrating by parts, we get

$$
\begin{aligned}
& E\left[\psi^{'}(X'(s,n,\overline{m},k))|X'(r,n,\overline{m},k)=x\right] \\
& = -\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)(\gamma_i+\theta\nu+\lambda-1) \\
& \times\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy \\
& = -\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\gamma_i\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy \\
& -\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)(\theta\nu+\lambda-1) \\
& \times\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy.
\end{aligned}
\tag{30}
$$

Using $a_i^{(r)}(s-1)=(\gamma_s-\gamma_i)a_i^{(r)}(s)$ and $\gamma_s C_{s-2}=C_{s-1}$, we get

$$\begin{aligned}
&E\left[\psi'(X'(s,n,\overline{m},k))|X'(r,n,\overline{m},k)=x\right]\\
&=\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\gamma_s\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}(\theta\nu+\lambda-1)\frac{C_{s-1}}{C_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\\
&\times\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy, \qquad (31)
\end{aligned}$$

then

$$\begin{aligned}
&E\left[\psi'(X'(s,n,\overline{m},k))|X'(r,n,\overline{m},k)=x\right]\\
&=\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\gamma_s\grave{C}_{r-1,s-2}\frac{C^\star_{s-2}}{C^\star_{r-1}}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}\gamma_s\grave{C}_{r-1,s-1}\frac{C^\star_{s-1}}{C^\star_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu F(x)^{(\theta\nu+\lambda-1)}(\theta\nu+\lambda-1)\grave{C}_{r-1,s-1}\frac{C^\star_{s-1}}{C^\star_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\\
&\times\int_a^x\psi(y)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\frac{f(y)}{F(y)}dy.
\end{aligned}$$

After simplifying, Eq. (28) is proved.

Theorem 4. For $1\le r\le n-1$ and for any measurable function $\psi(x)$, a random variable X has a distribution function with cdf given by Eq. (4) *iff* Eq. (9) holds.

Proof: The necessary part is proved in Theorem (1). To prove the sufficient part, let Eq.(9) holds, then we obtain

$$\begin{aligned}
E\left[\psi'(X'(r,n,\overline{m},k))\right]&=\alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{r-1}\sum_{i=1}^{r-1}a_i(r-1)\int_a^b\psi(x)\left(F(x)\right)^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_r C_{r-1}\sum_{i=1}^{r}a_i(r)\int_a^b\psi(x)\left(F(x)\right)^{\gamma_i+\theta\nu+\lambda-2}f(x)dx\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)C_{r-1}\sum_{i=1}^{r}a_i(r)\\
&\times\int_a^b\psi(x)\left(F(x)\right)^{\gamma_i+\theta\nu+\lambda-2}f(x)dx \qquad (32)
\end{aligned}$$

Since $a_i(r-1)=(\gamma_r-\gamma_i)a_i(r)$, then we obtain

$$
\begin{aligned}
E\left[\psi^{'}\left(X'(r,n,\overline{m},k)\right)\right] &= -\alpha C_{r-1}\sum_{\nu=0}^{\beta}\eta_\nu\sum_{i=1}^{r}a_i(r)\gamma_i\int_a^b\psi(x)\left(F(x)\right)^{\gamma_i+\theta\nu+\lambda-2}f(x)dx \\
&- \alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)C_{r-1}\sum_{i=1}^{r}a_i(r) \\
&\times\int_a^b\psi(x)\left(F(x)\right)^{\gamma_i+\theta\nu+\lambda-2}f(x)dx \\
&= -\alpha C_{r-1}\sum_{\nu=0}^{\beta}\eta_\nu\sum_{i=1}^{r}a_i(r)(\gamma_i+\theta\nu+\lambda-1) \\
&\times\int_a^b\psi(x)\left(F(x)\right)^{\gamma_i+\theta\nu+\lambda-2}f(x)dx. \qquad (33)
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
E\left[\psi^{'}\left(X'(r,n,\overline{m},k)\right)\right] &= C_{r-1}\sum_{i=1}^{r}a_i(r)\int_a^b\psi'(x)\left(F(x)\right)^{\gamma_i-1}f(x)dx \\
&= \alpha C_{r-1}\sum_{\nu=0}^{\beta}\eta_\nu\sum_{i=1}^{r}a_i(r)\int_a^b\psi'(x)\left(F(x)\right)^{\gamma_i+\theta\nu+\lambda-1}dx. \qquad (34)
\end{aligned}
$$

After simplifying we obtain

$$
C_{r-1}\sum_{i=1}^{r}a_i(r)\left[\int_a^b\psi'(x)(F(x))^{\gamma_i-1}\left(f(x)-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\left(F(x)\right)^{\theta\nu+\lambda}\right)\right]=0. \qquad (35)
$$

Using Definition 1, we get

$$
\left(f(x)-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\left(F(x)\right)^{\theta\nu+\lambda}\right)=0,\Rightarrow\quad f(x)=\alpha\sum_{\nu=0}^{\beta}\eta_\nu\left(F(x)\right)^{\theta\nu+\lambda}.
$$

Thus, the theorem is proved.

Theorem 5. For $1\le r<s\le n-1$ and for any measurable function $\psi(x,y)$, a random variable X has a distribution function with cdf given by Eq. (4) *iff* Eq. (15) holds.

Proof: The necessary part is proved in Theorem (2). To prove the sufficient part, let Eq. (15) holds, then we obtain

$$\begin{aligned}
E\left[\psi'(X'(r,s,n,\overline{m},k))\right] &= \alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{s-1}\int_a^b\int_a^x \psi(x,y)\\
&\times\left[\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\\
&\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\gamma_s C_{s-1}\int_a^b\int_a^x \psi(x,y)\\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\\
&\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)C_{s-1}\int_a^b\int_a^x \psi(x,y)\\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\\
&\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx. \qquad (36)
\end{aligned}$$

Using $a_i^{(r)}(s-1)=(\gamma_s-\gamma_i)a_i^{(r)}(s)$, we get

$$\begin{aligned}
E\left[\psi'(X'(r,s,n,\overline{m},k))\right] &= -\alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{s-1}\int_a^b\int_a^x \psi(x,y)\\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\gamma_i\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\\
&\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx\\
&-\alpha\sum_{\nu=0}^{\beta}\eta_\nu(\theta\nu+\lambda-1)C_{s-1}\int_a^b\int_a^x \psi(x,y)\\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\\
&\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx\\
&= -\alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{s-1}\int_a^b\int_a^x \psi(x,y)\\
&\times\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)(\gamma_i+\theta\nu+\lambda-1)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\\
&\times\left[\sum_{i=1}^{r}a_i(r)\left(F(x)\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx. \qquad (37)
\end{aligned}$$

Integrating by parts, we get

$$E\left[\psi^{'}(X'(r,s,n,\overline{m},k))\right] = C_{s-1}\int_a^b\int_a^x \psi'(x,y)\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}\right]$$
$$\times\left[\sum_{i=1}^{r} a_i(r)\,(F(x))^{\gamma_i}\right]\frac{f(x)f(y)}{F(x)F(y)}dydx$$
$$= \alpha\sum_{\nu=0}^{\beta}\eta_\nu C_{s-1}\int_a^b\int_a^x \psi'(x,y)\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{(\gamma_i+\theta\nu+\lambda-1)}\right]$$
$$\times\left[\sum_{i=1}^{r} a_i(r)\,(F(x))^{(\gamma_i+\theta\nu+\lambda-1)}\right]\frac{f(x)}{F(x)}dydx. \tag{38}$$

After simplifying we obtain

$$\int_a^b\int_a^x \psi'(x,y)\frac{f(x)}{F(x)}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i-1}\right]$$
$$\times\left[\sum_{i=1}^{r} a_i(r)\,(F(x))^{\gamma_i-1}\right]\left(f(y)-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\,(F(y))^{\theta\nu+\lambda}\right)dydx = 0. \tag{39}$$

By using Definition 1, Eq. (39) reduce to

$$f(y)-\alpha\sum_{\nu=0}^{\beta}\eta_\nu\,(F(y))^{\theta\nu+\lambda} = 0, \Rightarrow f(y) = \alpha\sum_{\nu=0}^{\beta}\eta_\nu\,(F(y))^{\theta\nu+\lambda}.$$

Thus, the theorem is proved.

5 Applications

1.Uniform(a,b)

We have $\alpha=\frac{1}{b-a}$, $\lambda=0$, $\theta=1$, $\beta=0$, $C^{\star}_{r-1}=\prod_{i=1}^{r}(\gamma_i+\nu-1)$ and $\grave{k}=k+m_{n-1}+\nu$. Then the relations (14) and (20) reduce, respectively, to

$$\mu'^{(\ell-1)}(r,n,\overline{m},k) = \frac{1}{\ell(b-a)}\sum_{\nu=0}^{\beta}\eta_\nu[\gamma_r \grave{C}_{r-2}\mu'^{(\ell)}(r-1,n-1,\widetilde{m},\grave{k})$$
$$-\ \gamma_r \grave{C}_{r-1}\mu'^{(\ell)}(r,n-1,\widetilde{m},\grave{k})$$
$$-\ (\nu-1)\grave{C}_{r-1}\mu'^{(\ell)}(r,n-1,\widetilde{m},\grave{k})], \tag{40}$$

and

$$\mu'^{(\ell,\tau-1)}(r,s,n,\overline{m},k) = \frac{1}{\tau(b-a)}\sum_{\nu=0}^{\beta}\eta_\nu[\gamma_s \grave{C}_{s-2}\mu'^{(\ell,\tau)}(r,s-1,n-1,\widetilde{m},\grave{k})$$
$$-\ \gamma_s \grave{C}_{s-1}\mu'^{(\ell,\tau)}(r,s,n-1,\widetilde{m},\grave{k})$$
$$-\ (\nu-1)\grave{C}_{s-1}\mu'^{(\ell,\tau)}(r,s,n-1,\widetilde{m},\grave{k})]. \tag{41}$$

In the case of oOSs ($m_1=...=m_{n-1}=0$, $k=1$, $C^{\star}_{r-1}=\prod_{i=1}^{r}(n-i+\nu)$ and $\gamma_r=n-r+1$), the relations (40) and (41) reduce, respectively, to

$$\mu'^{(\ell-1)}_{r:n} = \frac{1}{\ell(b-a)}\sum_{\nu=0}^{\beta}\eta_\nu[(n-r+1)\grave{C}_{r-2}\mu'^{(\ell)}_{r-1:n+\nu}-(n-r+1)\grave{C}_{r-1}\mu'^{(\ell)}_{r:n+\nu}$$
$$-\ (\nu-1)\grave{C}_{r-1}\mu'^{(\ell)}_{r:n+\nu}],$$

and

$$\mu'^{(\ell,\tau-1)}_{r,s:n} = \frac{1}{\tau(b-a)} \sum_{\nu=0}^{\beta} \eta_\nu [(n-s+1) \grave{C}_{s-2} \mu'^{(\ell,\tau)}_{r,s-1:n+\nu} - (n-s+1) \grave{C}_{s-1} \mu'^{(\ell,\tau)}_{r,s:n+\nu} - (\nu-1) \grave{C}_{s-1} \mu'^{(\ell,\tau)}_{r,s:n+\nu}].$$

In the case of k-th record values ($m=-1$, $k \geq 2-\nu$, $\gamma_r = k$ and $\grave{C}_{r-1} = (\frac{k}{k+\nu-1})^r$), the relations (40) and (41) reduce, respectively, to

$$\mu'^{(\ell-1)}_{r:k} = \frac{1}{\ell(b-a)} \sum_{\nu=0}^{\beta} \eta_\nu [k \left(\frac{k}{k+\nu-1}\right)^{r-1} \mu'^{(\ell)}_{r-1:k+\nu-1} - k \left(\frac{k}{k+\nu-1}\right)^{r} \mu'^{(\ell)}_{r:k+\nu-1} - (\nu-1) \left(\frac{k}{k+\nu-1}\right)^{r} \mu'^{(\ell)}_{r:k+\nu-1}],$$

and

$$\mu'^{(\ell,\tau-1)}_{r,s:k} = \frac{1}{\tau(b-a)} \sum_{\nu=0}^{\beta} \eta_\nu [k \left(\frac{k}{k+\nu-1}\right)^{s-1} \mu'^{(\ell,\tau)}_{r,s-1:k+\nu-1} - k \left(\frac{k}{k+\nu-1}\right)^{s} \mu'^{(\ell,\tau)}_{r,s:k+\nu-1} - (\nu-1) \left(\frac{k}{k+\nu-1}\right)^{s} \mu'^{(\ell,\tau)}_{r,s:k+\nu-1}].$$

2.Exponential(γ)

We have $\alpha = \gamma$, $\lambda = 0$, $\theta = 1$, $\beta = 1$, $C^{\star}_{r-1} = \prod_{i=1}^{r}(\gamma_i + \nu - 1)$ and $\grave{k} = k + m_{n-1} + \nu$. Then the relations (14) and (20) reduce, respectively, to

$$\mu'^{(\ell-1)}(r,n,\overline{m},k) = \frac{\gamma}{\ell} \sum_{\nu=0}^{\beta} \eta_\nu [\gamma_r \grave{C}_{r-2} \mu'^{(\ell)}(r-1,n-1,\widetilde{m},\grave{k}) - \gamma_r \grave{C}_{r-1} \mu'^{(\ell)}(r,n-1,\widetilde{m},\grave{k}) - (\nu-1) \grave{C}_{r-1} \mu'^{(\ell)}(r,n-1,\widetilde{m},\grave{k})], \tag{42}$$

and

$$\mu'^{(\ell,\tau-1)}(r,s,n,\overline{m},k) = \frac{\gamma}{\tau} \sum_{\nu=0}^{\beta} \eta_\nu [\gamma_s \grave{C}_{s-2} \mu'^{(\ell,\tau)}(r,s-1,n-1,\widetilde{m},\grave{k}) - \gamma_s \grave{C}_{s-1} \mu'^{(\ell,\tau)}(r,s,n-1,\widetilde{m},\grave{k}) - (\nu-1) \grave{C}_{s-1} \mu'^{(\ell,\tau)}(r,s,n-1,\widetilde{m},\grave{k})]. \tag{43}$$

In the case of oOSs, the relations (42) and (43) reduce, respectively, to

$$\mu'^{(\ell-1)}_{r:n} = \frac{\gamma}{\ell} \sum_{\nu=0}^{\beta} \eta_\nu [(n-r+1) \grave{C}_{r-2} \mu'^{(\ell)}_{r-1:n+\nu} - (n-r+1) \grave{C}_{r-1} \mu'^{(\ell)}_{r:n+\nu} - (\nu-1) \grave{C}_{r-1} \mu'^{(\ell)}_{r:n+\nu}],$$

and

$$\mu'^{(\ell,\tau-1)}_{r,s:n} = \frac{\gamma}{\tau} \sum_{\nu=0}^{\beta} \eta_\nu [(n-s+1) \grave{C}_{s-2} \mu'^{(\ell,\tau)}_{r,s-1:n+\nu} - (n-s+1) \grave{C}_{s-1} \mu'^{(\ell,\tau)}_{r,s:n+\nu} - (\nu-1) \grave{C}_{s-1} \mu'^{(\ell,\tau)}_{r,s:n+\nu}].$$

In the case of k-th records, the relations (42) and (43), reduce, respectively, to

$$\mu_{r:k}^{\prime(\ell-1)} = \frac{\gamma}{\ell}\sum_{\nu=0}^{\beta}\eta_\nu[k\left(\frac{k}{k+\nu-1}\right)^{r-1}\mu_{r-1:k+\nu-1}^{\prime(\ell)} - k\left(\frac{k}{k+\nu-1}\right)^{r}\mu_{r:k+\nu-1}^{\prime(\ell)}$$
$$-(\nu-1)\left(\frac{k}{k+\nu-1}\right)^{r}\mu_{r:k+\nu-1}^{\prime(\ell)}],$$

and

$$\mu_{r,s:k}^{\prime(\ell,\tau-1)} = \frac{\gamma}{\tau}\sum_{\nu=0}^{\beta}\eta_\nu[k\left(\frac{k}{k+\nu-1}\right)^{s-1}\mu_{r,s-1:k+\nu-1}^{\prime(\ell,\tau)}$$
$$-k\left(\frac{k}{k+\nu-1}\right)^{s}\mu_{r,s:k+\nu-1}^{\prime(\ell,\tau)}$$
$$-(\nu-1)\left(\frac{k}{k+\nu-1}\right)^{s}\mu_{r,s:k+\nu-1}^{\prime(\ell,\tau)}].$$

6 Conclusion

In this paper, we set up recurrence relations for the single and product moments from the generalized S-distribution (GS-distribution) of lower generalized order statistics. Characterizations for the GS-distribution are obtained. We use these relations in many inference applications such as best linear unbiased estimation.

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