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Two Simple Algorithms for Testing the Entanglement Status of an N-qubit Pure Quantum State

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Abstract: An N-qubit pure quantum state is "separable" if and only if it can be factored into N 1-qubit factors. Otherwise, the state is an "entangled" state. In this paper we develop two simple algorithms, one "classical" and the other "quantum", to determine the entanglement status of an N-qubit pure quantum state. First we develop a "classical" algorithm to determine entanglement status which is of the exponential order, $O(2^N)$. We then develop an efficient "quantum" algorithm for doing the same task and it is of the polynomial order, $O(N^2)$. This new "quantum" algorithm makes use of the "quantum" speedup available for finding the "inner product" of two vectors [5]. The new "quantum" algorithm also assumes the availability of sufficiently many copies of the N-qubit pure quantum state to be tested for determining its entanglement status.

Keywords: Entanglement and Separability, Partial Measurement, Inner Product

1 Introduction

In [1] a "classical" algorithm for the factorization of N-qubit pure quantum states was developed. This "factorization algorithm" consists of a systematic procedure for extracting all possible factors of the given multipartite pure quantum state. This algorithm finally expresses the given quantum state as a product of factor states such that these factor states are not further factorisable. The complexity of the factorization algorithm was then obtained. It was shown in [2] that the factorization algorithm requires steps of exponential order to factorize an N-qubit pure quantum state and its "complexity" is $O(N2^N)$. An interesting application of factorization was developed in [3] where it was shown that one can exponentially speedup the "synthesis" of a pure quantum state in the laboratory if the state under consideration has large many factors under the application of the factorization algorithm to that state.

One of the central issues in quantum information theory is to determine whether a given multipartite pure quantum state is separable or entangled [4,7,8,9,10]. By definition an N-qubit pure quantum state under consideration is separable if and only if it can be expressed as the tensor product of N 1-qubit states. Otherwise the state is entangled.

In this paper we first propose a "classical" algorithm for finding the entanglement status of an N-qubit pure quantum state. We show that for an arbitrary N-qubit pure quantum state the order of this algorithm is $O(2^N)$. We then develop a "quantum" algorithm for doing the same task and show that it is exponentially faster than the "classical" one mentioned above and has the order $O(N^2)$. This new "quantum" algorithm makes use of the result [5] that the time required for finding inner product of two post-processed N-dimensional vectors is of the order $O(Log(N))$ on a quantum computer and the well-known Cauchy-Schwarz inequality [6]. Also, this new quantum algorithm assumes the availability of sufficiently many copies of the given N-qubit pure quantum state under test.

Cauchy-Schwarz Inequality: For all vectors $|P\rangle$ and $|Q\rangle$ of inner product space if $||P\rangle| = +\sqrt{\langle P|P\rangle}$, $||Q\rangle| = +\sqrt{\langle Q|Q\rangle}$ and let $\langle P|Q\rangle$ denotes the inner product of $|P\rangle$ and $|Q\rangle$ then $|\langle P|Q\rangle| \leq ||P\rangle|.||Q\rangle|$ and the equality holds if and only if vectors $|P\rangle$ and $|Q\rangle$ are proportional to each other, i.e. if and only if $|P\rangle = \alpha|Q\rangle$ for some $\alpha \in \mathbb{C}$, the underlying field of complex numbers.

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2 Notation and Definitions:

Let $|\psi\rangle$ be an N-qubit pure quantum state expressed in terms of the computational basis:

$$|\psi\rangle = \sum_{i_1,i_2,\dots i_N \in \{0,1\}} a_{i_1 i_2 \dots i_N} |i_1 i_2 \dots i_N\rangle$$

such that

$$\sum_{i_1,i_2,\dots i_N \in \{0,1\}} |a_{i_1 i_2 \dots i_N}|^2 = 1$$

Thus each of $i_1, i_2 \dots, i_N$ takes values in $\{0,1\}$ and each $a_{i_1 i_2 \dots i_N} \in \mathbb{C}$, the field of complex numbers.

Few definitions are in order:

Let A be a $p \times q$ matrix over the field $\mathbb{C}$.

Definition 1. The row of a matrix A is called a non-zero row if all its elements are not equal to zero, i.e. it contains at least one non zero element. Otherwise it is called a zero row or a row of zeros.

Definition 2. Two non-zero rows of A, say $[a_1 \cdots a_p]$ and $[b_1 \cdots b_p]$, are said to be proportional if their non-zero elements correspond i.e. $a_i \neq 0$ if and only if $b_i \neq 0$, $1 \leq i \leq p$ and these elements have *the same* constant ratio i.e. there is a constant $k \neq 0$ such that $a_i / b_i = k$ whenever $a_i \neq 0$, $1 \leq i \leq p$.

3 A "Classical" Algorithm for Finding Entanglement Status of an N-qubit Pure Quantum State:

Algorithm 1:

(This Algorithm is guided by the basic definition that if an N-qubit state is a separable state then it can be factored into a tensor product of N 1-qubit states. Otherwise it is an entangled state.)

Step 1: We proceed with checking the existence of 1-qubit factors to the N-qubit state, $|\psi\rangle$, given above. For this to achieve we express the given state as

$$|\psi\rangle = |0\rangle \sum_{i_2,\dots,i_N \in \{0,1\}} a_{0 i_2 \dots i_N} |i_2 \dots i_N\rangle + |1\rangle \sum_{i_2,\dots,i_N \in \{0,1\}} a_{1 i_2 \dots i_N} |i_2 \dots i_N\rangle.$$

Step 2: We then form the following $2^1 \times 2^{N-1}$ matrix A associated to $|\psi\rangle$, namely,

$$A = \begin{bmatrix} a_{00\cdots 0} & a_{00\cdots 01} & a_{00\cdots 010} & \cdots & a_{01\cdots 11} \\ a_{10\cdots 0} & a_{10\cdots 01} & a_{10\cdots 010} & \cdots & a_{11\cdots 11} \end{bmatrix}.$$

where the first row is made up of the coefficients in the first term of $|\psi\rangle$ given in Step 1 above and the second row is made up of the coefficients in the second term of $|\psi\rangle$ given in Step 1 above.

Step 3: We then examine the rows of the above given $2^1 \times 2^{N-1}$ matrix A associated to $|\psi\rangle$ for their proportionality:

(i) If both the rows of A are made up of zeros then $|\psi\rangle = 0$. We declare that $|\psi\rangle$ is separable and stop.

(ii) If the first row of A is a nonzero row and the second row of A is made up of zeros then

$$|\psi\rangle = |0\rangle \sum_{i_2,\dots,i_N} a_{0 i_2 \dots i_N} |i_2 \dots i_N\rangle = |0\rangle |\psi_0\rangle.$$

Here we go back to Step 1 with $|\psi\rangle = |\psi_0\rangle$.

(iii) If the second row of A is a nonzero row and the first row of A is made up of zeros then

$$|\psi\rangle = |1\rangle \sum_{i_2,\dots,i_N} a_{1 i_2 \dots i_N} |i_2 \dots i_N\rangle = |1\rangle |\psi_1\rangle.$$

Here we go back to Step 1 with $|\psi\rangle = |\psi_1\rangle$.

(iv) If both the rows of A are nonzero rows then the following two cases arise. Either these rows will be proportional (Definition 2) to each other or they will not be proportional.

Case I: Suppose the rows of A are proportional. In this case we will have $a_{1 i_2 \dots i_N} = k(a_{0 i_2 \dots i_N})$ for all $a_{1 i_2 \dots i_N}$, for some number $k \in \mathbb{C}$. Therefore, we can express

$$|\psi\rangle = (|0\rangle + k|1\rangle)|\psi_2\rangle$$

where

$$|\psi_2\rangle = \sum_{i_2,\dots,i_N} a_{0 i_2 \dots i_N} |i_2 \dots i_N\rangle.$$

In this case we now go back to Step 1 with $|\psi\rangle = |\psi_2\rangle$.

Case II: Suppose the rows of A are not proportional. Then there will not exist some unique number $k \in \mathbb{C}$ such that $a_{1 i_2 \dots i_N} = k(a_{0 i_2 \dots i_N})$ for all $a_{1 i_2 \dots i_N}$. Therefore, in this case there will not exist a 1-qubit factor $(|0\rangle + k|1\rangle)$ as in Case I above on the left side to $|\psi\rangle$. In this case We declare that $|\psi\rangle$ is an entangled state and stop.

Remark 1: Algorithm 1 will either factorize the given N-qubit state, $|\psi\rangle$, into a tensor product of N 1-qubit states when this state is separable or it will terminate somewhere in between when the given N-qubit state, $|\psi\rangle$, is not factorisable into tensor product of N 1-qubit states and so is an entangled state.

Remark 2: It is easy to check that the rows of the above matrix A contain $2^{(N-1)}$ elements and all of them could be nonzero. Therefore, to check for the existence of a 1-qubit factor we may in the worst case require to evaluate $2^{(N-1)}$ ratios and to check whether they are all identical. Thus, we requires $2^{(N-1)}$ arithmetical operations to check for the existence of a 1-qubit factor. Thus, in order to find the first 1-qubit factor one requires $2^{(N-1)}$ arithmetical operations. To find the second 1-qubit factor one requires $2^{(N-2)}$ arithmetical operations, and so on. By adding the arithmetical operations required in each step we find the total required arithmetical operations as

$$2^{(N-1)} + 2^{(N-2)} + 2^{(N-3)} + \cdots + 2^0 = \frac{2^N - 1}{2 - 1}$$

Thus the overall complexity of Algorithm 1 is of the order $O(2^N)$.

4 A "Quantum" Algorithm for Finding Entanglement Status of an N-qubit Pure Quantum State:

In the above section we have developed a "classical" algorithm for finding entanglement status of an N-qubit pure quantum state and we saw that it is of the order $O(2^N)$. In the current section we will develop a "quantum" algorithm for testing the entanglement status of the given N-qubit pure quantum state and show that it is of the order $O(N^2)$.

An Important Observation:

It is easy to observe that the rows of the $2^1 \times 2^{N-1}$ matrix A given above in Step 2 of the Algorithm 1 are actually made up of the coefficients of the the quantum states that result from the partial measurements done on $|\psi\rangle$. The first row of the matrix A above associated with $|\psi\rangle$ is actually made up of the coefficients of the state that results after the partial measurement of $|\psi\rangle$ on the state $|0\rangle$, i.e. when the result of measuring the first qubit of the state $|\psi\rangle$ is $|0\rangle$. The second row of the matrix A is formed by the coefficients of the state that results after the partial measurement of the state $|\psi\rangle$ on the state $|1\rangle$, i.e. when the result of measuring the first qubit of the state $|\psi\rangle$ is $|1\rangle$.

Theorem 4.1: Let $|\psi_0\rangle$ and $|\psi_1\rangle$ be the two states that arise due to partial measurements of $|\psi\rangle$ on the states $|0\rangle$ and $|1\rangle$ respectively.

$|\langle\psi_0|\psi_1\rangle| = ||\psi_0\rangle|.|\psi_1\rangle|$ if and only if $|\psi\rangle$ has a 1-qubit factor on its left side.

Proof: If $|\langle\psi_0|\psi_1\rangle| = ||\psi_0\rangle|.||\psi_1\rangle|$ then the states $|\psi_0\rangle$ and $|\psi_1\rangle$ will form proportional vectors. Let these vectors be $[a_{00\cdots0}, a_{00\cdots01}, a_{00\cdots010}, \cdots, a_{01\cdots11}]$ and $[a_{10\cdots0}, a_{10\cdots01}, a_{10\cdots010}, \cdots, a_{11\cdots11}]$ then from the proportionality of these vectors $|\psi_0\rangle$ and $|\psi_1\rangle$ there will exist a constant, α say, such that

$$[a_{10\cdots0}, a_{10\cdots01}, a_{10\cdots010}, \cdots, a_{11\cdots11}]$$
$$= \alpha.[a_{00\cdots0}, a_{00\cdots01}, a_{00\cdots010}, \cdots, a_{01\cdots11}]$$

This in turn implies the existence of a 1-qubit factor on the left side to the given N-qubit state, $|\psi\rangle$, under test, i.e. $|\psi\rangle = (|0\rangle + \alpha|1\rangle)|\psi_0\rangle$. The converse is straightforward.

Example 1: Let $|\psi\rangle = (2|00\rangle + 7|01\rangle + 3|10\rangle + \frac{21}{2}|11\rangle)$. Find out the 1-qubit (linear) factor to $|\psi\rangle$, if exists. Is $|\psi\rangle$ a separable state?

Solution: The result of the partial measurements of the state $|\psi\rangle$ on the state $|0\rangle$ gives the state $|\psi_0\rangle = 2|0\rangle + 7|1\rangle$. The result after the partial measurements of the state $|\psi\rangle$ on the state $|1\rangle$ gives the state $|\psi_1\rangle = 3|0\rangle + \frac{21}{2}|1\rangle = \frac{3}{2}(2|0\rangle + 7|1\rangle)$. It is easy to check that $|\langle\psi_0|\psi_1\rangle| = ||\psi_0\rangle|.||\psi_1\rangle|$ and therefore $|\psi_0\rangle$ and $|\psi_1\rangle$ are proportional vectors and as per Theorem 4.1 $|\psi\rangle$ has a 1-qubit (linear) factor on its left side. Thus

$$|\psi\rangle = (|0\rangle + \frac{3}{2}|1\rangle) \otimes (2|0\rangle + 7|1\rangle).$$

Clearly, the given 2-qubit state has 2 1-qubit factors and so it is a separable state.

Example 2: Let $|\psi\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$. Find out the 1-qubit (linear) factor to $|\psi\rangle$, if exists. Is $|\psi\rangle$ a separable state?

Solution: The result of the partial measurements of the state $|\psi\rangle$ on the state $|0\rangle$ gives the state $|\psi_0\rangle = \frac{1}{\sqrt{2}}|1\rangle$. The result of the partial measurements of the state $|\psi\rangle$ on the state $|1\rangle$ gives the state $|\psi_1\rangle = -\frac{1}{\sqrt{2}}|0\rangle$. It is easy to check that $|\langle\psi_0|\psi_1\rangle| = 0$ and $||\psi_0\rangle|.||\psi_1\rangle| = \frac{1}{2}$. Therefore $|\psi_0\rangle$ and $|\psi_1\rangle$ are not proportional vectors and as per the above Theorem 4.1 $|\psi\rangle$ does not have a 1-qubit (linear) factor on its left side. Therefore the state $|\psi\rangle$ is an not separable and it is on the contrary an entangled state.

Algorithm 2

(We assume that we are given sufficiently many identical copies of the given N-qubit pure state $|\psi\rangle$ to be tested.)

Step 1: We proceed with checking the existence of 1-qubit factors to the N-qubit state, $|\psi\rangle$ given above on its left side. For this to achieve we measure the first (the leftmost) qubit of $|\psi\rangle$.

Step 2: During the partial measurement of the first qubit (made on the copies of $|\psi\rangle$) if we always find the first qubit to be in the state $|0\rangle$ then we have $|\psi\rangle = |0\rangle|\psi_0\rangle$. In this case we set $|\psi\rangle = |\psi_0\rangle$ and go back to Step 1.

Step 3: During the partial measurement of the first qubit (made on the copies of $|\psi\rangle$) if we always find the first qubit to be in the state $|1\rangle$ then we have $|\psi\rangle = |1\rangle|\psi_1\rangle$. In this case we set $|\psi\rangle = |\psi_1\rangle$ and go back to Step 1.

Step 4: During the partial measurement of the first qubit (made on the copies of $|\psi\rangle$) if we find for some measurements the first qubit to be in the state $|0\rangle$ and for some other measurements if we find the first qubit to be in the state $|1\rangle$ then it meant that when we measure first qubit the state $|\psi\rangle \rightarrow |0\rangle|\psi_0\rangle$ for some partial measurements of the first qubit and the state $|\psi\rangle \rightarrow |1\rangle|\psi_1\rangle$ for some other partial measurements of the first qubit.

In this case

(i) We find the inner product of $|\psi_0\rangle$ with itself $\langle\psi_0|\psi_0\rangle$ in $log(2^{(N-1)}) = (N-1)log(2)$ time [5].

(ii) We find the inner product of $|\psi_1\rangle$ with itself $\langle\psi_1|\psi_1\rangle$ in $log(2^{(N-1)}) = (N-1)log(2)$ time [5].

(iii) Now, we find the inner product $|\langle\psi_0|\psi_1\rangle|$ in $log(2^{(N-1)}) = (N-1)log(2)$ time [5].

Case I: If $|\langle\psi_0|\psi_1\rangle| = ||\psi_0\rangle|.||\psi_1\rangle|$ then $|\psi_0\rangle$ and $|\psi_1\rangle$ are proportional [6] and as a consequence $|\psi\rangle$ will have a linear factor on its left side as per Theorem 4.1.

Case II: On the contrary if $|\langle\psi_0|\psi_1\rangle| < ||\psi_0\rangle|.||\psi_1\rangle|$ then $|\psi_0\rangle$ and $|\psi_1\rangle$ are not proportional [6] and as a consequence $|\psi\rangle$ will not have a linear factor on its left side. We declare in this case that the state $|\psi\rangle$ under consideration is an "entangled" state and stop.

Step 5: If Case I is true than go back to Step 1 with $|\psi\rangle = |\psi_0\rangle$ and again continue up to Step 9.

Step 6: If Case I remains true till $|\psi\rangle$ becomes a single qubit state then we conclude that the originally given N-qubit state $|\psi\rangle$ is a "separable" state. Otherwise, the given N-qubit state $|\psi\rangle$ under consideration is an "entangled" state.

Complexity: To determine the first 1-qubit (linear) factor to the N-qubit pure quantum state under test one requires $(N-1)log2$ time. To determine the second 1-qubit (linear) factor one requires $(N-2)log2$ time, and so on. The total time to find out all factors one requires $((N-1)+(N-2)+(N-3)+\cdots+1)log2 = \frac{(N-1)(N)}{2}log2$ time. Therefore the total time required is of the order $O(N^2)$.

5 Conclusion

The overall complexity of the "Classical" Algorithm 1 for entanglement testing of an N-qubit pure quantum state is of the order $O(2^N)$ while the overall complexity of the "Quantum" Algorithm 2 is of the order $O(N^2)$ for doing the same task.

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