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On the Decay of Solutions for a Nonlinear Petrovsky Equation

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Abstract: This work studies the initial boundary value problem for the Petrovsky equation $u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m-1} u_t = |u|^{p-1} u$. Under suitable conditions decay estimates of the solution are proved by using Nakaos inequality.

Keywords: Decay, Global Existence, Petrovsky Equation

1 Introduction

In this work we study the following initial-boundary value problem

$$\begin{cases} u_{tt} + \triangle^2 u - \triangle u_t + |u_t|^{m-1} u_t = |u|^{p-1} u, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \partial_\upsilon u(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n \geq 1$; ν is the outer normal.

In the absence of the strong damping term $\triangle u_t$, the interaction between the nonlinear damping and source term were established by many authors [1,4]. Recently, Li et. al [5] investigated problem (1) and showed the global existence, energy decay and blow up of the solution.

In this paper, we analyze the influence of the damping terms and source terms on the solutions of problem (1). We obtained the global existence result by potential well method. The exponential decay, for $m = 1$ and the polynomial decay, for $m > 1$ were established by using Nakao's inequality.

This paper is organized as follows. In section 2, we present some lemmas, and the local existence theorem. In section 3, the global existence and the decay of the solution are given.

2 Preliminaries

In this section, we shall give some assumptions and lemmas which will be used throughout this paper. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

Lemma 1.*(Sobolev-Poincare inequality) [2]. If $2 \leq p \leq \frac{2n}{n-4}$ $(2 \leq p < \infty$ if $n = 1,2,3,4)$, then*

$$\|u\|_p \leq C_* \|\triangle u\| \text{ for } u \in H_0^2(\Omega)$$

holds with some constant C_*.

Lemma 2.*[3]. Let $\phi(t)$ be nonincreasing and nonnegative function defined on $[0,T]$, $T > 1$, satisfying*

$$\phi^{1+\alpha}(t) \leq w_0 (\phi(t) - \phi(t+1)),\ t \in [0,T]$$

for w_0 is a positive constant and α is a nonnegative constant. Then we have, for each $t \in [0,T]$,

$$\begin{cases} \phi(t) \leq \phi(0) e^{-w_1[t-1]^+}, & \alpha = 0, \\ \phi(t) \leq \left(\phi(0)^{-\alpha} + w_0^{-1}\alpha [t-1]^+\right)^{-\frac{1}{\alpha}}, & \alpha > 0, \end{cases}$$

where $[t-1]^+ = \max\{t-1, 0\}$, and $w_1 = \ln\left(\frac{w_0}{w_0 - 1}\right)$.

Next, we state the local existence theorem which is proved in [1].

Theorem 1.*(Local existence). Suppose that m, p satisfies*

$$\begin{cases} 1 < m < \infty, & n \leq 4; \\ 1 < m \leq \frac{n+4}{n-4}, & n > 4, \end{cases} \tag{2}$$

$$\begin{cases} 1 < p < \infty, & n \leq 4; \\ 1 < p \leq \frac{n}{n-4}, & n > 4, \end{cases} \tag{3}$$

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and further $u_0 \in H_0^2(\Omega)$ and $u_1 \in L^2(\Omega)$ such that problem (1) has a unique local solution

$$u \in C\left([0,T);H_0^2(\Omega)\right) \text{ and}$$
$$u_t \in C\left([0,T);L^2(\Omega)\right) \cap L^{m+1}(\Omega \times [0,T)) .$$

Moreover, at least one of the following statements holds true:

i) $T = \infty$,
ii) $\|u_t\|^2 + \|\triangle u\|^2 \longrightarrow \infty$ as $t \longrightarrow T^-$.

3 Global existence and decay of solutions

In this section, we discuss the global existence and decay of the solution for problem (1).

We define

$$J(t) = \frac{1}{2}\|\triangle u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{4}$$

and

$$I(t) = \|\triangle u\|^2 - \|u\|_{p+1}^{p+1}. \tag{5}$$

We also define the energy function as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\triangle u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \tag{6}$$

Finally, we define

$$W = \left\{u : u \in H_0^2(\Omega),\ I(u) > 0\right\} \cup \{0\}. \tag{7}$$

The next lemma shows that our energy functional (6) is a nonincreasing function along the solution of (1).

Lemma 3. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = -\left(\|u_t\|_{m+1}^{m+1} + \|\nabla u_t\|^2\right) \leq 0. \tag{8}$$

Proof. Multiplying the equation of (1) by u_t and integrating over Ω, using integrating by parts and summing up the product results, we get

$$E(t) - E(0) = -\int_0^t \left(\|u_\tau\|_{m+1}^{m+1} + \|\nabla u_\tau\|^2\right) d\tau \text{ for } t \geq 0. \tag{9}$$

Lemma 4. *Suppose that (2) holds. Let $u_0 \in W$ and $u_1 \in H_0^m(\Omega)$ such that*

$$\beta = C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} < 1, \tag{10}$$

then $u \in W$ for each $t \geq 0$.

Proof. Since $I(0) > 0$, it follows the continuity of $u(t)$ that

$$I(t) > 0,$$

for some interval near $t = 0$. Let $T_m > 0$ be a maximal time, when (5) holds on $[0, T_m]$.

From (4) and (3), we have

$$\begin{aligned} J(t) &= \frac{1}{p+1}I(t) + \frac{p-1}{2(p+1)}\|\triangle u\|^2 \\ &\geq \frac{p-1}{2(p+1)}\|\triangle u\|^2 \end{aligned} \tag{11}$$

By using (11), (6) and Lemma 3, we get

$$\begin{aligned} \|\triangle u\|^2 &\leq \frac{2(p+1)}{p-1}J(t) \\ &\leq \frac{2(p+1)}{p-1}E(t) \\ &\leq \frac{2(p+1)}{p-1}E(0). \end{aligned} \tag{12}$$

By recalling Lemma 1 and (12), we have

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq C_*\|\triangle u\|^{p+1} \\ &= C_*\|\triangle u\|^{p-1}\|\triangle u\|^2 \\ &\leq C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}}\|\triangle u\|^2 \\ &= \beta\|\triangle u\|^2 \\ &< \|\triangle u\|^2 \text{ on } t \in [0, T_m]. \end{aligned} \tag{13}$$

Therefore, by using (5), we conclude that $I(t) > 0$ for all $t \in [0, T_m]$. By repeating the procedure, T_m is extended to T. The proof of Lemma 4 is completed.

Lemma 5. *Let assumptions of Lemma 4 holds. Then there exists $\eta_1 = 1 - \beta$ such that*

$$\|u\|_{p+1}^{p+1} \leq (1 - \eta_1)\|\triangle u\|^2.$$

Proof. From (13), we get

$$\|u\|_{p+1}^{p+1} \leq \beta\|\triangle u\|^2.$$

Let $\eta_1 = 1 - \beta$, then we have the result.

Remark. From Lemma 5, we can deduce that

$$\|\triangle u\|^2 \leq \frac{1}{\eta_1}I(t). \tag{14}$$

Theorem 2. *Suppose that (2) holds. Let $u_0 \in W$ satisfying (10). Then the solution of problem (1) is global.*

Proof. It is sufficient to show that $\|u_t\|^2 + \|\triangle u\|^2$ is bounded independently of t. To achieve this we use (5) and (6) to obtain

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\triangle u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}$$
$$= \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|\triangle u\|^2 + \frac{1}{p+1}I(t)$$
$$\geq \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|\triangle u\|^2$$

since $I(t) \geq 0$. Therefore

$$\|u_t\|^2 + \|\triangle u\|^2 \leq CE(0),$$

where $C = \max\left\{2, \frac{2(p+1)}{p-1}\right\}$. Then by Theorem 1, we have the global existence result.

Theorem 3.*Suppose that (2) and (10) holds, and further $u_0 \in W$. Thus, we have following decay estimates:*

$$E(t) \leq \begin{cases} E(0)e^{-w_1[t-1]^+}, & \text{if } m=1, \\ \left(E(0)^{-\alpha} + C_7^{-1}\alpha[t-1]^+\right)^{-\frac{1}{\alpha}}, & \text{if } m>1, \end{cases}$$

where w_1, α and C_7 are positive constants which will be defined later.

*Proof.*By integrating (8) over $[t, t+1]$, $t>0$, we have

$$E(t) - E(t+1) = D^{m+1}(t), \tag{15}$$

where

$$D^{m+1}(t) = \int_t^{t+1}\left(\|u_\tau\|_{m+1}^{m+1} + \|\nabla u_\tau\|^2\right)d\tau. \tag{16}$$

By virtue of (16) and Hölder inequality, we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2\,dxdt \leq |\Omega|^{\frac{r+1}{r+2}}D^2(t) = CD^2(t). \tag{17}$$

Hence, from (17), there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|u_t(t_i)\| \leq CD(t),\ i=1,2 \tag{18}$$

Multiplying the equation of (1) by u, and integrating it over $\Omega \times [t_1, t_2]$, we get

$$\int_{t_1}^{t_2} I(t)\,dt = -\int_{t_1}^{t_2}\int_\Omega uu_{tt}\,dxdt - \int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u\,dxdt$$
$$-\int_{t_1}^{t_2}\int_\Omega |u_t|^{m-1}u_t u\,dxdt. \tag{19}$$

By using (1) and integrating by parts and Cauchy-Schwarz inequality in the first term, and Hölder inequality in the second term of the right hand side of (19), we obtain

$$\int_{t_1}^{t_2} I(t)\,dt \leq \|u_t(t_1)\|\|u(t_1)\| + \|u_t(t_2)\|\|u(t_2)\|$$
$$+\int_{t_1}^{t_2}\|u_t(t)\|^2 dt + \int_{t_1}^{t_2}\|\nabla u_t\|\|\nabla u\|\,dt$$
$$-\int_{t_1}^{t_2}\int_\Omega |u_t|^{m-1}u_t u\,dxdt. \tag{20}$$

Now, our goal is to estimate the last term in the right-hand side of inequality (20). By using Hölder inequality, we obtain

$$\int_{t_1}^{t_2}\int_\Omega |u_t|^{m-1}u_t u\,dxdt \leq \int_{t_1}^{t_2}\|u_t(t)\|_{m+1}^m\|u(t)\|_{m+1}dt \tag{21}$$

By applying the Sobolev-Poincare inequality and (12), we find

$$\int_{t_1}^{t_2}\|u_t(t)\|_{m+1}^m\|u(t)\|_{m+1}dt$$
$$\leq C_*\int_{t_1}^{t_2}\|u_t(t)\|_{m+1}^m\|\triangle u(t)\|\,dt$$
$$\leq C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\int_{t_1}^{t_2}\|u_t(t)\|_{m+1}^m E^{\frac{1}{2}}(s)\,dt$$
$$\leq C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)\int_{t_1}^{t_2}\|u_t(t)\|_{m+1}^m dt$$
$$\leq C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D^m(t). \tag{22}$$

$$\int_{t_1}^{t_2}\|\nabla u_t\|\|\nabla u\|\,dt$$
$$\leq C_*\int_{t_1}^{t_2}\|\nabla u_t\|\|\triangle u(t)\|\,dt$$
$$\leq C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\int_{t_1}^{t_2}\|\nabla u_t\|E^{\frac{1}{2}}(s)\,dt$$
$$\leq C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)\int_{t_1}^{t_2}\|\nabla u_t\|\,dt,$$

which implies

$$\int_{t_1}^{t_2}\|\nabla u_t\|\,dt \leq \left(\int_{t_1}^{t_2}1dt\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\|\nabla u_t\|^2 dt\right)^{\frac{1}{2}}$$
$$\leq CD(t)$$

Then

$$\int_{t_1}^{t_2}\|\nabla u_t\|\|\nabla u\|\,dt \leq CC_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D(t) \tag{23}$$

From (12), (18) and Sobolev-Poincare inequality, we have

$$\|u_t(t_i)\|\|u(t_i)\| \leq C_1 D(t)\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s), \tag{24}$$

where $C_1 = 2C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}$. Then by (20)-(24) we have

$$\int_{t_1}^{t_2} I(t)\,dt \leq C_1\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D(t) + D^2(t)$$
$$+CC_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D(t)$$
$$+C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D^m(t). \tag{25}$$

On the other hand, from (5), (6) and Remark, we obtain

$$E(t) \leq \frac{1}{2}\|u_t\|^2 + C_3 I(t), \tag{26}$$

where $C_3 = \frac{1}{\eta_1}\frac{p-1}{2(p+1)} + \frac{1}{p+1}$.

By integrating (26) over $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} E(t)\,dt \leq \frac{1}{2}\int_{t_1}^{t_2}\|u_t\|^2 dt + C_3\int_{t_1}^{t_2} I(t)\,dt. \tag{27}$$

Then by (21) and (27), we get

$$\begin{aligned}\int_{t_1}^{t_2} E(t)\,dt \leq{}& \frac{1}{2}CD^2(t) + C_3C_2\Bigg[\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s)D(t) + D^2(t)\\ &+ CC_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s)D(t)\\ &+ C_*\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s)D^m(t)\Bigg]\end{aligned} \tag{28}$$

By integrating $\frac{d}{dt}E(t)$ over $[t, t_2]$, we obtain

$$E(t) = E(t_2) + \int_t^{t+1}\left(\|u_\tau\|_{m+1}^{m+1} + \|\nabla u_\tau\|^2\right)d\tau. \tag{29}$$

Therefore, since $t_2 - t_1 \geq \frac{1}{2}$, we conclude that

$$\int_{t_1}^{t_2} E(t)\,dt \geq (t_2 - t_1)E(t_2) \geq \frac{1}{2}E(t_2).$$

That is,

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(t)\,dt. \tag{30}$$

Consequently, exploiting (15), (28)-(30), and since $t_1, t_2 \in [t, t+1]$, we get

$$\begin{aligned}E(t) &\leq 2\int_{t_1}^{t_2} E(t)\,dt + \int_t^{t+1}\left(\|u_\tau\|_{m+1}^{m+1} + \|\nabla u_\tau\|^2\right)d\tau\\ &= 2\int_{t_1}^{t_2} E(t)\,dt + D^{m+1}(t).\end{aligned} \tag{31}$$

Then, by (28), we have

$$\begin{aligned}E(t) \leq{}& \left(\frac{1}{2}C + C_3C_2\right)D^2(t) + D^{m+1}(t)\\ &+ C_4\left[D(t) + D^m(t)\right]E^{\frac{1}{2}}(t).\end{aligned}$$

Hence, by Young inequality, we obtain

$$E(t) \leq C_5\left[D^2(t) + D^{m+1}(t) + D^{2m}(t)\right]. \tag{32}$$

Case 1: When $m = 1$, from (32), we obtain

$$E(t) \leq 3C_5D^2(t) = 3C_5\left[E(t) - E(t+1)\right].$$

By Lemma 2, we get

$$E(t) \leq E(0)\,e^{-w_1[t-1]^+},$$

where $w_1 = \ln\frac{3C_5}{3C_5-1}$.

Case 2: When $m > 1$, from (32), we obtain

$$E(t) \leq C_5D^2(t)\left(1 + D^{m-1}(t) + D^{2(m-1)}(t)\right).$$

Then since $E(t) \leq E(0)$, $\forall t \geq 0$, we see from (15)

$$\begin{aligned}E(t) &\leq C_5\left(1 + E^{\frac{m-1}{m+1}}(0) + E^{\frac{2(m-1)}{m+1}}(0)\right)D^2(t)\\ &\leq C_6D^2(t), \quad t \geq 0.\end{aligned}$$

Then we obtain

$$\begin{aligned}E(t)^{\frac{m+1}{2}} &\leq C_7D^{m+1}(t)\\ &\leq C_7\left(E(t) - E(t+1)\right).\end{aligned} \tag{33}$$

Thus, from (33) and Lemma 2, we have

$$E(t) \leq \left(E(0)^{-\alpha} + C_7^{-1}\alpha\,[t-1]^+\right)^{-\frac{1}{\alpha}}.$$

The proof of Theorem 3 is completed.

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