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Matroidal and Lattices Structures of Rough Sets and Some of Their Topological Characterizations

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Abstract: Matroids, rough set theory and lattices are efficient tools of knowledge discovery. Lattices and matroids are studied on preapproximations spaces. Li et al. proved that a lattice is Boolean if it is clopen set lattice for matroids. In our study, a lattice is Boolean if it is closed for matroids. Moreover, a topological lattice is discussed using its matroidal structure. Atoms in a complete atomic Boolean lattice are completely determined through its topological structure. Finally, a necessary and sufficient condition for a predefinable set is proved in preapproximation spaces. The value k for a predefinable set in lattice of matroidal closed sets is determined.

Keywords: Matroids, lattices, preapproximation spaces, predefinable sets

1 Introduction

Matroids initiated by Whitney [1] and seem in several combinatorial and algebraic contexts [2,3,4,5,6,7]. Rough set theory were initiated by Pawlak [8] through the approximation space in eighties, many authors have turned their attention to the generalization rough sets [9, 10,11,12,13,14]. Lattices are mathematical objects that have been used to solve some problems in computer science, approximation spaces [15,16,17,18,19,20,21, 22,23]. The class of preopen sets is applied in general topology by researchers in [24], to investigate preapproximation spaces. Some algebraic applications were studied on rough (resp. prerough) sets and named Ω (resp. Ω_p). For example, each of rough and prerough sets as lattices, as congruences. The approximations were used to calculate the accuracy [25]. Some new results on rough (resp. prerough) sets were presented. Also, new order relations on lattices [26,27] were defined. The concept of lattice constructed based on approximate operators were introduced and studied in [28,29]. Also, Yao [30] introduced a different concept for lattice and compared it with another notions in data analysis. Recently, topological structures have been used to study graphs as in [31,32,33,34,35]. Also, many researchers suggested topological models in biology [36,37,38], medicine [39, 40,41], physics [42,43,44,45] and smart city [46].

In terms of preapproximations and prerough sets, some topological lattice models throughout this paper are presented and studied. Some algebraic properties for Abd El Monsef's preapproximation space, such as a complete Boolean lattice is investigated. It will be created new types of upper preapproximation and lower preapproximation in the preapproximation space. Eventually, the value of k in which $P\mathscr{D}$ $(\overline{\mathfrak{apr}}_{\Omega_p})$ $\subseteq$ $\{\overline{\mathfrak{apr}}^k_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$ and $P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p})$ $\subseteq \{\underline{\mathfrak{apr}}^k_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$ is determined. A comparison between $\overline{\mathfrak{apr}}_{\Omega}$ (resp. $\underline{\mathfrak{apr}}_{\Omega}$) and $\overline{\mathfrak{apr}}_{\Omega_p}$ (resp. $\underline{\mathfrak{apr}}_{\Omega_p}$), respectively is discussed. Finally, we prove that $\underline{\mathfrak{apr}}^n_{\Omega}$ is the $\mathscr{M}$ matroidal closure. This means that this set will be predefinable in lattice matroidal closed sets and the value k is necessary condition for the predefinability for any subset of the universal set $\mathfrak{U}$.

2 Preliminary Results

Definition 1. *[14] The pair* $(X, \mathtt{int})$ *is a topological space if* $\forall A \subseteq X$*, there is an operator* $\mathtt{int}(A)$*, say, the interior of A, s.t. the conditions are satisfied*
(i) $\mathtt{int}(A) \subseteq A$*;*
(ii) $\mathtt{int}(\mathtt{int}(A)) = \mathtt{int}(A)$*;*

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(iii) $\mathfrak{int}(X) = X$;
(iv) $\mathfrak{int}(A \cap B) = \mathfrak{int}(A) \cap \mathfrak{int}(B)$, *for any* $A, B \subseteq X$.
Each set in $(X, \mathfrak{int})$ *is open and its complement is closed.*

Definition 2. *[47] A is preopen w.r.to* τ *if* $A \subseteq \mathfrak{int}(\mathfrak{cl}(A))$.

Definition 3. *[48] Consider* $\bigcap_{i\in I} X_i \in \mathscr{L} \subseteq \mathscr{P}(\mathfrak{U})\ \forall\ \{X_i : i \in I\} \subseteq \mathscr{L}$. *Then,* $\mathscr{L}$ *is called a closure system. A closure system with ordered lattice is named complete in which* $\wedge_{i\in I} X_i = \bigcap_{i\in I} X_i$ *and* $\vee_{i\in I} X_i = \bigcap\{Y \in \mathscr{P}(\mathfrak{U}) : \bigcap_{i\in I} X_i \subseteq Y\}$.

Definition 4. *[2,5] Let E be the ground set and* $\mathscr{I}$ *be a subclass of E.* $\mathscr{M} = (E, \mathscr{I})$ *is a matroid if the conditions hold*
(I1) $\phi \in \mathscr{I}$.
(I2) If $I \in \mathscr{I}$ *and* $I' \subseteq I$, *then* $I' \in \mathscr{I}$.
(I3) If $I, J \in \mathscr{I}$ *and* $|I| < |J|$, *then* $\exists\ j \in J - I$ *s.t.* $I \cup \{j\} \in \mathscr{I}$ *where* $|I|$ *denotes the cardinality of I.*
Each element in $\mathscr{I}$ *is called an independent set. Any subset of* $\mathscr{P}(E) - \mathscr{I}$ *is called dependent, where* $\mathscr{P}(E)$ *is the power set of E.*

Definition 5. *[4] Let* $\mathscr{M} = (E, \mathscr{I})$ *be a matroid. Then,*
(i) Each element in $\mathscr{I}$ *is said to be an independent set. Otherwise, it was called dependent.*
(ii) A base element is the maximal set in $\mathscr{I}$ *in the sense of inclusion. The minimal set is called a circuit of the matroid* $\mathscr{M}$ *and is denoted by* $\mathscr{C}(\mathscr{M})$.
(iii) The singleton circuit is called a loop. If $\{a, b\}$ *is a circuit, then a and b are said to be parallel.*
(iv) $\forall\ A \subseteq E$, *the closure operator* $\mathfrak{cl}_{\mathscr{M}}(A)$ *of a matroid* $\mathscr{M}$ *is defined as* $\mathfrak{cl}_{\mathscr{M}}(A) = \{a \in E : f(A) = f(A \cup \{a\})\}$ *and* $\mathfrak{cl}_{\mathscr{M}}(A)$ *is called the closure of A in* $\mathscr{M}$. *When there is no confusion, the symbol* $\mathfrak{cl}(X)$ *is used for abbreviation. A is called a flat or a closed set if* $\mathfrak{cl}(A) = A$.

Proposition 1. *[5] The following properties are hold for* $\mathfrak{cl}_{\mathscr{M}}$:
(i) $\forall\ X \subseteq \mathfrak{U}$, $X \subseteq \mathfrak{cl}_{\mathscr{M}}(X)$.
(ii) $\mathfrak{cl}_{\mathscr{M}}(X) \subseteq \mathfrak{cl}_{\mathscr{M}}(Y)$ *if* $X \subseteq Y$.
(iii) $\mathfrak{cl}_{\mathscr{M}}(\mathfrak{cl}_{\mathscr{M}}(X)) = \mathfrak{cl}_{\mathscr{M}}(X)$.
(iv) $\forall\ X \subseteq \mathfrak{U}$ *and* $x \in \mathfrak{U}$, *if* $y \in \mathfrak{cl}_{\mathscr{M}}(X \cup \{x\}) - \mathfrak{cl}_{\mathscr{M}}(X)$, *then* $x \in \mathfrak{cl}_{\mathscr{M}}(X \cup \{y\})$.

Lemma 1.7.3 in [5] proved that the class of lattice matroidal closed sets is lattice and is denoted by $\mathscr{CL}(\mathscr{M})$. In this lattice, $A \wedge B = \mathfrak{cl}_{\mathscr{M}}(A \cap B)$ and $A \vee B = \mathfrak{cl}_{\mathscr{M}}(A \cup B), \forall\ A, B \in \mathscr{CL}(\mathscr{M})$.

Proposition 2. *[3]* $r_{\mathscr{M}}(A) = |A|$ *iff* $A \in \mathscr{I}$, $\forall\ A \subseteq E$.

Definition 6. *[3] The closure operator* $\mathfrak{cl}_{\mathscr{M}}(A) = \{u \in E : r_{\mathscr{M}}(A) = r_{\mathscr{M}}(A \cup \{u\})\}, \forall\ A \subseteq E$. $\mathfrak{cl}_{\mathscr{M}}(A)$ *is said to be the closure of A w.r.to* $\mathscr{M}$.

3 Main Results

Throughout this section, consider $\overline{\mathfrak{apr}}_{\Omega_p}$ and $\underline{\mathfrak{apr}}_{\Omega_p}$ are denoted to the upper and lower approximation w.r.to the preapproximation space $(\mathfrak{U}, \Omega_p)$.

3.1 Prerough sets and some algebraic properties

Definition 7. *Let* $\mathfrak{U}$ *be a finite nonempty set and* $(\mathfrak{U}, \Omega)$ *is a generalized approximation space, where* Ω *is a relation which will be a subbase for a topological space, say,* τ. *Then, a class of preopen sets called* $\mathscr{PO}(\mathfrak{U}, \tau)$ *from* τ *is generated. If* Ω_p *is a relation on* $\mathscr{PO}(\mathfrak{U}, \tau)$, *then* $\mathscr{PO}(\mathfrak{U}, \Omega_p)$ *is said to be a preapproximation space.*

From Definition 7, $\mathfrak{U}/\Omega_p = \{[x]_{\Omega_p} : x \in \mathfrak{U}\}$ s.t. $[x]_{\Omega_p} = \{y \in \mathfrak{U} : x\Omega_p y\}$ is satisfied.

Definition 8. *Let* $(\mathfrak{U}, \Omega_p)$ *be a preapproximation space. A prelower and preupper approximation of X is*
$\underline{\mathfrak{apr}}_{\Omega_p}(X) = \{x \in \mathfrak{U} : \Omega_p(x) \subseteq X\}$, *and*
$\overline{\mathfrak{apr}}_{\Omega_p}(X) = \{x \in \mathfrak{U} : \Omega_p(x) \cap X \neq \phi\}$, *respectively. This can be shown in Figure 1.*

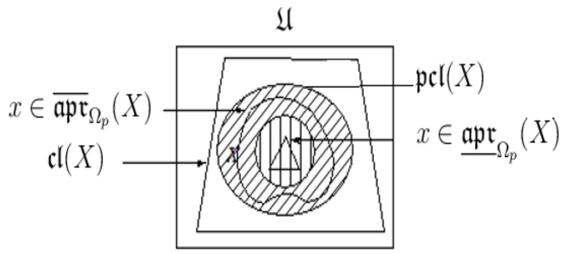

Fig. 1: A prerough approximations.

X is a lower predefinable in $(\mathfrak{U}, \Omega_p)$ *if* $\underline{\mathfrak{apr}}_{\Omega_p}(X) = X$ *and is denoted by* $P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p})$. *Similarly, X is an upper predefinable set in* $(\mathfrak{U}, \Omega_p)$ *if* $\overline{\mathfrak{apr}}_{\Omega_p}(X) = X$ *is denoted by* $P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$. *Hence, X is predefinable if* $\underline{\mathfrak{apr}}_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}(X) = X$ *and is denoted by* $P\mathscr{D}(\mathfrak{U}, \Omega_p)$.

In Definition 9, $\mathfrak{pint}(X)$ (resp. $\mathfrak{pcl}(X)$) denotes to preinterior (resp. preclosure) operators w.r.to the preapproximation space $(\mathfrak{U}, \Omega_p)$.

Definition 9. *Let* $(\mathfrak{U}, \Omega_p)$ *be a preapproximation space and* $X \subseteq \mathfrak{U}$. *Then,*
(i) X is a preexact if $\mathfrak{pint}(X) = \mathfrak{pcl}(X)$.
(ii) X is a prerough if $\mathfrak{pint}(X) \neq \mathfrak{pcl}(X)$.

By analogous of results of Zhu in [49], it is easy to prove propositions 3 and 4.

Proposition 3. *If $(\mathfrak{U}, \Omega_p)$ is a preapproximation space, where the relation Ω_p is serial and $X, Y \subseteq \mathfrak{U}$, then the following are verified:*
(i) $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{U}) = \mathfrak{U}$.
(ii) $\underline{\mathfrak{apr}}_{\Omega_p}(X \cap Y) = \underline{\mathfrak{apr}}_{\Omega_p}(X) \cap \underline{\mathfrak{apr}}_{\Omega_p}(Y)$.
(iii) $X \subseteq Y \Rightarrow \underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y)$.
(vi) $X \subseteq Y \Rightarrow \overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$.
(v) $\overline{\mathfrak{apr}}_{\Omega_p}(X \cup Y) = \overline{\mathfrak{apr}}_{\Omega_p}(X) \cup \overline{\mathfrak{apr}}_{\Omega_p}(Y)$.
(iv) $\overline{\mathfrak{apr}}_{\Omega_p}(\phi) = \phi$.
(vii) $\underline{\mathfrak{apr}}_{\Omega_p}(X^c) = (\overline{\mathfrak{apr}}_{\Omega_p}(X))^c$.

Proposition 4. *For a relation Ω_p on $\mathfrak{U}$, we get*
(i) Ω_p is reflexive iff $\underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq X$ iff $X \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(X)$.
(ii) Ω_p is transitive iff $\underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X))$ iff $\overline{\mathfrak{apr}}_{\Omega_p}(\overline{\mathfrak{apr}}_{\Omega_p}(X)) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(X)$, $\forall\, X \subseteq \mathfrak{U}$.

Remark 1. *According to Proposition 3, $\mathfrak{U} \in P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p})$, $\phi \in P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$. This means that $P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p})$ and $P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$ are nonempty in some cases, while $P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p}) \cap P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$ may be empty other cases.*

Remark 2. *Since each open set is preopen, then a definable set is predefinable [48]. Generally, the inverse direction is not hold.*

Example 1. Let $\mathfrak{U} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ and $\mathfrak{U}/\Omega = \{\{\mathfrak{a}\}, \{\mathfrak{b}, \mathfrak{c}\}\}$ be a subbase for τ. If $X = \{\mathfrak{a}, \mathfrak{b}\}$ be a rough set, then the expansion of given approximation space is $\tau_\Omega = \mathscr{PO}(\mathfrak{U}, \tau) = \{\mathfrak{U}, \phi, \{\mathfrak{a}\}, \{\mathfrak{b}\}, \{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{b}, \mathfrak{c}\}\}$. The subsets $\{\mathfrak{a}\}$ and $\{\mathfrak{b}, \mathfrak{c}\}$ are predefinable, but neither of them is definable.

For computing the families $P\mathscr{D}$ $(\underline{\mathfrak{apr}}_{\Omega_p})$ and $P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$, the following notions are introduced $\underline{\mathfrak{apr}}^0_{\Omega_p}(X) = X$, $\underline{\mathfrak{apr}}^1_{\Omega_p}(X) = \underline{\mathfrak{apr}}_{\Omega_p}(X)$, $\underline{\mathfrak{apr}}^2_{\Omega_p}(X) = \underline{\mathfrak{apr}}_{\Omega_p}((\underline{\mathfrak{apr}}_{\Omega_p})(X))$, $\underline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X) = \underline{\mathfrak{apr}}_{\Omega_p}((\underline{\mathfrak{apr}}^k_{\Omega_p})(X))$; $\overline{\mathfrak{apr}}^0_{\Omega_p}(X) = X$, $\overline{\mathfrak{apr}}^1_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}(X)$, $\overline{\mathfrak{apr}}^2_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}((\overline{\mathfrak{apr}}_{\Omega_p})(X))$, $\overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}((\overline{\mathfrak{apr}}^k_{\Omega_p})(X))$.

Lemma 1. *In a space $(\mathfrak{U}, \Omega_p)$, if $\overline{\mathfrak{apr}}_{\Omega_p}(X) = X$, then $\overline{\mathfrak{apr}}^k_{\Omega_p}(X) = X$, $\forall\, k \in \mathbb{N}$, for $X \subseteq \mathfrak{U}$.*

Proof. The relation is true for $k = 1$. For $k > 1$, $\overline{\mathfrak{apr}}^2_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}(X) = X$, implies $\overline{\mathfrak{apr}}^3_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}(X) = X$ and so on to $\overline{\mathfrak{apr}}^k_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}(X) = X$.

Example 2. Let $\mathfrak{U} = \{1,2,3,4,5,6\}$ with $\mathfrak{U}/\Omega_p = \{\{1\}, \{2\}, \{3\}, \{1,4\}, \{4,5\}\}$. By Definition 12, $\tau_{\Omega_p} = \{\mathfrak{U}, \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,4\}, \{4,5\}, \{1,2\}\}$. So, $\mathscr{PO}(\mathfrak{U}, \tau_{\Omega_p}) = \{\mathfrak{U}, \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{4,5\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3,4\}, \{1,2,4,5\}\}$. By Definition 8, $\overline{\mathfrak{apr}}_{\Omega_p}(\{1\}) = \{1,6\}$, $\overline{\mathfrak{apr}}^2_{\Omega_p}(\{1\}) = \overline{\mathfrak{apr}}_{\Omega_p}(\{1, 6\}) = \{1,6\}$. Then, $\overline{\mathfrak{apr}}_{\Omega_p}(\{1\}) \in P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$. Also, $\underline{\mathfrak{apr}}_{\Omega_p}(\{1,4,6\}) = \{1,4\}$, $\overline{\mathfrak{apr}}^2_{\Omega_p}(\{1,4,6\}) = \overline{\mathfrak{apr}}_{\Omega_p}(\{1,4\}) = \{1,4\}$. Then, $\overline{\mathfrak{apr}}_{\Omega_p}(\{1,4,6\}) \in P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p})$.

By a mathematical induction, it is easy to prove Proposition 5 and so the proof is omitted.

Proposition 5. *Given $(\mathfrak{U}, \Omega_p)$ and $k \in \mathbb{N}$. Then, $\forall\, X, Y \in \mathscr{P}(\mathfrak{U})$,*
(L1) $\underline{\mathfrak{apr}}^k_{\Omega_p}(\mathfrak{U}) = \mathfrak{U}$.
(U1) $\overline{\mathfrak{apr}}^k_{\Omega_p}(\mathfrak{U}) = \mathfrak{U}$.
(L2) $\underline{\mathfrak{apr}}^k_{\Omega_p} = \phi$.
(U2) $\overline{\mathfrak{apr}}^k_{\Omega_p}(\phi) = \phi$.
(L3) $\underline{\mathfrak{apr}}^k_{\Omega_p}(X) = (\overline{\mathfrak{apr}}^k_{\Omega_p}(X^c))^c$.
(U3) $\overline{\mathfrak{apr}}^k_{\Omega_p}(X) = (\underline{\mathfrak{apr}}^k_{\Omega_p}(X^c))^c$.
(L4) $\underline{\mathfrak{apr}}^k_{\Omega_p}(X \cap Y) = \underline{\mathfrak{apr}}^k_{\Omega_p}(X) \cap \underline{\mathfrak{apr}}^k_{\Omega_p}(Y)$.
(U4) $\overline{\mathfrak{apr}}^k_{\Omega_p}(X \cup Y) = \overline{\mathfrak{apr}}^k_{\Omega_p}(X) \cup \overline{\mathfrak{apr}}^k_{\Omega_p}(Y)$.
(L5) If $X \subseteq Y$, then $\underline{\mathfrak{apr}}^k_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}^k_{\Omega_p}(Y)$.
(U5) If $X \subseteq Y$, then $\overline{\mathfrak{apr}}^k_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^k_{\Omega_p}(Y)$.
(L6) If Ω_p is reflexive, then $\underline{\mathfrak{apr}}^k_{\Omega_p}(X) \subseteq X$.
(U6) If Ω_p is reflexive, then $X \subseteq \overline{\mathfrak{apr}}^k_{\Omega_p}(X)$.

Definition 10. *The sets X and Y in $(\mathfrak{U}, \Omega_p)$ are called*
(i) preroughly bottom equal $X \eqsim_p Y$ if $\underline{\mathfrak{apr}}_{\Omega_p}(X) = \underline{\mathfrak{apr}}_{\Omega_p}(Y)$.
(ii) preroughly top equal $X \simeq_p Y$ if $\overline{\mathfrak{apr}}_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}(Y)$.
(iii) preroughly equal $X \approx_p Y$ if $X \eqsim_p Y$ and $X \simeq_p Y$.

Remark. The equivalence class of $\approx_p$, for $X \subseteq \mathfrak{U}$, has the form $[X]_{\approx_p} = \{A \subseteq \mathfrak{U} : \underline{\mathfrak{apr}}_{\Omega_p}(A) = \underline{\mathfrak{apr}}_{\Omega_p}(X)$ and $\overline{\mathfrak{apr}}_{\Omega_p}(A) = \overline{\mathfrak{apr}}_{\Omega_p}(X)\}$.

Definition 11. *For any $[X]_{\approx_p}$ and $[Y]_{\approx_p}$ in $\Omega_p(\mathfrak{U})$, a relation $[X]_{\approx p} \leq [Y]_{\approx p}$ if $\underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y)$ and $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$.*

Six types of approximations in terms of bottom (resp. prebottom) rough are given if $X \eqsim Y$ (resp. $X \eqsim_p Y$). Similarly, top (resp. pretop) rough if $X \simeq Y$ (resp. $X \simeq_p Y$). Then, $\approx = \eqsim \cap \simeq$ and $\approx_p = \eqsim_p \cap \simeq_p$. Each of relations $\eqsim$, $\simeq$, $\eqsim_p$ and $\simeq_p$ is equivalence.

Lemma 2. *The relation $\simeq$ (resp. $\eqsim$) is a congruence on $(\mathscr{P}(\mathfrak{U}), \cup)$ (resp. $(\mathscr{P}(\mathfrak{U}), \cap)$).*

Proof. Let $\simeq$ and $\eqsim$ be equivalence relations on $\mathscr{P}(\mathfrak{U})$. Then, for A, B, C, D are subsets of $\mathscr{P}(\mathfrak{U})$, we have
(i) If $A \simeq B$ and $C \simeq D$, then $\overline{\mathfrak{apr}}_{\Omega_p}(A) = \overline{\mathfrak{apr}}_{\Omega_p}(B)$ and $\overline{\mathfrak{apr}}_{\Omega_p}(C) = \overline{\mathfrak{apr}}_{\Omega_p}(D)$. Since $\overline{\mathfrak{apr}}_{\Omega_p}(A \cup C) = \overline{\mathfrak{apr}}_{\Omega_p}(A) \cup \overline{\mathfrak{apr}}_{\Omega_p}(C) = \overline{\mathfrak{apr}}_{\Omega_p}(B) \cup \overline{\mathfrak{apr}}_{\Omega_p}(D) = \overline{\mathfrak{apr}}_{\Omega_p}(B \cup D)$, then $A \cup C \simeq B \cup D$ and so $\simeq$ is a congruence on $(\mathscr{P}(\mathfrak{U}), \cup)$.

(ii) If $A \eqsim B$ and $C \eqsim D$, then $\underline{\mathfrak{apr}}_{\Omega_p}(A) = \underline{\mathfrak{apr}}_{\Omega_p}(B)$ and $\underline{\mathfrak{apr}}_{\Omega_p}(C) = \underline{\mathfrak{apr}}_{\Omega_p}(D)$. Now, since $\underline{\mathfrak{apr}}_{\Omega_p}(A \cap C) = \underline{\mathfrak{apr}}_{\Omega_p}(A) \cap \underline{\mathfrak{apr}}_{\Omega_p}(C) = \underline{\mathfrak{apr}}_{\Omega_p}(B) \cap \underline{\mathfrak{apr}}_{\Omega_p}(D) = \underline{\mathfrak{apr}}_{\Omega_p}(B \cap D)$. Thus, $A \cap C \eqsim B \cap D$. Therefore, $\eqsim$ is a congruence on $(\mathscr{P}(\mathfrak{U}), \cap)$.

Remark 3. *Relations $\eqsim_p$ and $\simeq_p$ are not usually congruences. Because of $\underline{\mathfrak{apr}}_{\Omega_p}(X \cap Y) = \underline{\mathfrak{apr}}_{\Omega_p}(X) \cap \underline{\mathfrak{apr}}_{\Omega_p}(Y)$ is not truthful, in general and $\overline{\mathfrak{apr}}_{\Omega_p}(X \cup Y) \neq \overline{\mathfrak{apr}}_{\Omega_p}(X) \cup \overline{\mathfrak{apr}}_{\Omega_p}(Y)$.*

Lemma 3. *Let $(\mathfrak{U}, \Omega_p)$ be a preapproximation space. Then,*
(i) If $\eqsim$ is a congruence on $(\mathscr{P}(\mathfrak{U}), \cap)$ and $X \eqsim Y$, then $X \wedge Z \eqsim Y \wedge Z$.
(ii) If $\simeq$ is a congruence on $(\mathscr{P}(\mathfrak{U}), \cup)$ and $X \simeq Y$, then $X \vee Z \simeq Y \vee Z$.
(iii) If $X \eqsim Z$ and $X \leq Z \leq Y$, then $X \eqsim Z$.
(iv) If $X \simeq Z$ and $X \leq Z \leq Y$, then $Y \simeq Z$, $\forall\ X, Y, Z \in \mathscr{P}(\mathfrak{U})$.

Proof. (i) Assume that $\eqsim$ is a congruence on $(\mathscr{P}(\mathfrak{U}), \cap)$. If $X \eqsim Y$, then $Z \eqsim Z$ and so $X \wedge Z \eqsim Y \wedge Z$, because $X \eqsim Y$. Hence, $\underline{\mathfrak{apr}}_{\Omega}(X) = \underline{\mathfrak{apr}}_{\Omega_p}(Y)$ and so $\underline{\mathfrak{apr}}_{\Omega_p}(Z) = \underline{\mathfrak{apr}}_{\Omega_p}(Z)$, $\underline{\mathfrak{apr}}_{\Omega_p}(X \wedge Z) = \underline{\mathfrak{apr}}_{\Omega_p}(X) \wedge \underline{\mathfrak{apr}}_{\Omega_p}(Z) = \underline{\mathfrak{apr}}_{\Omega_p}(Y) \wedge \underline{\mathfrak{apr}}_{\Omega_p}(Z) = \underline{\mathfrak{apr}}_{\Omega_p}(Y \wedge Z)$. Then, $X \wedge Z \eqsim Y \wedge Z$.
(ii) Similar to (i).
(iii) Since $X \leq Z \leq Y$, then $X = X \wedge Z$ and $Z = Y \wedge Z$. If $X \eqsim Y$, then $X \wedge Z \eqsim Y \wedge Z$. Therefore, $X \eqsim Z$.
(iv) The proof is true for $\simeq$ by replacing every $\wedge$ by $\vee$ in (iii).

Theorem 4. *Let $\simeq$ be a congruence on $(\mathscr{P}(\mathfrak{U}), \cup)$. Then,*
(i)If $(\mathscr{P}(\mathfrak{U})/\approx, \vee)$ is a join semilattice, then a quotient map q from $\mathscr{P}(\mathfrak{U})$ into $\mathscr{P}(\mathfrak{U})/\approx$ and is defined by $q(A) = [A]_{\Theta}$ is a join homomorphism.
(ii)If congruence Θ is a bottom rough, then q from $\mathscr{P}(\mathfrak{U})$ into $\mathscr{P}(\mathfrak{U})/\Theta$ is a meet homomorphism.

Proof. (i) It is clear that $(\mathscr{P}(\mathfrak{U})/\approx, \vee)$ is a join semilattice. The map q is a join homomorphism of $\mathscr{P}(\mathfrak{U})$ onto $\mathscr{P}(\mathfrak{U})/\approx$, for A, B in $P(\mathfrak{U})$, $q(A) = [A]_{\approx}$, $q(B) = [B]_{\approx}$, $q(A \vee B) = [A \vee B]_{\approx} = [A]_{\approx} \vee [B]_{\approx} = q(A) \vee q(B)$. Thus, q is a join homomorphism.
(ii) is similar to (i).

3.2 Relation between prerough inclusion and lattices

There are six types of inclusion based on upper and lower approximations that applied on preapproximation spaces.

Definition 12. *$\forall\, A, B \subseteq \mathfrak{U}$, the relations are*
(i) $A \underset{\sim}{\subset} B$ if $\underline{\mathfrak{apr}}_{\Omega}(A) \subseteq \underline{\mathfrak{apr}}_{\Omega}(B)$.
(ii) $A \overset{\sim}{\subset} B$ if $\overline{\mathfrak{apr}}_{\Omega}(A) \subseteq \overline{\mathfrak{apr}}_{\Omega}(B)$.
(iii) $A \equiv B$ if $\underline{\mathfrak{apr}}_{\Omega}(A) \subseteq \underline{\mathfrak{apr}}_{\Omega}(B)$ and $\overline{\mathfrak{apr}}_{\Omega}(A) \subseteq \overline{\mathfrak{apr}}_{\Omega}(B)$.
(iv) $A \underset{\sim}{\subset}_p B$ if $\underline{\mathfrak{apr}}_{\Omega_p}(A) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(B)$.
(v) $A \overset{\sim}{\subset}_p B$ if $\overline{\mathfrak{apr}}_{\Omega_p}(A) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(B)$.
(vi) $A \equiv_p B$ if $\underline{\mathfrak{apr}}_{\Omega_p}(A) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(B)$ and $\overline{\mathfrak{apr}}_{\Omega_p}(A) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(B)$.

To avoid a confusion in Definition 12, Ω is a Pawlak equivalence relation and Ω_p is a relation that forms a preapproximation space.

Remark 5. *If $(\mathfrak{U}, \Omega_p)$ be a preapproximation space, then the relations in Definition 12 are partially ordered in $\mathscr{P}(\mathfrak{U})$. Moreover, each of $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subset})$, $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$, $(\mathscr{P}(\mathfrak{U}), \equiv)$, $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subset}_p)$, $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset}_p)$ and $(\mathscr{P}(\mathfrak{U}), \equiv_p)$ is a lattice.*

Proposition 6. *Each of lattices $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subset})$ and $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$ are sublattices of $(\mathscr{P}(\mathfrak{U}), \subseteq)$.*

Proof. Firstly, for any $X, Y \subseteq \mathscr{P}(\mathfrak{U})$, suppose that $\underline{\mathfrak{apr}}_{\Omega_p}(X)$, $\underline{\mathfrak{apr}}_{\Omega_p}(Y)$ are subsets of $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subset})$. Then, $\underline{\mathfrak{apr}}_{\Omega_p}(X) \wedge \underline{\mathfrak{apr}}_{\Omega_p}(Y) = \underline{\mathfrak{apr}}_{\Omega_p}(X \wedge Y)$ which implies $\underline{\mathfrak{apr}}_{\Omega_p}(X) \wedge \underline{\mathfrak{apr}}_{\Omega_p}(Y) \in (\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subset})$. Now, we show that $\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y) = \underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y))$, $\underline{\mathfrak{apr}}_{\Omega_p}(X) \leq \underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y)$ and $\underline{\mathfrak{apr}}_{\Omega_p}(X) = \underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y))$. Similarly, $\underline{\mathfrak{apr}}_{\Omega_p}(Y) \leq \underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y))$ is proved. Thus, $\underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y))$ is an upper bound of $\underline{\mathfrak{apr}}_{\Omega_p}(X)$ and $\underline{\mathfrak{apr}}_{\Omega_p}(Y)$. Therefore, $\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y) \leq \underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y))$. Secondly, since $\underline{\mathfrak{apr}}_{\Omega_p}(X) \leq X$, then $\underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y)) \leq \underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y)$. Then, $\underline{\mathfrak{apr}}_{\Omega_p}(\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y)) = \underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y)$ and so $\underline{\mathfrak{apr}}_{\Omega_p}(X) \vee \underline{\mathfrak{apr}}_{\Omega_p}(Y) \in (\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subset})$. In the same manner, $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$ is sublattices of $(\mathscr{P}(\mathfrak{U}), \subseteq)$.

Example 3. Let $\mathfrak{U} = \{\alpha, \beta, \gamma\}$ with a relation Ω defined as $\Omega = \{(\alpha, \alpha), (\beta, \alpha), (\beta, \gamma), (\gamma, \gamma)\}$. Then, the topology which associated with R is $\tau = \{\phi, \{\alpha\}, \{\gamma\}, \{\alpha, \gamma\}, \mathfrak{U}\}$. The lattice of $(\mathscr{P}(\mathfrak{U}), \subseteq)$ is shown in Figure 2. From Table 1 and Figures 3 and 4, each of lattices $(P(\mathfrak{U}), \underset{\sim}{\subset})$ and $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$ is sublattices of $(\mathscr{P}(\mathfrak{U}), \subseteq)$. Also, from Figures 3 and 4, we show that $X \underset{\sim}{\subset} Y$ if $\underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y)$ and $X \overset{\sim}{\subset} Y$ if $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$ (cf. Definition 12).

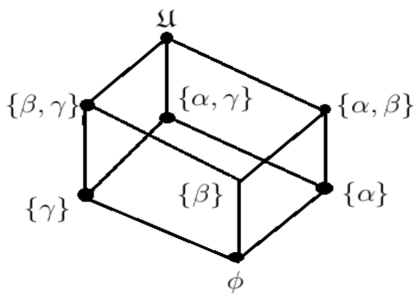

Fig. 2: The lattice of $(\mathscr{P}(\mathfrak{U}),\subseteq)$.

Table 1: The approximations of $\mathscr{P}(\mathfrak{U})$

A	$\overline{\mathfrak{apr}}_\Omega(A)$	$\underline{\mathfrak{apr}}_\Omega(A)$
$\{\alpha\}$	$\{\alpha,\beta\}$	$\{\alpha\}$
$\{\beta\}$	$\{\beta\}$	ϕ
$\{\gamma\}$	$\{\beta,\gamma\}$	$\{\gamma\}$
$\{\alpha,\beta\}$	$\{\alpha,\beta\}$	$\{\alpha\}$
$\{\alpha,\gamma\}$	$\mathfrak{U}$	$\{\alpha,\beta\}$
$\{\beta,\gamma\}$	$\{\beta,\gamma\}$	$\{\gamma\}$
ϕ	ϕ	ϕ
$\mathfrak{U}$	$\mathfrak{U}$	$\mathfrak{U}$

Remark 6. *Each of relations $\approx$ and $\approx_{\Omega_p}$ is equivalence, but not usually congruences on $(\mathscr{P}(\mathfrak{U}),\cup)$. This can be shown in Figures 3 and 4 in Example 3.*

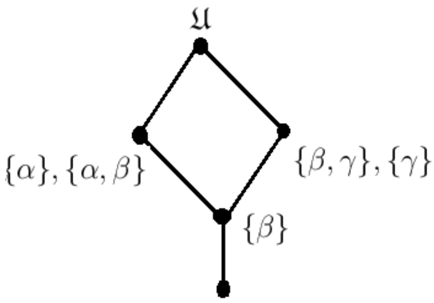

Fig. 3: A sublattice on $\mathscr{P}(\mathfrak{U})$ if $\underline{\mathfrak{apr}}_\Omega(X)\subseteq\underline{\mathfrak{apr}}_\Omega(Y)$.

Example 4. Consider a universal set $\mathfrak{U}=\{\mathfrak{x},\mathfrak{y},\mathfrak{z}\}$ with a relation $\Omega_p=\{(\mathfrak{x},\mathfrak{x}),\ (\mathfrak{y},\mathfrak{x}),(\mathfrak{y},\mathfrak{y})\}$. Then, the topology will be $\tau=\{\{\mathfrak{x}\},\{\mathfrak{x},\mathfrak{y}\},\ \mathfrak{U},\ \phi\}$. By Table 2, the lattices which are given from relations $\underset{\sim}{\subsetneq}$, $\overset{\sim}{\subset}$, $\underset{\sim}{\subsetneq}_p$ and $\overset{\sim}{\subset}_p$ are deduced. Since there are some elements which have the same approximation (upper or lower), then we give only one chain. So, there are four cases:

Case 1: $X\overset{\sim}{\subset}Y$ if $\overline{\mathfrak{apr}}_\Omega(X)\subseteq\overline{\mathfrak{apr}}_\Omega(Y)$ and all congruences on chain lattice are shown in Figure 5. Theses congruences are ordered by normal inclusion such that $\theta_i\leq\theta_j$ iff $\theta_i\subseteq\theta_j$, for $i\neq j$ and $i,j\in\{1,2\cdots,6\}$. This can be shown in Figure 6.

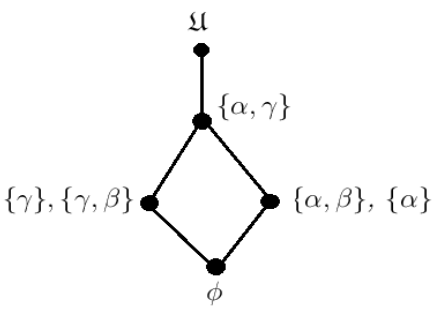

Fig. 4: A sublattice on $\mathscr{P}(\mathfrak{U})$ if $\overline{\mathfrak{apr}}_\Omega(X)\subseteq\overline{\mathfrak{apr}}_\Omega(Y)$.

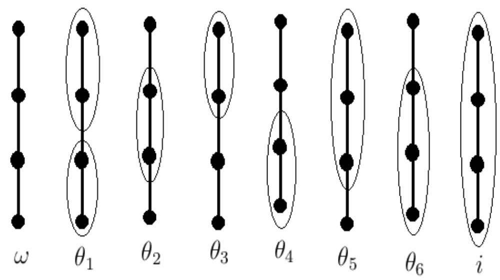

Fig. 5: Congruence lattices.

Case 2: $X\underset{\sim}{\subsetneq}Y$ iff $\overline{\mathfrak{apr}}_\Omega(X)\subseteq\overline{\mathfrak{apr}}_\Omega(Y)$. By similarity, chain lattice and congruence lattices are also shown in Figure 5.

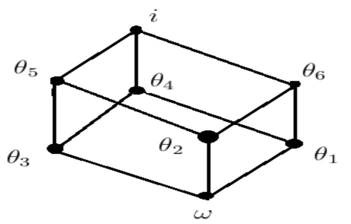

Fig. 6: Congruence with normal inclusion.

Table 2: The preapproximations of $\mathscr{P}(\mathfrak{U})$

A	$\overline{\mathfrak{apr}}_{\Omega}(A)$	$\underline{\mathfrak{apr}}_{\Omega}(A)$	$\overline{\mathfrak{apr}}_{\Omega_p}(A)$	$\underline{\mathfrak{apr}}_{\Omega_p}(A)$
$\{\mathfrak{x}\}$	$\mathfrak{U}$	$\{\mathfrak{x}\}$	$\mathfrak{U}$	$\{\mathfrak{x}\}$
$\{\mathfrak{y}\}$	$\{\mathfrak{y},\mathfrak{z}\}$	ϕ	$\{\mathfrak{y}\}$	ϕ
$\{\mathfrak{z}\}$	$\{\mathfrak{z}\}$	ϕ	$\{\mathfrak{z}\}$	ϕ
$\{\mathfrak{x},\mathfrak{y}\}$	$\mathfrak{U}$	$\{\mathfrak{x},\mathfrak{y}\}$	$\mathfrak{U}$	$\{\mathfrak{x},\mathfrak{y}\}$
$\{\mathfrak{x},\mathfrak{z}\}$	$\mathfrak{U}$	$\{\mathfrak{x}\}$	$\mathfrak{U}$	$\{\mathfrak{x},\mathfrak{z}\}$
$\{\mathfrak{y},\mathfrak{z}\}$	$\{\mathfrak{y},\mathfrak{z}\}$	ϕ	$\{\mathfrak{y},\mathfrak{z}\}$	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ
$\mathfrak{U}$	$\mathfrak{U}$	$\mathfrak{U}$	$\mathfrak{U}$	$\mathfrak{U}$

Case 3: $X \underset{\sim}{\subseteq}_p Y$ if $\underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y)$.

Case 4: $X \overset{\sim}{\subset}_p Y$ if $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$.

Theorem 7. *$(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$ is a sublattice of $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq}_p)$.*

Proof. Suppose that $\underline{\mathfrak{apr}}_{\Omega_p}(X)$ and $\underline{\mathfrak{apr}}_{\Omega_p}(Y)$ are subsets of $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$. Obviously, $\underline{\mathfrak{apr}}_{\Omega_p}(X) \wedge \underline{\mathfrak{apr}}_{\Omega_p}(Y) = \underline{\mathfrak{apr}}_{\Omega_p}(X \wedge Y)$ which implies that $\underline{\mathfrak{apr}}_{\Omega_p}(X) \wedge \underline{\mathfrak{apr}}_{\Omega_p}(Y) \in (\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$. Now, we prove that each of $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$ and $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq}_p)$ is dually order isomorphic. This means that there is a lattice isomorphism $\cong_f$, where f is an order isomorphism.

The proof of Theorem 8 similar to Theorem 7. Hence, the proof is omitted.

Theorem 8. *$(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$ is a sublattice of $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq}_p)$.*

From Theorems 7 and 8, Proposition 7 is given.

Proposition 7. *Let $(\mathfrak{U}, \Omega_p)$ be a preapproximation space. Then, $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq}) \cong (\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$.*

Proof. We prove that $f : \overline{\mathfrak{apr}}_{\Omega_p}(X) \longrightarrow \underline{\mathfrak{apr}}_{\Omega_p}(X')$, where X' is the complement of X in $\mathscr{P}(\mathfrak{U})$, is a dual order isomorphism. Firstly, It is clear that f is onto, so we prove that f is embedding. Consider $X \overset{\sim}{\subset} Y$ s.t. $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$ and so $\mathfrak{cl}(X) \subseteq \mathfrak{cl}(Y)$. This means that $M \cap X \neq \phi$ and so $M \cap Y \neq \phi$, $\forall\, M \in \tau$. Now, assume that $\underline{\mathfrak{apr}}_{\Omega_p}(Y') \not\subseteq \underline{\mathfrak{apr}}_{\Omega_p}(X')$. Then, $\exists$ an open set $N \in \tau$ s.t. $N \subseteq X'$ (take $N = \mathfrak{int}(X')$). So, $N \subseteq X'$, but $N \not\subseteq \underline{\mathfrak{apr}}_{\Omega_p}(X')$ which is equivalent to $M \cap X \neq \phi$ and so $N \cap Y \neq \phi$. This means that $N \not\subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y')$, which gives a contradiction. Hence, $\underline{\mathfrak{apr}}_{\Omega_p}(Y') \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(X')$ and so $Y' \underset{\sim}{\subseteq} X'$. Secondly, assume that $\underline{\mathfrak{apr}}_{\Omega_p}(Y') \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(X')$, which means that $\mathfrak{int}(Y') \subseteq \mathfrak{int}(X')$. Suppose that $\overline{\mathfrak{apr}}_{\Omega_p}(X) \not\subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$, which means that $\exists\ M \in \tau$ s.t. $M \cap X \neq \phi$ and $M \cap Y = \phi$, but this implies that $M \subseteq Y'$ and $M \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y') \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(X')$. Then, $M \subseteq X'$, this equivalent to $M \cap X = \phi$, which give a contradiction with our assumption. Therefore, $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$ and so $X \overset{\sim}{\subset} Y$.

By Proposition 7, $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$ and $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$ are called dually isomorphic.

Example 5. (Continued for Example 3)
The lattices $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$ are dual order isomorphic. Also, the interior of any set is equal to its preinterior and also the closure of any subset is the preclosure. Then, the lattices $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$ and $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq}_p)$ are coincide. Similarly, $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$ and $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset}_p)$ are the same. It is noted that $X \overset{\sim}{\subset} Y$ if $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$ is the same with $X \overset{\sim}{\subset}_p Y$ if $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$. Also, $X \underset{\sim}{\subseteq} Y$ if $\underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y)$ is the same with $X \underset{\sim}{\subseteq}_p Y$ if $\underline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(Y)$. This can be shown in Figures 3 and 4. The lattices are equal.

Corollary 1. *If $\mathfrak{int}(A) = \mathfrak{pint}(A)$ and $\mathfrak{cl}(A) = \mathfrak{pcl}(A)$, for any $A \subseteq \mathfrak{U}$ in any preapproximation space, then the lattices $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$ and $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq}_p)$ are the same and also the lattices $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$ and $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset}_p)$.*

Corollary 2. *The lattices $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$, $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$, $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq}_p)$ and $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset}_p)$ are distributive. But, it is not Boolean lattices.*

Proposition 8. *(i) Every ideal in $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$ is an ideal in $(\mathscr{P}(\mathfrak{U}), \subseteq)$.*
(ii) Every filter in $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset}_p)$ is a filter in $(\mathscr{P}(\mathfrak{U}), \subseteq)$.

Proof. (i) Let $\mathscr{I}_0$ be an ideal in $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$. If $X \in \mathscr{I}_0$, $Y \leq X$ in $(\mathscr{P}(\mathfrak{U}), \subseteq)$, then we prove that $Y \in \mathscr{I}_0$, since $Y \leq X$ in $(\mathscr{P}(\mathfrak{U}), \subseteq)$, i.e. $Y \subseteq X$. Then, $\underline{\mathfrak{apr}}_{\Omega_p}(Y) \subseteq \underline{\mathfrak{apr}}_{\Omega_p}(X)$. Thus, $Y \underset{\sim}{\subseteq} X \in I_0$, but $\mathscr{I}_0$ is an ideal in $(\mathscr{P}(\mathfrak{U}), \underset{\sim}{\subseteq})$. Therefore, $\mathscr{I}_0$ is an ideal $(\mathscr{P}(\mathfrak{U}), \subseteq)$.

(ii) Let $\mathscr{F}_0$ be a filter in $(\mathscr{P}(\mathfrak{U}), \overset{\sim}{\subset})$. If $x \in \mathscr{F}_0$ and $Y \geq X$ in $(\mathscr{P}(\mathfrak{U}), \subseteq)$, then $Y \supseteq X$. We prove that $Y \in \mathscr{F}_0$. Since $X \subseteq Y$, $\overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(Y)$, $X \in \mathscr{F}_0$ and $\mathscr{F}_0$ is a filter, then $Y \in \mathscr{F}_0$. Therefore, $\mathscr{F}_0$ is a filter in $(P(\mathfrak{U}), \subseteq)$.

3.3 The matroid representation of a Boolean lattice

Definition 13. *The interior operator on a lattice* $(\mathscr{L},\wedge,\vee)$ *is* $\mathfrak{int}_{\mathscr{L}}(\mathfrak{x})=\vee\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}<\mathfrak{x}\}$. *The following for any* $\mathfrak{x},\mathfrak{y}\in\mathscr{L}$ *hold*
(i) $\mathfrak{int}_{\mathscr{L}}(\mathfrak{x}\wedge\mathfrak{y})=\mathfrak{int}_{\mathscr{L}}(\mathfrak{x})\wedge\mathfrak{int}_{\mathscr{L}}(\mathfrak{y})$.
(ii) $\mathfrak{int}_{\mathscr{L}}(\mathfrak{x})\leq\mathfrak{x}$.
(iii) $\mathfrak{int}_{\mathscr{L}}(\mathfrak{x})=\mathfrak{int}_{\mathscr{L}}\,(\mathfrak{int}_{\mathscr{L}}(\mathfrak{x}))$.

Definition 14. *The closure operator in* $(\mathscr{L},\ \wedge,\vee)$ *is* $\mathfrak{cl}_{\mathscr{L}}(\mathfrak{x})=(\mathfrak{int}_{\mathscr{L}}(\mathfrak{x}^c))^c$ *where* $\mathfrak{x}^c$ *is a complement of x w.r.to* $\mathscr{L}$. *Thus,* $\mathfrak{cl}_{\mathscr{L}}(\mathfrak{x})=(\mathfrak{int}_{\mathscr{L}}(\mathfrak{x}^c))^c=(\vee\{\mathfrak{a}\in L|\mathfrak{a}<\mathfrak{x}^c\})^c=\wedge\{\mathfrak{a}\in\mathscr{L}|\mathfrak{a}>\mathfrak{x}\}$.

Example 6. Let $\mathscr{L}=M_3=1\oplus\overline{3}\oplus 1$ be shown in Figure 7. Then, $\mathfrak{int}_{\mathscr{L}}(\mathfrak{a})=\vee\{0\}=\{0\}$, $\mathfrak{int}_{\mathscr{L}}(\mathfrak{b})=\{0\}$, $\mathfrak{int}_{\mathscr{L}}(\mathfrak{c})=\{0\}$, $\mathfrak{cl}_{\mathscr{L}}(\mathfrak{a})=\wedge\{1\}=\{1\}$, $\mathfrak{cl}_{\mathscr{L}}(\mathfrak{b})=\mathfrak{cl}_{\mathscr{L}}(\mathfrak{c})=\{1\}$, $\mathfrak{int}_{\mathscr{L}}(0)=\mathfrak{cl}_{\mathscr{L}}(0)=\{0\}$ and $\mathfrak{int}_{\mathscr{L}}\{1\}=\mathfrak{cl}_{\mathscr{L}}\{1\}=\{1\}$.

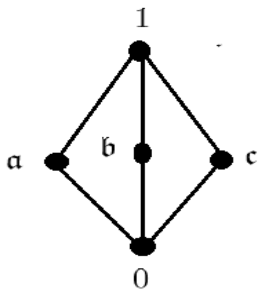

Fig. 7: Interior and closure operators on a lattice.

Definition 15. *The lower and upper preapproximation of* $\mathfrak{a}\in\mathscr{L}$ *is*
$\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{a})=\mathfrak{int}_{\mathscr{L}}(\mathfrak{a})=\vee\{\mathfrak{a}\in\mathscr{L}|\mathfrak{a}<\mathfrak{x}\}$,
$\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{a})=\mathfrak{cl}_{\mathscr{L}}(\mathfrak{a})=\wedge\{\mathfrak{a}\in\mathscr{L}|\mathfrak{a}>\mathfrak{x}\}$*), respectively.*

Example 7. In Figure 8, let $\mathfrak{U}=\{1,2,3\}$ and $\mathscr{L}=(\mathscr{P}(\mathfrak{U}),\ \subseteq)$ be the house diagram lattice. Then, $\underline{\mathfrak{apr}}_{\Omega_p}(\{1\})=\phi$, $\overline{\mathfrak{apr}}_{\Omega_p}(\{1\})=\{1\}$, $\underline{\mathfrak{apr}}_{\Omega_p}(\{2\})=\phi$, $\overline{\mathfrak{apr}}_{\Omega_p}(\{2\})=\{2\}$, $\underline{\mathfrak{apr}}_{\Omega_p}(\{3\})=\phi$, $\overline{\mathfrak{apr}}_{\Omega_p}(\{3\})=\{3\}$, $\underline{\mathfrak{apr}}_{\Omega_p}(\{1,2\})=\{1,2\}$, $\overline{\mathfrak{apr}}_{\Omega_p}(\{1,2\})=\mathfrak{U}$, $\underline{\mathfrak{apr}}_{\Omega_p}(\{1,3\})=\{1,3\}$, $\overline{\mathfrak{apr}}_{\Omega_p}(\{1,3\})=\mathfrak{U}$, $\underline{\mathfrak{apr}}_{\Omega_p}(\{2,3\})=\{2,3\}$, $\overline{\mathfrak{apr}}_{\Omega_p}(\{2,3\})=\mathfrak{U}$, $\underline{\mathfrak{apr}}_{\Omega_p}(\{\phi\})=\phi$ and $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{U})=\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{U})=\mathfrak{U}$.

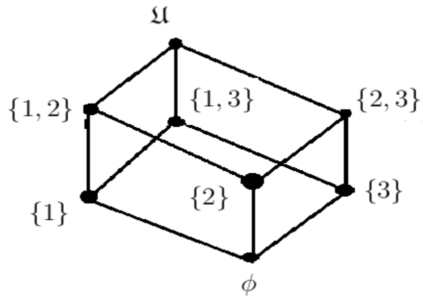

Fig. 8: A house diagram lattice.

Definition 16. $\mathfrak{a}\in\mathscr{L}$ *is called to be preexact if* $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{a})=\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{a})$. *Otherwise, it is called prerough.*

Example 8. In a lattice in Figure 8 and Example 4, ϕ and $\mathfrak{U}$ are preexact elements. Other elements are prerough.

Remark 9. *From Definition 12,*
(i) if $\underline{\mathfrak{apr}}_{\Omega}(X)=\underline{\mathfrak{apr}}_{\Omega}(Y)$*, then each set in* $\mathscr{L}$ *is preopen.*
(ii) if $\overline{\mathfrak{apr}}_{\Omega}(X)=\overline{\mathfrak{apr}}_{\Omega}(Y)$*, then each set in* $\mathscr{L}$ *is preclosed.*
(ii) if $\underline{\mathfrak{apr}}_{\Omega}(X)=\underline{\mathfrak{apr}}_{\Omega}(Y)$ *and* $\overline{\mathfrak{apr}}_{\Omega}(X)=\overline{\mathfrak{apr}}_{\Omega}(Y)$*, then each set in* $\mathscr{L}$ *is both preopen and preclosed. Moreover, all elements of lattices are preexact.*

Lemma 4. *Let* $\mathscr{L}$ *be a complete Boolean lattice. Then, for any* $\mathfrak{x},\mathfrak{y}\in\mathscr{L}$
(i) $\underline{\mathfrak{apr}}_{\Omega_p}(0)=\overline{\mathfrak{apr}}_{\Omega_p}(0)=0$ *and* $\underline{\mathfrak{apr}}_{\Omega_p}(1)=\overline{\mathfrak{apr}}_{\Omega_p}(1)=1$.
(ii) $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})\leq\mathfrak{x}\leq\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})$.
(iii) If $\mathfrak{x}\leq\mathfrak{y}$*, then* $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})\leq\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{y})$.

Proof. (i) Since 0 is the least element in $\mathscr{L}$, then the $\underline{\mathfrak{apr}}_{\Omega_p}(0)=0$. Also, since $\overline{\mathfrak{apr}}_{\Omega_p}(0)=\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}>0\}=0$, then $\overline{\mathfrak{apr}}_{\Omega_p}(0)=0$. The second part of (i) have the same manner.
(ii) Let $\alpha\in\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})$. Then, $\alpha\in\vee\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}<\mathfrak{x}\}$. Thus, $\exists\ \mathfrak{a}_0\in\mathscr{L}$ s.t. $\alpha\leq\mathfrak{a}_0$, but $\mathfrak{a}_0<x$ and so $\alpha\leq\mathfrak{x}$. Hence, $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})\leq\mathfrak{x}$. Also, since $\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})=\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}>\mathfrak{x}\}$, then $\mathfrak{x}<\mathfrak{a}$, $\forall\ \mathfrak{a}\in\mathscr{L}$. Therefore, $\mathfrak{x}\leq\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}>\mathfrak{x}\}=\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})$. Hence, $\mathfrak{x}\leq\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})$.
(iii) Let $\mathfrak{x}\leq\mathfrak{y}$. Then, $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})=\vee\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}<\mathfrak{x}\}$, but $\mathfrak{x}<\mathfrak{y}$. Then, $\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}<\mathfrak{x}\}\leq\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}<\mathfrak{y}\}$. Therefore, $\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})\leq\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{y})$. Also, $\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{y})=\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}>\mathfrak{y}\}$, but $\mathfrak{x}<\mathfrak{y}$, and so $\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}>\mathfrak{y}\}\geq\wedge\{\mathfrak{a}\in\mathscr{L}:\mathfrak{a}>\mathfrak{x}\}$. Hence, $\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{y})\geq\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x})$. By Proposition 7, it is noted that the $\underline{\mathfrak{apr}}_{\Omega_p}$ and $\overline{\mathfrak{apr}}_{\Omega_p}$ are order preserving, $\forall\,A\subseteq\mathscr{L}$, since $\underline{\mathfrak{apr}}_{\Omega_p}(A)=\{\underline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x}):\mathfrak{x}\in A\}$ and $\overline{\mathfrak{apr}}_{\Omega_p}(A)=\{\overline{\mathfrak{apr}}_{\Omega_p}(\mathfrak{x}):\mathfrak{x}\in A\}$.

Proposition 9. *Let* $\mathscr{B}$ *be a complete Boolean lattice. Then,*
(i) $\vee\overline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})=\overline{\mathfrak{apr}}_{\Omega_p}(\vee\mathscr{S})$, $\forall\,\mathscr{S}\subseteq\mathscr{B}$,

(ii) $\wedge\underline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S}) = \underline{\mathfrak{apr}}_{\Omega_p}(\wedge\mathscr{S})\ \forall\ \mathscr{S}\subseteq\mathscr{B}$.

Proof. (i) Firstly, let $\mathscr{S}\subseteq\mathscr{B}$. A function $\overline{\mathfrak{apr}}_{\Omega_p}: \mathscr{B}\to\mathscr{B}$ is in order preserving, since $\mathscr{S}\leq\vee\mathscr{S}$. Thus, $\overline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})\subseteq\overline{\mathfrak{apr}}_{\Omega_p}(\vee\mathscr{S})$, and so $\vee\overline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})\subseteq\overline{\mathfrak{apr}}_{\Omega_p}(\vee\mathscr{S})$. On the other hand, $\overline{\mathfrak{apr}}_{\Omega_p}(\vee\mathscr{S}) = \wedge\{\alpha\in\mathscr{B}:\alpha>\vee\mathscr{S}\}\leq\wedge\{\bigcup_{x\in\mathscr{S}}\{\alpha\in\mathscr{B}:\alpha>x\}\}$ $=\bigvee_{x\in\mathscr{S}}\{\wedge\{\alpha\in\mathscr{B}:\alpha>x\}\} = \vee\{\overline{\mathfrak{apr}}_{\Omega_p}(x): x\in\mathscr{S}\} = \vee\overline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})$. Therefore, $\overline{\mathfrak{apr}}_{\Omega_p}(\vee\mathscr{S}) = \vee\overline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})$.
(ii) Let $\mathscr{S}\subseteq\mathscr{B}$ and a map $\underline{\mathfrak{apr}}_{\Omega_p}:\mathscr{B}\to\mathscr{B}$ be preserving. Since $\wedge\mathscr{S}\leq\mathscr{S}$, $\forall\ \mathscr{S}\subseteq\mathscr{B}$, then $\underline{\mathfrak{apr}}_{\Omega_p}(\wedge\mathscr{S})\leq\underline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})$. Thus, $\underline{\mathfrak{apr}}_{\Omega_p}(\wedge\mathscr{S})\leq\underline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})$. On the other hand, $\underline{\mathfrak{apr}}_{\Omega_p}(\wedge\mathscr{S}) = \vee\{\alpha\in\mathscr{B}:\alpha<\wedge\mathscr{S}\}$ $\geq\vee\{\bigcap_{x\in\mathscr{S}}\{\alpha\in\mathscr{B}:\alpha<x\}\} = \bigwedge_{x\in\mathscr{S}}\{\vee\{\alpha\in\mathscr{B}:\alpha<x\}\}$ $=\wedge\{\underline{\mathfrak{apr}}_{\Omega_p}(x), x\in\mathscr{S}\} = \wedge\underline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})$. Therefore, $\underline{\mathfrak{apr}}_{\Omega_p}(\wedge\mathscr{S}) = \wedge\underline{\mathfrak{apr}}_{\Omega_p}(\mathscr{S})$.

Definition 17. *Let a, b be two elements in $\mathscr{L}$. Define*
(i) $a\preccurlyeq b$ if $\underline{\mathfrak{apr}}_{\Omega}(a)\subseteq\underline{\mathfrak{apr}}_{\Omega}(b)$ and $\preccurlyeq$ is called rough bottom order.
(ii) $a\curlyeqprec b$ if if $\overline{\mathfrak{apr}}_{\Omega}(a)\subseteq\overline{\mathfrak{apr}}_{\Omega}(b)$ and $\curlyeqprec$ is called rough top order.
(iii) $a = b$ if $\underline{\mathfrak{apr}}_{\Omega}(a)\subseteq\underline{\mathfrak{apr}}_{\Omega}(b)$ and $\overline{\mathfrak{apr}}_{\Omega}(a)\subseteq\overline{\mathfrak{apr}}_{\Omega}(b)$, and $=$ is called rough order.
(iv) $a\preccurlyeq_p b$ if $\underline{\mathfrak{apr}}_{\Omega_p}(a)\subseteq\underline{\mathfrak{apr}}_{\Omega_p}(b)$ and $\preccurlyeq_p$ is called prerough bottom order.
(v) $a\curlyeqprec_p b$ if $\overline{\mathfrak{apr}}_{\Omega_p}(a)\subseteq\overline{\mathfrak{apr}}_{\Omega_p}(b)$ and $\curlyeqprec_p$ is called prerough top order.
(vi) $a =_p b$ if $\underline{\mathfrak{apr}}_{\Omega_p}(a)\subseteq\underline{\mathfrak{apr}}_{\Omega_p}(b)$ and $\overline{\mathfrak{apr}}_{\Omega_p}(a)\subseteq\overline{\mathfrak{apr}}_{\Omega_p}(b)$, and $=_p$ is called prerough order.

Proposition 10. *Let $(B,\subseteq)$ be a complete Boolean lattice. Then, the following hold*
(i) Each of $(\mathscr{P}(B),\wedge)$ and $(\mathscr{P}(B),\vee)$ is a complete lattice.
(ii) A relation $\simeq$ (resp. $\overline{\simeq}$) of a map $\underline{\mathfrak{apr}}_{\Omega}$(resp. $\overline{\mathfrak{apr}}_{\Omega}$): $B\to B$ is a congruence on $(B,\wedge)$ (resp. $(B,\vee)$).

Proof. (i) Follows by Proposition 9 (i) and (ii).
(ii) It is seen that $\simeq$ is an equivalence on B. If $a,b,c,d\in B$ and assume that $a\simeq b$ and $c\simeq d$, then $\underline{\mathfrak{apr}}_{\Omega}(a\wedge c) = \underline{\mathfrak{apr}}_{\Omega}(a)\wedge\underline{\mathfrak{apr}}_{\Omega}(c) = \underline{\mathfrak{apr}}_{\Omega}(b)\wedge\underline{\mathfrak{apr}}_{\Omega}(d) = \underline{\mathfrak{apr}}_{\Omega}(b\wedge d)$. Thus, $\simeq$ is a congruence on $(B,\wedge)$. $\overline{\simeq}$ has a similar proof.

Remark 10. *The proofs of Propositions 9, 10 and 7 are true on topological lattices which are generated by preinterior or preclosure operators $\mathscr{L}$.*

Definition 18. *Let 0 be the least in $\mathscr{L}$. $\mathfrak{a}$ is an atom in $\mathscr{L}$ if $0<\mathfrak{a}$ and the class of atoms is named $\mathscr{A}(\mathscr{L})$. $\mathscr{L}$ is called atomic if $\forall\ \mathfrak{x}\in\mathscr{L}$ is a spermium of all atoms. The pair $(\mathscr{P}(\mathfrak{U}),\subseteq)$ is a complete atomic Boolean lattice in which each atom can be approached to an element of $\mathfrak{U}$. The map $\varphi:\mathfrak{U}\to\mathscr{P}(\mathfrak{U})$ with $\mathfrak{x}\to[x]_{\approx}$ is called rough equality and also has $\varphi:\mathscr{A}(B)\to B$, where $B = (\mathscr{P}(\mathfrak{U}),\subseteq)$.*

Example 9. Let $B = \{0,\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d}, \mathfrak{e},\ \mathfrak{f},\ 1\}$ with an ordered relation $\leq$in Figure 9. The atom set is $\{\mathfrak{a},\ \mathfrak{b}, \mathfrak{c}\}$. Let $\varphi:\mathscr{A}(B)\to B$ be $\varphi(\mathfrak{a}) = \mathfrak{d}$, $\varphi(\mathfrak{b}) = \mathfrak{b}$ and $\varphi(\mathfrak{c}) = \mathfrak{f}$. The approximations are in Table 3. The duality order isomorphic sets $(B,\subseteq)$ and $(B,\curlyeqprec)$ are in Figure 10.

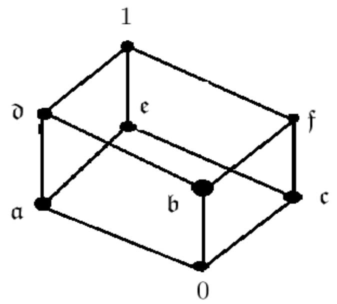

Fig. 9: Complete atomic Boolean lattice.

Table 3: Atoms of a complete atomic Boolean lattice for $\mathscr{B}$

x	$\underline{\mathfrak{apr}}_{\Omega}(x)$	$\overline{\mathfrak{apr}}_{\Omega}(x)$
0	0	0
$\mathfrak{a}$	0	$\mathfrak{a}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{a}\vee\mathfrak{b}\vee\mathfrak{c}=1$
$\mathfrak{c}$	0	$\mathfrak{c}$
$\mathfrak{d}$	$\mathfrak{a}\vee\mathfrak{b}=\mathfrak{d}$	$\mathfrak{a}\vee\mathfrak{b}\vee\mathfrak{c}=1$
$\mathfrak{e}$	0	$\mathfrak{a}\vee\mathfrak{c}=\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{b}\vee\mathfrak{c}=\mathfrak{f}$	$\mathfrak{a}\vee\mathfrak{b}\vee\mathfrak{c}=1$
1	$\mathfrak{a}\vee\mathfrak{b}\vee\mathfrak{c}=1$	$\mathfrak{a}\vee\mathfrak{b}\vee\mathfrak{c}=1$

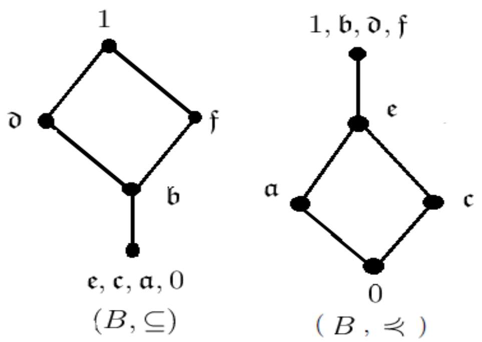

Fig. 10: Duality order isomorphic sets.

Remark 11. *If our approach is used to determine lower and the upper approximations, then the results are given in*

Table 4. The duality order isomorphisms $(B, \subseteq)$ and $(B, \preccurlyeq)$ illustrate in Figure 11.

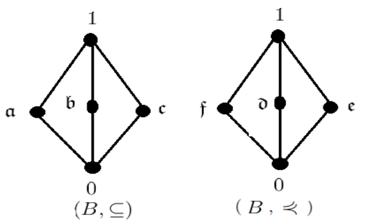

Fig. 11: Duality order isomorphic sets by another approach.

Table 4: Duality order isomorphic sets by another approach

x	$\underline{\mathfrak{apr}}_{\Omega}(x)$	$\overline{\mathfrak{apr}}_{\Omega}(x)$
0	0	0
$\mathfrak{a}$	0	$\mathfrak{a}$
$\mathfrak{b}$	0	$\mathfrak{b}$
$\mathfrak{c}$	0	$\mathfrak{c}$
$\mathfrak{d}$	$\mathfrak{d}$	1
$\mathfrak{e}$	$\mathfrak{e}$	1
$\mathfrak{f}$	$\mathfrak{f}$	1
1	1	1

In the following, the representation of closure is given for matroids that is induced by complete Boolean lattices using the fact in Remark 12.

Remark 12. *In [29], researchers proved that a lattice is a Boolean lattice if it is the open and closed set lattice of matroids. A lattice is a Boolean lattice if it is only closed set lattice of matroids.*

Lemma 5. *Let Ω_p is either reflexive or transitive. Then, $\overline{\mathfrak{apr}}^{n+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$ and $\underline{\mathfrak{apr}}^{n+1}_{\Omega_p}(X) = \underline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$, $\forall X \in \mathscr{P}(\mathfrak{U})$.*

Proof. Firstly, using Proposition 5, we prove that $\overline{\mathfrak{apr}}^{n+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$, $\forall X \in \mathscr{P}(\mathfrak{U})$. Since Ω_p is reflexive, then by Proposition 4(ii), $X \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(X)$. By Proposition 4(i), $X \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{2}_{\Omega_p}(X) \subseteq \cdots \overline{\mathfrak{apr}}^{n-1}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cdots$. Since $|\mathfrak{U}| = n$, then $\exists$ a $k \in \mathbb{N}$ s.t. $\overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k}_{\Omega_p}(X)$. Choose at least $k \leq n$ s.t. $X \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{2}_{\Omega_p}(X) \subseteq \cdots \overline{\mathfrak{apr}}^{k-1}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{k}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X)$ Therefore, $|\overline{\mathfrak{apr}}^{k}_{\Omega_p}(X)| \geq k$ and so $k \leq |\overline{\mathfrak{apr}}^{k}_{\Omega_p}(X)| \leq n$. By a successive of the iteration, $\overline{\mathfrak{apr}}^{k+2}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X)$, $\overline{\mathfrak{apr}}^{k+3}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k+2}_{\Omega_p}(X)$ and so on. By induction for $k \leq n$, $\overline{\mathfrak{apr}}^{n+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$. Secondly, Since Ω_p is transitive and by Proposition 4(ii), then it is sufficient to show that $\overline{\mathfrak{apr}}^{n+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$, $\forall X \in \mathscr{P}(\mathfrak{U})$. Since $\overline{\mathfrak{apr}}_{\Omega_p}(\overline{\mathfrak{apr}}_{\Omega_p}(X)) = \overline{\mathfrak{apr}}^{2}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(X)$. By Proposition 4(i), $\cdots \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{n-1}_{\Omega_p}(X) \subseteq \cdots \subseteq \overline{\mathfrak{apr}}^{3}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{2}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{1}_{\Omega_p}(X)$. Since $|\mathfrak{U}| = n$, then $\exists$ a $k \in \mathbb{N}$ s.t. $\overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k}_{\Omega_p}(X)$. Choose at least $k \leq n$ s.t. $\overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}_{\Omega_p}(X) \subseteq \cdots \subseteq \overline{\mathfrak{apr}}^{3}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{2}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{1}_{\Omega_p}(X)$. If $\overline{\mathfrak{apr}}_{\Omega_p}(X) = \mathfrak{U}$, then $\overline{\mathfrak{apr}}^{2}_{\Omega_p}(X) = \overline{\mathfrak{apr}}_{\Omega_p}(X) = \mathfrak{U}$. Take $k = 1 \leq |\mathfrak{U}| = n$. Otherwise, if $\overline{\mathfrak{apr}}_{\Omega_p}(X) \neq \mathfrak{U}$, then $|\overline{\mathfrak{apr}}_{\Omega_p}(X)| \leq |\mathfrak{U}| = n$ and also $k - 1 \leq |\overline{\mathfrak{apr}}_{\Omega_p}(X)|$. Therefore, $k - 1 \leq |\overline{\mathfrak{apr}}_{\Omega_p}(X)| < |\mathfrak{U}| = n$, that is $k \leq n$ and so $\exists$ $k \in \mathbb{N}$ with $k \leq n$ s.t. $\overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k}_{\Omega_p}(X)$. By a successive of the iteration, $\overline{\mathfrak{apr}}^{k+2}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k+1}_{\Omega_p}(X)$, $\overline{\mathfrak{apr}}^{k+3}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{k+2}_{\Omega_p}(X)$ and so on. By induction for $k \leq n$, $\overline{\mathfrak{apr}}^{n+1}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$.

It is directly deduce Corollary 3 from a successive of iteration $\overline{\mathfrak{apr}}_{\Omega_p}$.

Corollary 3. *Let Ω_p is either reflexive or transitive. Then, $\forall$ $m \geq n$ and $X \subseteq \mathfrak{U}$, $\overline{\mathfrak{apr}}^{m}_{\Omega_p}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$ and $\underline{\mathfrak{apr}}^{m}_{\Omega_p}(X) = \underline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$.*

Proposition 11. *If $(\mathfrak{U}, \Omega_p)$ and $k \in \mathbb{N}$, $k \geq 1$, then $P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p}) \subseteq \{\overline{\mathfrak{apr}}^{k}_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$ and $P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p}) \subseteq \{\underline{\mathfrak{apr}}^{k}_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$.*

Proof. By a definition of $P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$, if $\forall$ $A \in P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$, then $\overline{\mathfrak{apr}}_{\Omega_p}(A) = A$. By Lemma 1, $A = \overline{\mathfrak{apr}}^{k}_{\Omega_p}(A) \in \{\overline{\mathfrak{apr}}^{k}_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$ and so $P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p}) \subseteq \{\overline{\mathfrak{apr}}^{k}_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$. Using the duality, the second part is hold.

Theorem 13. *Let Ω_p is either reflexive or transitive. Then, $P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p}) = \{\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$ and $P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p}) = \{\underline{\mathfrak{apr}}^{n}_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\}$*

Proof. For Ω_p is reflexive and $X \in \mathscr{P}(\mathfrak{U})$, take $A = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$, by Lemma 5, $\overline{\mathfrak{apr}}_{\Omega_p}(A) = A$. Thus, $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) = A \in P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$. This gives $\{\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) : X \in \mathscr{P}(\mathfrak{U})\} \subseteq P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$. The other side is cleared by Proposition 11. Also, for Ω_p is transitive, the proof is straightforward from Lemma 5 and Proposition 11.

Proposition 12. *Let Ω_p is reflexive and $P\mathscr{D}$ $(\underline{\mathfrak{apr}}_{\Omega_p})$ is lattice matroidal closed sets of $\mathscr{M}$, then $\underline{\mathfrak{apr}}^{n}_{\Omega_p} = \mathfrak{cl}_{\mathscr{M}}$.*

Proof. By Theorem 13, we have $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \in P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$. So, $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$ is a closed set of $\mathscr{M}$ and so $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$

$\cap \mathfrak{cl}_{\mathscr{M}}(X)$ is a closed set of $\mathscr{M}$. Therefore, $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X) \in P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$. By Theorem 13, $\exists\ A \subseteq \mathfrak{U}$ s.t. $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(A)$. From Propositions 2 and 5, $X \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X)$. Also, $X \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(A)$. Thus, by Proposition 5 and Corollary 3, $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(\overline{\mathfrak{apr}}^{n}_{\Omega_p}(A)) = \overline{\mathfrak{apr}}^{2n}_{\Omega_p}(A) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(A) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X)$, that is, $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X)$. Therefore, $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \subseteq \mathfrak{cl}_{\mathscr{M}}(X)$. On the other hand, by Proposition 2, $\mathfrak{cl}_{\mathscr{M}}(X) \subseteq \mathfrak{cl}_{\mathscr{M}}(\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X))$. Since $\overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X)$ is a closed set of $\mathscr{M}$, then $\mathfrak{cl}_{\mathscr{M}}(X) \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X) \cap \mathfrak{cl}_{\mathscr{M}}(X)$ and so $\mathfrak{cl}_{\mathscr{M}}(X) \subseteq \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$. Therefore, $\mathfrak{cl}_{\mathscr{M}}(X) = \overline{\mathfrak{apr}}^{n}_{\Omega_p}(X)$. This is true, $\forall\ X \in \mathscr{P}(\mathfrak{U})$ and so $\underline{\mathfrak{apr}}^{n}_{\Omega_p} = \mathfrak{cl}_{\mathscr{M}}$.

4 Conclusions

The mathematical sciences of topology [50], lattice [26], and rough sets [51,8] are concerned with all issues directly or indirectly linked to preapproximations. As a result, lattice theory, rough sets, and topological spaces became the most significant mathematica disciplines. In rough set theory, the aim of study is to extend the lower preapproximation of a nonempty set to itself and to intend the upper preapproximation to the set itself. This means that the boundary region will be empty. There are a modification for Li's study in [29] and proved that a lattice is Boolean if it is only closed set lattice of matroids. So, the value of k that satisfies $\overline{\mathfrak{apr}}^{k}_{\Omega} \in P\mathscr{D}(\overline{\mathfrak{apr}}_{\Omega_p})$ is determined and $\underline{\mathfrak{apr}}^{k}_{\Omega} \in P\mathscr{D}(\underline{\mathfrak{apr}}_{\Omega_p})$. We prove that $\underline{\mathfrak{apr}}^{n}_{\Omega}$ is the closure of a matroid $\mathscr{M}$.

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Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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