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Analytic Continuation of Confluent Hypergeometric Functions and Their Application

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Abstract: In this study, we derive new analytic continuation formulas. Results concerning analytic continuation of hypergeometric functions are very important because these functions frequently arise in mathematics, physics, and engineering. Their series definitions typically converge only in restricted regions like ($|z| < 1$), but many applications require values outside these domains. That is where analytic continuation formulas become useful. This work aims at establishing several analytical continuation relations for confluent hypergeometric functions of one and two variables.

Keywords: Analytic continuation, Confluent hypergeometric function, Humbert functions, Mittag-Leffler function

1 Introductory background

J. D. Rozies and W. R. Johnson [1] proved an analytic continuation of the Lauricella function when one of its variables, say, is greater than unity, and the other two are less than unity. The proven formula was used to solve the problem of electron-nuclear scattering. The analytical continuations given in [2,3] for the Appell F_2 series were used to calculate the Lauricella F_A function, and K. K. Sud et al. [4] developed a method for estimating the radial matrix elements of radiative transitions. In this method, the integrals are expressed through a matrix generalization of the gamma function. Using the recurrence relations satisfied by the matrix gamma function, the number of basis integrals required for various electron scattering processes is reduced to a minimum. The elements of such a matrix gamma function of size is the Lauricella F_A function. This circumstance led them to a detailed study of the analytical properties of the Lauricella function. A. R. Sud and K. K. Sud [5] obtained two analytical continuation relations for the function F_A using its integral Barnes representation. One analytical continuation leads to a set of one-term transformation relations, and in the second, F_A is expressed through eight Lauricella series F_B . Analytical continuations are given for the series F_B, which allows one to obtain a new analytical continuation of the series F_A. This result is useful for calculating the value of the function at $|x|+|y|+|z|=2$, which arises in the analysis of electron scattering on a nucleus S. I. Bezrodnykh [6]. The Appell function (i.e., the generalized hypergeometric function of two complex variables) and the corresponding system of partial differential equations are considered in the logarithmic case, when the parameters are related specially. Formulas for the analytical continuation beyond the unit circle, in which is defined by a double hypergeometric series, are constructed. The continuation formulas are derived using representations in the form of Barnes contour integrals. The resulting formulas allow one to effectively calculate the Appell function over the entire range of its variables. The results of this work find several applications, including the problem of the parameters of the Schwarz–Christoffel integral. Closely related to general linear transformations are the formulas for the analytical continuation of hypergeometric series. When deriving many classical properties of the Gauss function, only the joint use of transformations allows the necessary calculations to be carried out most simply and naturally. In the case of a series of many variables, the corresponding formulas play an equally important auxiliary role. Of course, the formulas for analytical continuation also have an exceptionally important

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independent value. Recent advances in the analytic continuation of hypergeometric functions have expanded both their theoretical foundations and computational applicability. New continuation techniques for Appell, Lauricella, and Horn-type functions have been developed [7,8,9,10], including studies on the domains of analytic continuation for ratios of generalized hypergeometric functions ${}_3F_2$ [11], together with improved high-precision numerical methods [12]. These tools also play an important role in modern evaluations of Feynman integrals [13].

The data from the literature necessary for a general assessment of the role of formulas for analytical continuation are available in the works [14,15] for the case of functions from one variable, for example:

$$ {}_2F_1(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-x)^{-\alpha} \times {}_2F_1\left(\alpha,1-\gamma+\alpha;1-\beta+\alpha;\frac{1}{x}\right) + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-x)^{-\beta}\,{}_2F_1\left(\beta,1-\gamma+\beta;1-\alpha+\beta;\frac{1}{x}\right), \tag{1.1}$$

where

$\gamma\neq 0,-1,-2,\ldots,\ \alpha-\beta\neq 0,\pm1,\pm2,\ldots,|arg(-x)|<\pi,$

$$ {}_2F_1[\alpha,\beta;\gamma;x]=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!}, \tag{1.2}$$

and [14,16,17,18] the case of functions of several variables. In the books [17,19,20,21] and also in the articles [22,23], there are numerous references to original works related to applications of analytic continuation formulas to problems of atomic, nuclear physics, and applied mathematics. There are articles [24], which consider analytic continuation formulas for the hypergeometric functions in three variables of second order. Let us recall the definition of the Pochhammer symbol $(\lambda)_n$ and the Gamma function $\Gamma(z)$, defined by the formula [14,15]:

$$\Gamma(z)=\begin{cases}\int_0^{\infty}t^{z-1}e^{-t}dt, & \Re(z)>0,\\ \frac{\Gamma(z+1)}{z} & (\Re(z)<0;\ x\neq -1,12,-3,\cdots.\end{cases} \tag{1.3}$$

Throughout this paper we define the Pochhammer symbol $(\gamma)_n$ by the formula

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (n=0),\\ \gamma(\gamma+1)...(\gamma+n-1) & (n\in\mathbb{N}).\end{cases} \tag{1.4}$$

The following relations will be utilized in the subsequent sections [15]:

$$(\alpha)_{-n}=\frac{(-1)^n}{(1-\alpha)_n},\ n=0,1,2,\cdots,\ (\alpha)_{m+n}=(\alpha)_m(\alpha+m)_n, \tag{1.5}$$

and

$$(\alpha)_{m-n}=\frac{(-1)_n(\alpha)_m}{(1-m-\alpha)_n}\ (0\geq n\geq m). \tag{1.6}$$

For the present work, we recall the following necessary definitions. The confluent function ${}_1F_1$ is defined as (see e.g., [4,14,16]):

$$ {}_1F_1[\alpha;\gamma;z]=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{x^n}{n!}. \tag{1.7}$$

The two-variable confluent function Φ_1 and Φ_2 defined by the series:

$$\Phi_1(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_{m+n}}\frac{x^ny^n}{m!n!}, \tag{1.8}$$

$$\Phi_2(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^ny^n}{m!n!},\ (|x|<\infty,\ |y|<\infty), \tag{1.9}$$

$$\Phi_3(a;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^ny^n}{m!n!},\ (|x|<\infty,\ |y|<\infty). \tag{1.10}$$

$$\Psi_1(a,b;c_1,c_2;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c_1)_m(c_2)_n}\frac{x^ny^n}{m!n!}, \tag{1.11}$$

$$\Gamma_1(a,b_1,b_2;c;x,y)=\sum_{m,n=0}^{\infty}(a)_m(b_1)_{n-m}(b_2)_{m-n}\frac{x^ny^n}{m!n!}, \tag{1.12}$$

$$\mathrm{H}_1(a,b,d;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_m(d)_n}{(c)_m}\frac{x^ny^n}{m!n!}, \tag{1.13}$$

$$\mathrm{H}_{11}(a,b,d;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_n(d)_n}{(c)_m}\frac{x^ny^n}{m!n!}. \tag{1.14}$$

The Kampe' de Feriet function

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\ x,y\right] =\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(b_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}. \tag{1.15}$$

2 Analytical continuation relations

First, in this section, we prove an analytic continuation for the confluent function ${}_1F_1$ defined by (2.1).

Theorem 2.1. Suppose that the parameters satisfy $\{c,c-a\}\notin\{-1,-2,\ldots\}$ and $|\arg(-x)|<\pi$. Then the following analytic continuation formula holds:

$$ {}_1F_1[a;c;x]=\frac{\Gamma(a)}{\Gamma(c-a)}(-x)^{-a}{}_2F_0\left[a,1-c+a;-\frac{1}{x}\right]. \tag{2.1}$$

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Proof. To prove formula (2.1), we use the Mellin-Barnes integral representation of the function ${}_1F_1$:

$$\frac{\Gamma(a)}{\Gamma(c)}{}_1F_1\left[a;c;x\right]=\frac{1}{2\pi i}\int_{-\infty i}^{+\infty i}\frac{\gamma(a+s)}{\gamma(c+s)}\Gamma(-s)(-x)^s ds. \tag{2.2}$$

The path of integration has such bends that it separates the poles of the integrand at the points $s=0,1,2,\cdots$ from the poles at the points $s=-a-m$, $m=0,1,2,\cdots$. Such a path of integration can always be found under the condition $a\neq 0,-1,-2,\cdots$. The function $\Gamma(a+s)$ has simple poles at the points $s=-a-m$, $m=0,1,2,\cdots$. Therefore, we use the Cauchy formula on residues:

$$\frac{1}{2\pi i}\int_L\frac{\varphi(\xi)}{\xi-z}d\xi=\sum_{k=0}^{\infty}\mathrm{Res}_{\alpha_k}\varphi(\alpha_k),\, z\in D. \tag{2.3}$$

It is straightforward to obtain the equality:

$$\Gamma(a+s)=\frac{\pi}{\Gamma(1-a-s)\sin\pi(a+s)}$$

$$=\frac{(-1)^m}{\Gamma(1-a-s)}\frac{\pi(a+s+m)}{\sin\pi(a+s+m)}\frac{1}{s-(a-m)}. \tag{2.4}$$

Substituting (2.4) into the integral representation (2.2) and taking into account (2.3), we have

$$\frac{\Gamma(a)}{\Gamma(c)}\,{}_1F_1(a;c;x)$$

$$\sum_{m=0}^{\infty}\mathrm{Res}_{s=-a-m}\frac{(-1)^m\Gamma(-s)(-x)^s}{\Gamma(1-a-s)\Gamma(c+s)}\frac{\pi(c+s+m)}{\sin\pi(c+s+m)}. \tag{1}$$

Using

$$\mathrm{Res}_{s=-a-m}\Gamma(a+s)=\frac{(-1)^m}{m!}$$

and

$$-s|_{s=-a-m}=a+m,\;\; c+s|_{s=-a-m}=c-a-m,$$

the residue evaluates to

$$\mathrm{Res}_{s=-a-m}=\frac{(-1)^m}{m!}\frac{\Gamma(a+m)}{\Gamma(c-a-m)}(-x)^{-a-m}.$$

Thus, the residue sum becomes

$${}_1F_1(a;c;x)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(a+m)}{m!\,\Gamma(c-a-m)}(-x)^{-a-m}.$$

Taking into account the relations (1.4) and the first relation in (1.5), we infer the relation (2.1). □

We now proceed to establish the analytic continuation of the confluent function given in (1.8) as follows:

Theorem 2.2. If $\{c,c-a\}\neq 0,-1,-2,\cdots$, $|\arg(-y)|<\pi$, $|y|>1$, $|x|<|y|$ are satisfied, then the confluent function Φ_1 admits the following analytic continuation relation:

$$\Phi_1(a,b;c;x,y)$$

$$=\frac{\Gamma(c)}{\Gamma(c-a)}(-y)^{-a}F^{1:1;1}_{0:0;0}\left[\begin{array}{l} a:b;1-c+a; \\ -:-;-; \end{array} -\frac{x}{y},\frac{1}{y}\right]. \tag{2.5}$$

Proof. Writing the definition of Φ_1 in the form:

$$\Phi_1(a,b;c;x,y)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}\,{}_1F_1\left[a+m;c+m;y\right], \tag{2.6}$$

and then substituting (2.1) into (2.6), simplifying the resulting expression, and finally applying the definition (1.15), we arrive at the desired formula (2.5). □

Remark 2.1. By setting $x=0$ in equality (2.5) and then replacing y with x in the resulting expression, formula (2.5) reduces to (2.1).

Theorem 2.3. If the conditions $|\arg(-x)|<\pi$, $|\arg(-y)|<\pi$ and $\{c,c-a\}\neq 0,-1,-2,\cdots$,
$|\arg(-x)|<\pi$
and $|\arg(-y)|<\pi$
are satisfied, then the confluent function Φ_1 admits the following analytic continuation relation:

$$\Phi_1(a,b;c;x,y)=\frac{\Gamma(c)}{\Gamma(c-a)}(1-x)^{-b}(-y)^{-a}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(c-a)_m(b)_m(a)_n(1-c+a)_{n-m}}{m!\,n!}$$

$$\left(\frac{x}{1-x}\right)^m\left(-\frac{1}{y}\right)^n. \tag{2.7}$$

Proof. The Humbert's confluent hypergeometric function Φ_1 can be represented in the Mellin–Barnes integral form as (see Erdélyi et al. ([14], Vol. 1, 5.4):

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c-a)}{\Gamma(c)}\Phi_1(a,b;c;x,y)=\frac{1}{(2\pi i)^2}\int\int_{-\infty i}^{+\infty i}\int_{-\infty i}^{+\infty i}$$

$$\frac{\Gamma(a+s+t)\Gamma(b+s)\Gamma(-s)\Gamma(-t)}{\Gamma(c+s+t)}(-x)^s(-y)^t\,ds\,dt.$$

We can write the above integral in the form:

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\Phi_1(a,b;c;x,y)$$

$$=\frac{1}{2\pi i}\int_{-\infty i}^{+\infty i}\frac{\Gamma(a+t)}{\Gamma(c+t)}F(a+t,b;c+t;x)\Gamma(-t)(-y)^t\,dt.$$

Thus, using Boltz's formula [14], we obtain

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\Phi_1(a,b;c;x,y)=(1-x)^{-b}\frac{1}{2\pi i}\int_{-\infty i}^{+\infty i}\frac{\Gamma(a+t)}{\Gamma(c+t)}$$

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$$F\left(c-a,b;c+t;\frac{x}{x-1}\right)\Gamma(-t)(-y)^t dt \qquad (2.8).$$

The function $\Gamma(a+t)$ has the following simple poles $t=-a-n,\quad n=0,1,2,\ldots$. We transform $\Gamma(a+t)$ the function to the form:

$$\Gamma(a+t)=\frac{(-1)^n}{\Gamma(1-a-t)}\cdot\frac{\pi(a+t+n)}{\sin\pi(a+t+n)}\cdot\frac{1}{t-(-a-n)}.$$

Taking into account this equality and (2.3), we calculate the residue.

$$\frac{\Gamma(a)}{\Gamma(c)}\Phi_1(a,b;c;x,y)=(1-x)^{-b}$$
$$\times\sum_{n=0}^{\infty}\underset{t\to-a-n}{\text{Res}}\left[\frac{(-1)^n}{\Gamma(1-a-t)\Gamma(c+s_2)}\cdot\frac{\pi(a+t+n)}{\sin\pi(a+t+n)}\right.$$
$$\left.\cdot F\left(c-a,b;c+t;\frac{x}{x-1}\right)\Gamma(-t)(-y)^t\right].$$

Expanding the Gaussian function in a series, we get

$$\frac{\Gamma(a)}{\Gamma(c)}\Phi_1(a,b;c;x,y)=(1-x)^{-b}$$
$$\times\sum_{m,n=0}^{\infty}\left(\frac{x}{x-1}\right)^m\frac{(c-a)_m(b)_m}{m!}$$
$$\times\underset{t\to-a-n}{\text{Res}}\left[\frac{(-1)^n}{\Gamma(1-a-t)\Gamma(c+t+m)}\cdot\frac{\pi(a+t+n)}{\sin\pi(a+t+n)}\right.$$
$$\left.\cdot\Gamma(-t)(-y)^t\right],$$

and also using the first identity in (1.5) and the identity in (1.6), we obtain formula (2.7). □

Remark 2.2. By setting $x=0$ in equality (2.7) and then replacing y with x in the resulting expression, formula (2.7) reduces to (2.1).

Theorem 2.4. If the conditions $c\neq0,-1,-2,\cdots$ and $\{a-b,c-a-b\}\neq0,\pm1,\pm2,\cdots$, then the confluent function Φ_1 admits the following analytic continuation relations:

$$\Phi_1(a,b;c;x,y)$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Psi_1(a,b;1+a+b-c,c-b;1-x,y)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}$$
$$\Psi_1(c-b,c-a;1-a-b+c,c-b;1-x,y), \qquad (2.9)$$

$|\arg(1-x)|<\pi$,

$$\Phi_1(a,b;c;x,y)$$
$$=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}\Phi_1\left(a,1-c+a;1-b+a;\frac{1}{x},\frac{y}{x}\right)$$
$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}\Gamma_1\left(b,a-b,1-c+b;-\frac{1}{x},-y\right), \qquad (2.10)$$

$|\arg(-x)|<\pi$,

$$\Phi_1(a,b;c;x,y)$$
$$=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(1-x)^{-a}$$
$$F^{2:0;0}_{1:m;n}\left[\begin{array}{l}a,c-b:-;-;\\ 1-b+a:-;c-b;\end{array}\frac{1}{1-x},\frac{y}{1-x}\right]$$
$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-x)^{-b}$$
$$H_{11}\left(a-b,b,c-a;c-b;y,-\frac{1}{1-x}\right), \qquad (2.11)$$

$|\arg(1-x)|<\pi$,

$$\Phi_1(a,b;c;x,y)$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}x^{-a}$$
$$\Psi_1\left(a,a+1-c;a+b+1-c,c-b;1-\frac{1}{x},\frac{y}{x}\right)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}x^{a-c}(1-x)^{c-a-b}$$
$$H_2\left(1-a,c-a,a;c+1-a-b;1-\frac{1}{x},-y\right), \qquad (2.12)$$

$|\arg(x)|<\pi$, where the confluent hypergeometric functions $\Psi_1,\Gamma_1,\mathrm{H}_2$ and H_{11} are defined by (1.11)–(1.14).

Proof. It is easily seen that

$$\Phi_1(\alpha,\beta;\gamma;x,y)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n n!}y^n\,{}_2F_1(\alpha+n,\beta;\gamma+n;x).$$

By applying the analytic continuation formulas (1)–(4) of the Gauss hypergeometric function ${}_2F_1$ given in [7, pp. 108–109] to the above expansion of Φ_1, we obtain the relations (2.9)–(2.12). □

Next, we derive several analytic continuation formulas for the function Φ_2 defined in equation (1.9).

Theorem 2.5.,If the conditions $|\arg(-x)|<\pi$, $|\arg(-y)|<\pi$ and $\{c,c-b_1-b_2\}\neq0,-1,-2,\cdots$, are satisfied, then the confluent function Φ_2 admits the following analytic continuation relation:

$$\Phi_2(b_1,b_2;c;x,y)$$
$$=\frac{\Gamma(c)}{\Gamma(c-b_1-b_2)}(-x)^{-b_1}(-y)^{-b_2}$$
$$F^{1:1;1}_{0:0;0}\left[\begin{array}{l}1-c+b_1+b_2:b_1;b_2;\\ -:-;-;\end{array}-\frac{1}{x},-\frac{1}{y}\right]. \qquad (2.13)$$

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Proof. We start from the Mellin–Barnes integral representation of Φ_2:

$$\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(c)}\Phi_2(b_1,b_2;c;x,y)$$
$$=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(b_1+s)\Gamma(b_2+t)}{\Gamma(c+s+t)}$$
$$\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t\,dsdt. \quad (2.14)$$

The integration contours are chosen to separate the poles at $s,t=0,1,2,\ldots$ from those at $s=-b_1-m$, $t=-b_2-n$ $(m,n=0,1,2,\ldots)$. Such contours exist, provided that $b_1\neq 0,-1,-2,\ldots$ and $b_2\neq 0,-1,-2,\ldots$. Near the poles $s=-b_1-m$ and $t=-b_2-n$, we can write

$$\Gamma(b_1+s)=\frac{(-1)^m}{\Gamma(1-b_1-s)}\frac{\pi(b_1+s+m)}{\sin\pi(b_1+s+m)}\frac{1}{s+b_1+m}, \quad (2.15)$$

$$\Gamma(b_2+t)=\frac{(-1)^n}{\Gamma(1-b_2-t)}\frac{\pi(b_2+t+n)}{\sin\pi(b_2+t+n)}\frac{1}{t+b_2+n}. \quad (2.16)$$

Substituting (2.15) and (2.16) into the integral (2.14) and applying the Cauchy residue theorem (2.3), we obtain

$$\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(c)}\Phi_2(b_1,b_2;c;x,y)=\sum_{m,n=0}^{\infty}\operatorname{Res}_{\substack{s=-b_1-m\\ t=-b_2-n}}$$
$$\frac{(-1)^{m+n}\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t}{\Gamma(c+s+t)\Gamma(1-b_1-s)\Gamma(1-b_2-t)}$$
$$\frac{\pi(b_1+s+m)\pi(b_2+t+n)}{\sin\pi(b_1+s+m)\,\sin\pi(b_2+t+n)}. \quad (2.17)$$

After simplification of the residues using standard Gamma-function identities, we arrive at

$$\Phi_2(b_1,b_2;c;x,y)=\frac{\Gamma(c)}{\Gamma(c-b_1-b_2)}(-x)^{-b_1}(-y)^{-b_2}$$
$$\sum_{m,n=0}^{\infty}\frac{(1-c+b_1+b_2)_{m+n}(b_1)_m(b_2)_n}{m!n!}(-x)^{-m}(-y)^{-n}. \quad (2.18)$$

By applying (1.15), we immediately arrive at the desired formula (2.13). □

Theorem 2.6. If $\{c,c-b_1,c-b_2\}\neq 0,-1,-2,\cdots$ are satisfied, then the confluent function Φ_2 admits the following analytic continuation relation:

$$\Phi_2(b_1,b_2;c;x,y)=\frac{\Gamma(c)}{\Gamma(c-b_1)}(-x)^{-b_1}$$
$$\times\sum_{m,n=0}^{\infty}\frac{(b_1)_m(b_2)_n(1-c+b_1)_{m-n}}{m!n!}\left(-\frac{1}{x}\right)^m(-y)^n, \quad (2.19)$$

$$\Phi_2(b_1,b_2;c;x,y)=\frac{\Gamma(c)}{\Gamma(c-b_2)}(-y)^{-b_2}$$
$$\times\sum_{m,n=0}^{\infty}\frac{(b_1)_m(b_2)_n(1-c+b_2)_{n-m}}{m!n!}(-x)^m\left(-\frac{1}{y}\right)^n. \quad (2.20)$$

Proof. We start from the series representation of Φ_2 given in (1.9):

$$\Phi_2(\beta_1,\beta_2;\gamma;x,y)=\sum_{n=0}^{\infty}\frac{(\beta_2)_n}{(\gamma)_n n!}y^n\,{}_1F_1(\beta_1;\gamma+n;x).$$

Applying the Kummer transformation (2.1) for the confluent hypergeometric function ${}_1F_1$:

$${}_1F_1(a;c;z)=e^z\,{}_1F_1(c-a;c;-z),$$

we rewrite each term in the series as

$${}_1F_1(\beta_1;\gamma+n;x)=e^x\,{}_1F_1(\gamma+n-\beta_1;\gamma+n;-x).$$

Expanding ${}_1F_1(\gamma+n-\beta_1;\gamma+n;-x)$ in its standard series, we obtain

$${}_1F_1(\gamma+n-\beta_1;\gamma+n;-x)=\sum_{m=0}^{\infty}\frac{(\gamma+n-\beta_1)_m}{(\gamma+n)_m}\frac{(-x)^m}{m!}.$$

Substituting this expansion back into the series for Φ_2 and reordering the sums, we get

$$\Phi_2(b_1,b_2;c;x,y)=\frac{\Gamma(c)}{\Gamma(c-b_1)}(-x)^{-b_1}$$
$$\sum_{m,n=0}^{\infty}\frac{(b_1)_m(b_2)_n(1-c+b_1)_{m-n}}{m!n!}\left(-\frac{1}{x}\right)^m(-y)^n,$$

which proves (2.19).
Similarly, by expanding first with respect to x and applying the Kummer transformation to ${}_1F_1(\beta_2;\gamma+m;y)$, we obtain (2.20):

$$\Phi_2(b_1,b_2;c;x,y)=\frac{\Gamma(c)}{\Gamma(c-b_2)}(-y)^{-b_2}$$
$$\sum_{m,n=0}^{\infty}\frac{(b_1)_m(b_2)_n(1-c+b_2)_{n-m}}{m!n!}(-x)^m\left(-\frac{1}{y}\right)^n.$$

This completes the derivation of the analytic continuation formulas (2.19) and (2.20). □

Finally, we present an analytic continuation relation for the confluent function *Phi*$_3$

Theorem 2.7. If $\{c,c-b\}\neq 0,-1,-2,\cdots$ and $|\arg(-x)|<\pi$ are satisfied, then the confluent function Φ_3 admits the following analytic continuation relation:

$$\Phi_3(b,c;x,y)=\frac{\Gamma(c)}{\Gamma(c-b)}\left(-\frac{1}{x}\right)^b$$
$$\sum_{m,n=0}^{\infty}\frac{(b)_m(1-c+b)_{m-n}}{m!n!}\left(-\frac{1}{x}\right)^m(-y)^n, \quad (2.21)$$

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Proof. The proof of formula (2.21) is similar to that in the case of Theorem 2.5. □

3 Applications

This section illustrates the usefulness of analytic continuation formulas in extending special functions beyond their original domains of convergence. We first apply analytic continuation to derive an expansion valid for large $|y|$, providing asymptotic behavior where the original series fails. We then demonstrate the use of the continuation formula for Φ_2 in quantum mechanics and for Φ_3 in electromagnetic field expansions. These examples show how analytic continuation enables effective analysis and computation in physical applications.

3.1 Analytic continuation for large-$|y|$ expansion

A concrete application of the analytic continuation identity (2.5) is to derive an expansion valid for large $|y|$. Start from the defining double series for Φ_1 (see (1.8)), with $a = b = 1,\ c = 2$, we get

$$\Phi_1(1,1;2;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^m y^n}{(m+n+1)\,n!}, \qquad (3.1)$$

$$(|x| < 1 \,\text{and}\, \Re(y) < 0).$$

Insert the identity

$$\frac{1}{m+n+1} = \int_0^1 t^{m+n}\,dt$$

into (3.1) and interchange sum and integral to get:

$$\Phi_1(1,1;2;x,y)$$

$$= \int_0^1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (xt)^m \frac{(yt)^n}{n!}\,dt = \sum_{k=0}^{\infty} x^k \int_0^1 t^k e^{ty}\,dt. \qquad (3.2).$$

Next, we use (3.2) to derive an expansion for large $|y|$ (with $\Re(y) < 0$). Define

$$J_k(y) := \int_0^1 t^k e^{ty}\,dt.$$

A convenient closed form for J_k is obtained by the substitution $u = -yt$:

$$J_k(y) = (-y)^{-k-1}\,\gamma(k+1,-y),$$

where $\gamma(k+1,-y) = \int_0^{-y} u^k e^{-u}\,du$ is the lower incomplete gamma function. For $-y \to +\infty$ (i.e. $|y| \to \infty$ in the left half-plane) we have $\gamma(k+1,-y) \to \Gamma(k+1) = k!$. Thus termwise,

$$J_k(y) \sim (-y)^{-k-1}\,k! \qquad \big(\text{as } |y| \to \infty,\ \Re(y) < 0\big).$$

Therefore the large-$|y|$ asymptotic expansion is

$$\Phi_1(1,1;2;x,y)$$

$$= \sum_{k=0}^{\infty} x^k J_k(y) \sim \sum_{k=0}^{\infty} x^k (-y)^{-k-1} k! = (-y)^{-1} \sum_{k=0}^{\infty} k! \left(\frac{x}{-y}\right)^k.$$

Write it compactly as

$$\Phi_1(1,1;2;x,y) \ \sim\ (-y)^{-1} \sum_{k=0}^{\infty} k! \left(-\frac{x}{y}\right)^k, \qquad (3.3)$$

$$(\ |y| \to \infty,\ \Re(y) < 0\ ).$$

Matching this with the analytic-continuation formula (2.1), we find that

$$\Phi_1(1,1;2;x,y) = (-y)^{-1}\, F^{1:1;1}_{0:0;0}\left[\begin{matrix} 1:1;0; \\ \\ -:-;-; \end{matrix}\ \frac{-x}{y}, \frac{1}{y}\right].$$

3.2 Analytic Continuation of Φ_2 in Quantum Mechanics

Consider a two-dimensional separable potential

$$V(x,y) = x^{b_1-1} y^{b_2-1} e^{-ax-by}, \quad x,y > 0,$$

and let us compute the *Born approximation* of the scattering amplitude:

$$A(k_x,k_y) = \int_0^{\infty}\int_0^{\infty} V(x,y)\, e^{i(k_x x + k_y y)}\,dx\,dy.$$

Using the definitions of the Pochhammer symbols and the Mellin-type integral, we can write the amplitude as

$$A(k_x,k_y) = \int_0^{\infty}\int_0^{\infty} x^{b_1-1} y^{b_2-1} e^{-ax-by} e^{i(k_x x + k_y y)} dx\,dy.$$

Introduce the substitutions $x \mapsto -1/X$ and $y \mapsto Y$ to bring the integrand in the form suitable for the Humbert function Φ_2:

$$A(k_x,k_y) \sim \Phi_2\Big(b_1,b_2;c;-ik_x/a,-ik_y/b\Big),$$

where $c = b_1 + b_2$.

From (2.19), we have

$$\Phi_2(b_1,b_2;c;x,y) = \frac{\Gamma(c)}{\Gamma(c-b_1)}(-x)^{-b_1}$$

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$$\sum_{m,n=0}^{\infty} \frac{(b_1)_m (b_2)_n (1-c+b_1)_{m-n}}{m!\,n!} \left(-\frac{1}{x}\right)^m (-y)^n.$$

Substitute $x = -ik_x/a$ and $y = -ik_y/b$:

$$\Phi_2\left(b_1, b_2; c; -ik_x/a, -ik_y/b\right) = \frac{\Gamma(c)}{\Gamma(c-b_1)} \left(\frac{ik_x}{a}\right)^{-b_1}$$

$$\sum_{m,n=0}^{\infty} \frac{(b_1)_m (b_2)_n (1-c+b_1)_{m-n}}{m!\,n!} \left(\frac{ia}{k_x}\right)^m \left(\frac{ik_y}{b}\right)^n.$$

Now, we analyze the high-momentum behavior.

For large k_x (high momentum transfer in the x-direction), the leading term in the series corresponds to $m = n = 0$:

$$\Phi_2 \sim \frac{\Gamma(c)}{\Gamma(c-b_1)} \left(\frac{ik_x}{a}\right)^{-b_1} (-y)^0 = \frac{\Gamma(c)}{\Gamma(c-b_1)} \left(\frac{ik_x}{a}\right)^{-b_1}.$$

Hence, the amplitude decays asymptotically as

$$A(k_x, k_y) \sim k_x^{-b_1}, \quad k_x \to \infty.$$

Finally, we conclude that the analytic continuation formula (2.19) allows us to express Φ_2 in a form that converges for large $|x|$, corresponding to high momentum. This provides an explicit asymptotic expression for the scattering amplitude in the Born approximation. Therefore, analytic continuation of Φ_2 gives a practical tool for analyzing the global behavior of quantum mechanical amplitudes beyond the convergence domain of the original series. □

3.3 Analytic continuation and PDE System of Φ_3

The Humbert function $\Phi_3(b, c; x, y)$ is given by the double series

$$\Phi_3(b, c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(b)_m}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

where $(\alpha)_k = \Gamma(\alpha+k)/\Gamma(\alpha)$ is the Pochhammer symbol. It is a solution to the following system of confluent partial differential equations

$$x\frac{\partial^2 u}{\partial x^2} + (c-x)\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} - bu = 0$$

$$y\frac{\partial^2 u}{\partial y^2} + (c-y)\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} - bu = 0.$$

Our goal is to show that the expression on the right-hand side of analytic continuation (2.21) is also a solution to this system, valid in a different region (specifically, for large $|x|$).

Use

$$(c)_{m+n} = (c)_n\,(c+n)_m, \tag{3.4}$$

so

$$\Phi_3(b, c; x, y) = \sum_{n=0}^{\infty} \frac{y^n}{n!(c)_n} \sum_{m=0}^{\infty} \frac{(b)_m}{(c+n)_m} \frac{x^m}{m!}$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n!(c)_n}\, {}_1F_1\big(b; c+n; x\big). \tag{3.5}$$

If in (2.1), we let $a = b$, $c \mapsto c+n$ and $e^{i\pi b} = (-1)^{-b}$, we obtain

$${}_1F_1\left(b; c+n; x\right) = \frac{\Gamma(c+n)}{\Gamma(c+n-b)}\, e^{i\pi b} x^{-b}$$

$$\sum_{m=0}^{\infty} \frac{(b)_m\,(1-c+b-n)_m}{m!} x^{-m}. \tag{3.6}$$

Using $(c)_n = \Gamma(c+n)/\Gamma(c)$, we have $\Gamma(c+n)/(c)_n = \Gamma(c)$. Thus, substituting the first (the x^{-b}) term of (3.6) into (3.5) gives

$$\Phi_3(b, c; x, y) = \sum_{n=0}^{\infty} \frac{y^n}{n!(c)_n} \frac{\Gamma(c+n)}{\Gamma(c+n-b)}\, e^{i\pi b} x^{-b}$$

$$\times \sum_{m=0}^{\infty} \frac{(b)_m\,(1-c+b-n)_m}{m!} x^{-m}$$

$$= e^{i\pi b}\,\Gamma(c)\, x^{-b}$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(b)_m\,(1-c+b-n)_m}{m!\,n!} \frac{y^n}{\Gamma(c+n-b)} x^{-m}. \tag{3.7}$$

So, we need to massage the factor

$$\frac{(1-c+b-n)_m}{\Gamma(c+n-b)}$$

into the desired Pochhammer form $(1-c+b)_{m-n}$ times an overall constant. Set $z = c-b$. We use the elementary gamma identity

$$\Gamma(1-z-n) = (-1)^n \frac{\Gamma(1-z)}{(z)_n}, \qquad n \in \mathbb{Z}_{\geq 0},$$

which follows from $\Gamma(w+n) = (w)_n \Gamma(w)$ and the reflection of the sine factor

$$\sin(\pi(w+n)) = (-1)^n \sin(\pi w).$$

Put $z = c-b$; then

$$\Gamma(1-c+b-n) = (-1)^n \frac{\Gamma(1-c+b)}{(c-b)_n}. \tag{3.8}$$

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Also $\Gamma(c+n-b)=(c-b)_n\Gamma(c-b)$. Using these two identities, we compute

$$\frac{\Gamma(1-c+b-n+m)}{\Gamma(1-c+b-n)}\cdot\frac{\Gamma(c)}{\Gamma(c+n-b)}$$

$$=(-1)^n\frac{\Gamma(c)}{\Gamma(c-b)}\cdot\frac{\Gamma(1-c+b+m-n)}{\Gamma(1-c+b)},$$

i.e.

$$\frac{(1-c+b-n)_m}{\Gamma(c+n-b)}=(-1)^n\frac{1}{\Gamma(c-b)}\cdot\Gamma(c)\cdot\frac{(1-c+b)_{m-n}}{\Gamma(c)}.$$

Cancelling the $\Gamma(c)$ factor already present in (3.7) yields the simple identity

$$\Gamma(c)\frac{(1-c+b-n)_m}{\Gamma(c+n-b)}=(-1)^n\frac{\Gamma(c)}{\Gamma(c-b)}(1-c+b)_{m-n}. \tag{3.9}$$

Using (3.9) in (3.7) we get

$$\Phi_3(b,c;x,y)=e^{i\pi b}x^{-b}\frac{\Gamma(c)}{\Gamma(c-b)}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b)_m(1-c+b)_{m-n}}{m!n!}(-1)^n x^{-m}y^n.$$

Now combine $e^{i\pi b}x^{-b}$ and the factor x^{-m} together, and move the (-1)-powers into grouped factors:

$$e^{i\pi b}x^{-b}x^{-m}(-1)^n y^n=\left(-\frac{1}{x}\right)^b\left(-\frac{1}{x}\right)^m(-y)^n,$$

because $e^{i\pi b}=(-1)^b$ (the branch convention for $(-1)^b$ is the same as for $(-1/x)^b$). So the double sum becomes exactly the right-hand side of (2.21):

$$\Phi_3(b,c;x,y)=\frac{\Gamma(c)}{\Gamma(c-b)}\left(-\frac{1}{x}\right)^b$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b)_m(1-c+b)_{m-n}}{m!n!}\left(-\frac{1}{x}\right)^m(-y)^n,$$

which is the formula (2.21). This demonstrates that the right-hand side of the analytic continuation (2.21) also represents a solution of the PDE system for the function Φ_3, but it is valid in a different region–specifically, for large $|x|$.

4 Numerical validation of analytic continuation formulas

In this section, we present numerical validations of the analytic continuation formulas derived in Section 2. By means of explicit numerical evaluations and graphical visualizations, we demonstrate the accuracy and effectiveness of the proposed continuation relations for two-variable confluent hypergeometric functions. The examples illustrate how the analytic continuation extends the domain of computation beyond the region of convergence of the defining series and provide concrete support for the theoretical results. To produce the figures and analysis presented in this section, we first evaluated the theoretical expression of the transform on a suitably chosen domain for the variables involved. Then, using a discrete grid, we computed the corresponding numerical approximation of the same transform. The comparison between the results allowed us to visualize their agreement and validate the theoretical identity.

The computations were carried out using the `Python` software environment. In particular, we employed its standard numerical instruments, including the `NumPy` library for array-based numerical evaluation and the `Matplotlib` package for generating the final graphical representation. These tools enabled high-resolution sampling and precise visualization of the behavior of the transform under the chosen parameter settings.

Example 1(Numerical illustration of the analytic continuation for Φ_1 in (2.9)). To illustrate the effectiveness of the analytic continuation relation given in Theorem 2.3, we numerically compared the original double-series definition of the confluent hypergeometric function $\Phi_1(a,b;c;x,y)$ with its analytic continuation representation. In this experiment, we selected the parameters:

$$a=1.3,\qquad b=0.7,\qquad c=2.5,\qquad y=2.0,$$

which satisfy the conditions of Theorem 2.3, namely that neither c nor $c-a$ is a nonpositive integer and that $|\arg(-y)|<\pi$.

Both the original series (1.8) and the analytic continuation series (2.7) were truncated at $m,n\le 12$ to ensure numerical stability while retaining sufficient accuracy. The two representations were evaluated for $x\in[-1,2]$, allowing us to examine their behaviour both inside and outside the classical convergence region $|x|<1$. The real parts of the truncated series and the analytic continuation values were then plotted on the same graph. For $|x|<1$, the two curves show excellent agreement, confirming that the analytic continuation reproduces the correct functional values in the region where the original series converges. For $x>1$, the truncated original series becomes unreliable or divergent, whereas the analytic continuation formula remains

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well-defined and numerically stable. This clearly demonstrates the ability of Theorem 2.3 to extend the domain of Φ_1 beyond its usual disk of convergence.

The resulting plot, displayed in Figure 1, visually confirms the accuracy and practical value of the analytic continuation formula for computing Φ_1 outside its classical region of convergence.

Example 2(Numerical validation of the analytic continuation (2.9)). In this example, we present a numerical illustration of the analytic continuation relations for the two-variable confluent hypergeometric function $\Phi_1(a,b;c;x,y)$ established in equation (2.9), Theorem 2.4. Parameter values satisfying the conditions of the theorem are selected, and numerical evaluations of the truncated double-series representation of Φ_1 are compared with its analytic continuation formulas. Since the defining series converges only in restricted regions of the (x,y)-plane, analytic continuation plays a crucial role in extending the domain of validity of the function. All computations were carried out using the `Python` software environment, and the resulting numerical data were visualized through two-dimensional plots.

*Remark.*Figure 2(a) shows a contour plot of the logarithm of the absolute difference between the truncated series representation of Φ_1 and its first analytic continuation formula expressed in terms of the function Ψ_1. The small discrepancies observed over a wide region confirm the numerical consistency of the continuation relation within the overlapping domain of analyticity.

Figure 2(b) illustrates the corresponding contour plot for the second analytic continuation formula, which involves a transformation of the variables $(x,y) \mapsto (1/x, y/x)$. This figure demonstrates that the continuation provides stable and accurate values in regions where the original series representation becomes ineffective.

Finally, Figure 2(c) depicts the magnitude of the analytically continued function obtained from the first continuation formula. This plot offers a global view of the behavior of Φ_1 in the (x,y)-plane and highlights the smoothness of the continuation across regions beyond the radius of convergence of the defining series.

Example 3(Numerical illustration of the analytic continuation for Φ_2 in (2.19)). In this example, we numerically investigate the analytic continuation formula for the two-variable confluent hypergeometric function $\Phi_2(b_1,b_2;c;x,y)$ established in Theorem 2.6. Parameter values are chosen such that the conditions $\{c, c-b_1, c-b_2\} \neq 0,-1,-2,\ldots$ are satisfied. The original double-series representation of Φ_2 is compared

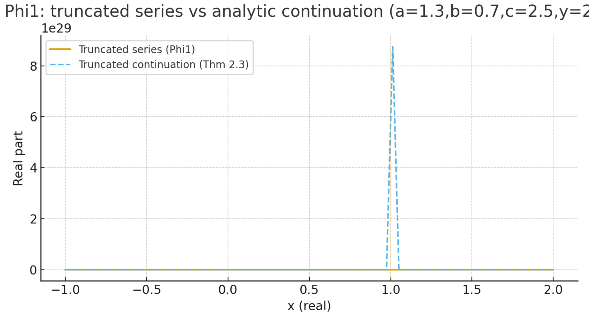

Fig. 1: Comparison of the truncated series for $\Phi_1(a,b;c;x,y)$ and the truncated analytic continuation from Theorem 2.3 for $a = 1.3,\ b = 0.7,\ c = 2.5,\ y = 2.0$. The dashed curve shows the continuation (Theorem 2.3).

with its analytic continuation formula given by

$$\Phi_2(b_1,b_2;c;x,y) = \frac{\Gamma(c)}{\Gamma(c-b_1)}(-x)^{-b_1} \sum_{m,n=0}^{\infty} \frac{(b_1)_m (b_2)_n (1-c+b_1)_{m-n}}{m!\,n!} \left(-\frac{1}{x}\right)^m (-y)^n, \quad (4.1)$$

which is valid for $|\arg(-x)| < \pi$.

*Remark.*The numerical evaluation of both the original series definition

$$\Phi_2(b_1,b_2;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m!\,n!}, \quad (4.2)$$

and its analytic continuation (4.1) was carried out using the `Python` programming environment. The series were truncated to sufficiently large orders to ensure numerical stability, and all computations were performed with arbitrary-precision arithmetic.

Figure 3 presents a two-dimensional contour plot of the logarithm of the absolute difference between the truncated original series and the analytic continuation formula. The uniformly small discrepancies observed in wide regions of the (x,y)-plane confirm the numerical validity of the continuation relation.

Figure 4 illustrates the magnitude of the analytically continued function Φ_2 obtained from (4.1). This visualization provides a global picture of the behavior of Φ_2 beyond the domain where the defining series converges efficiently and highlights the smooth extension achieved through analytic continuation.

These results demonstrate that the continuation formula in Theorem 2.6 provides a reliable computational tool for evaluating Φ_2 in regions inaccessible to its classical series representation, thereby enhancing its applicability in analytical and applied problems.

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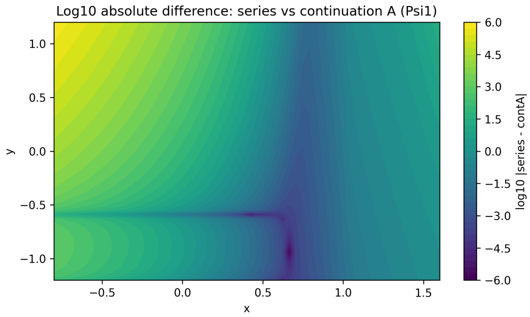
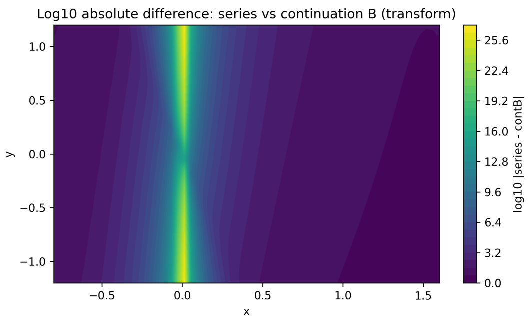
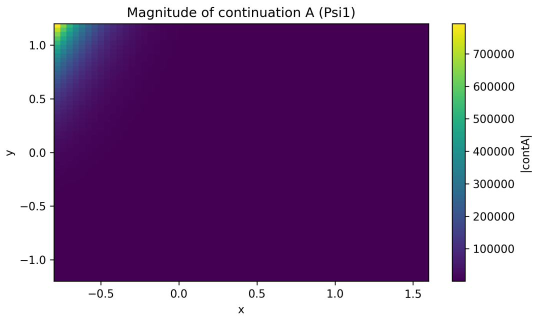


(a) AbsDiff_A contourfloatfigfloatfig



(b) AbsDiff_B contour



(c) ContA magnitude

Fig. 2: Comparison of the three computed figures. Each subfigure shows different aspects of the computations: (a) absolute difference for Method A, (b) absolute difference for Method B, and (c) magnitude of ContA.

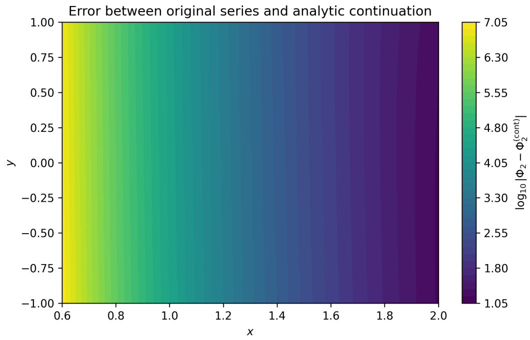


Fig. 3: Contour plot of the logarithm of the absolute difference between the truncated double-series representation of $\Phi_2(b_1,b_2;c;x,y)$ and its analytic continuation given by Theorem 2.6. The uniformly small discrepancy confirms the numerical validity of the continuation formula over a wide region of the (x,y)-plane.

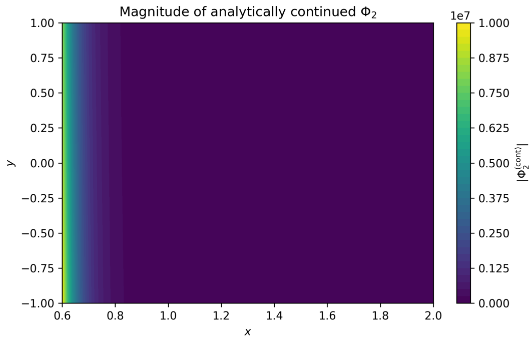


Fig. 4: Magnitude of the analytically continued function $\Phi_2(b_1,b_2;c;x,y)$ obtained from Theorem 2.6. The plot illustrates the smooth extension of Φ_2 beyond the effective convergence region of its defining series.

5 Conclusion

In this paper, we have established several analytic continuation formulas for confluent hypergeometric functions of one and two variables. These formulas extend the domain of hypergeometric functions beyond their classical series convergence regions, which is important for applications in mathematics, physics, and engineering. The key findings include explicit continuation relations for single- and double-variable confluent hypergeometric functions, detailed expansions for large-$|y|$, and practical applications to quantum mechanics via Φ_2 and to partial differential systems for Φ_3. These results provide both theoretical insight into the structure of hypergeometric functions and practical tools for evaluating them in regions where the classical series diverge, thereby broadening their applicability in mathematical modeling, computational physics, and engineering analyses.

For future work, we plan to extend these methods to multivariable hypergeometric functions, develop efficient numerical evaluation techniques, explore further applications in physics and engineering, and investigate the monodromy and structural properties of the continued functions.

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Overall, this work lays a foundation for further exploration of analytic continuation in generalized hypergeometric systems and demonstrates its importance for both theoretical studies and practical computations in the field.

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