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The Study of Asymptotic Behavior of Positive Solutions and its Stability for a New Class of Hyperbolic Differential System

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Abstract: This paper deals with the existence of positive solution and its asymptotic behavior for hyperbolic system of $(p(x),q(x))$-Laplacian system of partial differential equations using a sub and super solution according to some given boundary conditions, Our result is an extension solutions to other previous studies, which treated the stationary case, this study is new for evolutionary case of this kind of the problem. The purpose of our paper provides a framework for image restoration.

Keywords: Asymptotic behavior; Hyperbolic system, Sub-supersolutions method, Positive solution

1 Introduction

We consider the following hyperbolic system of partial differential equation: find $u(x,t)$ such that $u \in L^2\left(0,T,H_0^1(\Omega)\right)$, $u_t \in L^2\left(0,T,L^2(\Omega)\right)$, $u_{tt} \in L^2\left(0,T,L^2(\Omega)\right)$ solution of problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta_{p(x)} u$$

$$= \lambda^{p(x)}\left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right] \text{ in } \Omega_T = \Omega \times (0,T),$$

$$\frac{\partial^2 v}{\partial t^2} - \Delta_{q(x)} v$$

$$= \lambda^{q(x)}\left[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)\right] \text{ in } \Omega_T = \Omega \times (0,T),$$

$$\begin{cases} u = v = 0 \text{ on } \partial\Omega \times (0,T), \\ (u(.,0), v(.;0)) = (\varphi_1,\varphi_2) \text{ on } \Omega \times (t=0), \\ (u_t(x,0), v_t(x;0)) = (\varphi_3,\varphi_4) \text{ on } \Omega \times (t=0) \end{cases} \quad (1)$$

Where Ω part of $\mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and $1 < p(x), q(x) \in C^1(\overline{\Omega})$ are functions with

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \sup_{\Omega} p(x) < \infty,$$

$$1 < q^- := \inf_{x\in\Omega} q(x) \le q^+ := \sup_{x\in\Omega} q(x) < \infty$$

and $\Delta_{p(x)}$ is a $p(x)-$Laplacien defined as:

$$\Delta_{p(x)} u = div(|\nabla u|^{p(x)-2}\nabla u)$$

with positive parameters $\lambda, \lambda_1, \lambda_2, \mu_1, \mu_2$, and functions f, g, h, τ are monotone in in $[0,+\infty[$ such that :

$$\lim_{u\to+\infty} f(u) = +\infty \; \lim_{u\to+\infty} g(u) = +\infty,$$

$$\lim_{u\to+\infty} h(u) = +\infty, \; \lim_{u\to+\infty} \tau(u) = +\infty,$$

satisfying some natural growth condition at $u = \infty$.

In addition, we haven't considered any sign condition on $f(0), g(0), h(0), \tau(0)$.

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Where the existence of positive solution of the hyperbolic partial differential equation will be proved according to the conditions of symmetry, using super and sub solutions. The linear and nonlinear stationary equations with operators of quasilinear homogeneous type as p-Laplace operator, which can be carried out according to the standard Sobolev spaces theory of $W^{m,p}$, and thus we can find the weak solutions. The last spaces consist of functions having weak derivatives which verify some conditions of integrability. Thus, we can have the non-homogeneous case of $p(.)$-Laplace operators in this last condition. We use Sobolev spaces of the exponential variable in our standard framework, so that $L^{p(.)}(\Omega)$ is used instead of Lebesgue spaces $L^{p}(\Omega)$.

We denote new Sobolev space by$W^{m,p}(\Omega)$. If we replace $L^{p(.)}(\Omega)$ by $L^{p}(\Omega)$, the Sobolev spaces becomes $W^{m,p(.)}(\Omega)$. Several Sobolev spaces properties have been extended to spaces of Orlicz-Sobolev, particularly by O'Neill in the reference [15].

The spaces $W^{m,p(.)}(\Omega))$ and $L^{p(.)}(\Omega)$ have been carefully studied by many researchers team (see the references ([6], [7], [8] and [10]]).

Here, in our study we consider the boundedness condition in domain Ω, because many results for p-Laplacian theory are not usually verified for the $p(.)$-Laplacian theory; for that in ([28]) the quotient

$$\lambda_{p(x)} = \inf_{u\in\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx}$$

becomes 0 generally. Then $\lambda_{p(x)}$ can be positive only for some given conditions In fact, the first eigenvalue of $p(x)$-Laplacian and its associated eigenfunction cannot exist, the existence of the positive first eigenvalue λ_p and getting its eigenfunction are very important in the p-Laplacian problem study. Therefore, the study of existence of solutions of our problems has more meaning. Many studies of the experimental side have been studied on various materials that rely on this advanced theory, as they are important in electrical fluids, which state that viscosity relates to the electric field in a certain liquid.

The existence of positive solution for the following stationary problem was done in [23] as follows:

$$-\triangle_{p(x)}u = \lambda^{p(x)} f(v) \text{ in } \Omega \quad -\triangle_{p(x)}v = \lambda^{p(x)} g(u) \text{ in } \Omega \quad u = v = 0, \text{ on } \partial\Omega$$

where we have the following condition: For all $M \succ 0$

$$\lim_{u\to+\infty} \frac{f(M(g(u)))^{\frac{1}{p-1}}}{u^{p-1}} = 0,$$

This study didn't consider any condition of symmetric and without any initial sign condition on $f(0)$ and $g(0)$. Then they studied the existence of positively solution of the last stationary problem.

Recently, Boulaaras et al. ([2], [3], [4] and [12]) have studied the existence and asymptotic behavior of positive solutions, however, we've used the method sub-super solution theorem with respect to the symmetry condition to study the problem

$$\begin{cases} -\Delta_{p(x)}u \\ = \lambda^{p(x)}\left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right] \text{ in } \Omega, \\ -\Delta_{q(x)}v \\ = \lambda^{q(x)}\left[\lambda_2 b(x) g(u) + \mu_2 d(x)\tau(v)\right] \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

Recently, in ([3], [4], [11] and [12]), Boulaaras et al. have proved the existence of positive solutions of many classes of $(p(x), q(x))-$ Laplacian stationary problems by using the sub -super solution concept. The current results are an extensions of our previous stationary study to the hyperbolic case, where we follow-up the same procedures mathematical proofs similar to that in ([4], [11]) by using finite difference time scheme taking into consideration its stability analysis and the same conditions which were already studied before. Our result is an extension for the studied stationary case ([4], [11] and [12]) which studied the stationary case.

The outline of paper consists of the following : In first section we give some definitions, basic theorems and necessarily propositions in the functional analysis which will be used in our study. Then in section 2 and 3, we prove our main result.

2 Technical assumptions and main results

We give the following notation:

Let $P(\Omega)$ be the set of the all: $p : \Omega \to [1, +\infty[$ functions measurable

$$L^{p(x)}(\Omega) = \left\{ u \in P(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}$$

$L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx < 1 \right\}$$

and we define

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

the norm on $W^{1,p(x)}(\Omega)$ is

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \frac{|u|^{p(x)} + |\nabla u|}{\lambda^{p(x)}}^{p(x)} dx < 1\right\}$$

we denote $W_0^{1.p(x)}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{1.p(x)}$.

To explicate our ultimate approach to the combine with the system $(1,1)$, we consider the following hyperbolic problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta_{p(x)} u \\ = \lambda^{p(x)}[\lambda_1 a(x) f(u) + \mu_1 c(x) h(u)] \text{ in } \Omega_T = \Omega \times (0,T), \\ u = 0, \text{ on } \partial\Omega. \times (0,t), \\ u(.,0) = \varphi_1 \text{ on } \Omega \times (t=0). \end{cases} \tag{2}$$

Theorem 1. *There exists a weak positive solution of problem 2 on the interval $[0,T]$ for the fixed time $T > 0$.*

Proof. We pose for a function $u^n : [0,T] \to H_0^1(\Omega)$ of the form $u^n = \sum_{i=1}^{n} d_n^i w_i$ Here we hope to select the coefficients $d_n^i(t)$ such that

$$\begin{cases} \frac{d^2}{dt^2}(u^n, w_i) + B[u^n, w_i, t] \\ = (F(u^n), w_i) \\ d_i'(0) = \varphi_1 \end{cases}$$

taking

$$v = \sum_{i=1}^{n} v_i w_i(x)$$

thus

$$(u^n, v) = \sum_{i=1}^{n} d_n^i(t) v_i$$

then, we have

$$\begin{cases} \frac{d^2}{dt^2}(u^n, v) + B[u^n, v, t] \\ = (F(u^n), v) \\ (u^n, v)_{t=o} = (\varphi_1, v), \end{cases}$$

then for $i = 1, 2, ..n$, hence,

$$\begin{cases} \left(\sum_{i=1}^{n} \frac{d^2}{dt^2} d_n^i(t) + \sum_{i,j=1}^{n} k_{i,j} d_n^i(t)\right) v_i \\ = \left(F\left(\sum_{i=1}^{n} d_n^i w_i\right), w_j\right) v_i \text{ for all } v_i \\ (u^n, v)_{t=o} = \sum_{i=1}^{n} \varphi_1 v_i \end{cases}$$

then the system 2 becomes

$$\begin{cases} \left(\sum_{i=1}^{n} \frac{d^2}{dt^2} d_n^i(t) + \sum_{i,j=1}^{n} k_{i,j} d_n^i(t)\right) \\ = \left(F\left(\sum_{i=1}^{n} d_n^i w_i\right), w_j\right) \text{ for all } v_i, \\ \sum_{i=1}^{n} d_n^{i'}(0) = \sum_{i=1}^{n} \varphi_1 w_i \end{cases}$$

We choose $n \geq N$, multiply 2 by $d_n^{i'}(t)$ sum $i = 1, 2, ..N$ and than integrate with respect to t, we find

passing- to -weak limits, we have

for all function $v \in L^{p(x)}(0, T, H_0^1(\Omega))$

and

$$\int_0^T -\langle v, u\rangle + B[u, v, t]\, dt$$

$$= \int_0^T (F(u), v)\, dt + (u(0), v(0))$$

$$= \int_0^T (F(u), v)\, dt + (\varphi_1, v(0))$$

since $u^n(0) \to \varphi_1$ in $L^2(\Omega)$, as $v(0)$ is arbitrary, we deduce $u(0) = \varphi_1$

This approach is based on the method of sub-super solutions which are defined as follows:

Definition 1. *A pair of nonnegative functions $\underline{u}$, $\overline{u}$ $\in L^{p(x)}\left(0, T; W_0^{1,p(x)}(\Omega)\right) \cap L^{\infty}\left(0, T; L^2(\overline{\Omega})\right)$ are called order sub-super solutions of (2) if $\overline{u} \geq \underline{u}$ and if $\overline{u}$ satisfies:*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta_{p(x)} u \\ \geq \lambda^{p(x)}[\lambda_1 a(x) f(u) + \mu_1 c(x) h(u)] \text{ in } \Omega_T = \Omega \times (0,T), \\ u \geq 0, \text{ on } \partial\Omega. \times (0,t), \\ u(.,0) \geq \varphi_1 \text{ on } \Omega \times (t=0) \end{cases}$$

and $\underline{u}$ satisfies the above inequalities in reverse order .

Define $A : W^{1;p(x)} \cap C_0^{+} \to \left(W_0^{1;p(x)}(\Omega)\right)^{*}$

where

$$\langle Au, \varphi\rangle = \int_{\Omega} \left(u_{tt} - |\nabla u|^{p(x)-2} \nabla u \nabla\varphi - G(x,t,u)\nabla\varphi\right) dx,$$

where $u \in W^{1,p(x)} \cap L^{\infty}\left(0, T; L^2(\overline{\Omega})\right)$ *and* $\varphi \in W_0^{1,p(x)}(\Omega)$

Lemma 1.[Comparison principal] *Let* $u,v \in W^{1,p(x)} \cap L^{\infty}\left(0,T;L^{2}\left(\overline{\Omega}\right)\right)$ *be positive and satisfy* $Au - Av \geq 0$ *in* $\left(W_0^{1;p(x)}(\Omega)\right)^{*}$, *let*

$$\varphi(x) = \min\{u(x), v(x), 0\}$$

if $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ *(i.e.* $u \geq v$ *on* $\partial\Omega$*) than* $u \geq v$ *a.e in* Ω.

Definition 2. *A pair of nonnegative functions* $(\psi_1,\psi_2),(z_1,z_2)$ *in*

$$L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right) \cap L^{\infty}\left(0,T;L^{2}\left(\overline{\Omega}\right)\right)$$

$$\times L^{q(x)}\left(0,T;W_0^{1,q(x)}(\Omega)\right) \cap L^{\infty}\left(0,T;L^{2}\left(\overline{\Omega}\right)\right)$$

are called a weak subsolution and supersolution of (1) if they satisfy the following: for $(\psi_1,\psi_2) = (z_1,z_2) = (0,0)$ *on* $\partial\Omega$

$$\int_{\Omega}\left(\frac{\partial^2\psi_1}{\partial t^2}\omega_1 - |\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\omega_1\right)dx$$

$$\leq \int_{\Omega}\lambda^{p(x)}\left[\lambda_1 a(x)f(u) + \mu_1 c(x)h(v)\right]\omega_1 dx,$$

$$\int_{\Omega}\left(\frac{\partial^2\psi_2}{\partial t^2}\omega_2 - |\nabla\psi_2|^{p-2}\nabla\psi_2\nabla\omega_2\right)dx$$

$$\leq \int_{\Omega}\lambda^{p(x)}\left[\lambda_2 b(x)g(v) + \mu_2 d(x)\tau(u)\right]\omega_2 dx$$

and

$$\int_{\Omega}\left(\frac{\partial^2 z_1}{\partial t^2}\omega_1 - |\nabla z_1|^{p-2}\nabla z_1\nabla\omega_1\right)dx$$

$$\geq \int_{\Omega}\lambda^{p(x)}\left[\lambda_1 a(x)f(u) + \mu_1 c(x)h(v)\right]\omega_1 dx,$$

$$\int_{\Omega}\left(\frac{\partial^2 z_2}{\partial t^2}\omega_2 - |\nabla z_2|^{p-2}\nabla z_2\nabla\omega_2\right)dx$$

$$\geq \int_{\Omega}\lambda^{p(x)}\left[\lambda_1 a(x)f(u) + \mu_1 c(x)h(v)\right]\omega_2 dx$$

for all test functions

$$\omega_1(x) \in W_0^{1,p}(\Omega)$$

and

$$\omega_2(x) \in W_0^{1,p}(\Omega),$$

with $\omega_1,\omega_2 \geq 0$*.Then the following result holds:*

Lemma 2. *(ref. [6]) Suppose there exist sub and super-solutions* (ψ_1,ψ_2) *and* (z_1,z_2) *respectively of (1) such that* $(\psi_1,\psi_2) \leq (z_1,z_2)$. *Then (1) has a weak solution* (u,v) *such that*

$$(u,v) \in [(\psi_1,\psi_2),(z_1,z_2)].$$

In order to give the main result of this paper, we consider the following conditions holding:

$(H1)$ We have for every constant $K > 0$:

$$\lim_{s\to+\infty}\frac{f\left(K\left(h(s)^{\frac{1}{q-1}}\right)\right)}{s^{p-1}} = 0.$$

$(H2)$ $1 < p(x), q(x) \in C^1(\overline{\Omega})$ are functions with

$$1 < p(x), q(x) \in C^1(\overline{\Omega}) \text{ are functions with}$$

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < \infty,$$

$$1 < q^- := \inf_{x\in\Omega} q(x) \leq q^+ := \sup_{x\in\Omega} q(x) < \infty$$

$(H3)$

$$\lim_{u\to+\infty} f(u) = +\infty \lim_{u\to+\infty} g(u) = +\infty,$$

$$\lim_{u\to+\infty} h(u) = +\infty, \lim_{u\to+\infty}\tau(u) = +\infty,$$

and satisfying some natural growth condition at $u = \infty$without considering any sign condition on $f(0), g(0), h(0), \tau(0)$.

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Theorem 2. *Suppose that* $(H1)-(H3)$ *hold, then for every* $\lambda \in [A,B]$, *system (1) has at least one positive weak solution.*

Proof. For any $\upsilon = (u,v)$. Set

$$S = \begin{cases} \upsilon \in L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right) \cap L^{\infty}\left(0,T;L^{2}\left(\overline{\Omega}\right)\right): \\ \underline{\upsilon} \leq \upsilon \leq \overline{\upsilon} \end{cases}$$

and

$$S\times\overline{S} = \begin{cases} (\upsilon,\omega) \in \left(L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right) \cap L^{\infty}\left(0,T;L^{2}\left(\overline{\Omega}\right)\right)\right)^2: \\ (\underline{\upsilon},\underline{\omega}) \leq (\upsilon,\omega) \leq (\overline{\upsilon},\overline{\omega}) \end{cases}$$

Let

$$\underline{z}^{(1)} = \underline{\omega}^{(1)} - \underline{\omega}^{(0)}$$

and

$$\underline{z}_{tt}^{(1)} - B\underline{z}^{(1)} = F\left(.,\underline{u}^{(0)}\right) - \left[\underline{\omega}_{tt}^{(0)} - B\underline{\omega}^{(0)}\right]$$

since

$$Bz^{(1)} = G\left(.,\overline{\upsilon}^{(0)}\right) - G\left(.,\underline{\upsilon}^{(0)}\right) \geq 0$$

that

$$\overline{\upsilon}^{(1)} \geq \underline{\upsilon}^{(1)}$$

Now, let (υ^n,ω^n) denote either $(\overline{\upsilon}^n,\overline{\omega}^n)$or $(\underline{\upsilon}^n,\underline{\omega}^n)$.

For every n, each υ^n is a solution of the system

$$\frac{\partial^2\upsilon^n}{\partial t^2} + B(\upsilon^n) = F(x,t,\upsilon^n) \text{ in } \Omega_T$$

$$\upsilon(.,0) = G_i\left(x,t,\upsilon^{n-1}\right)$$

$$\upsilon(x,0) = \varphi_j, j = 3,4 \text{ in } \Omega.$$

There is a constant $\alpha > 0$ such that $u^n \in L^{p(x)}\left(0,T,W_0^{\alpha,p(x)}\left(\overline{\Omega}_T\right)\right)$ for all $n = 1,2,...,N$

Furthermore, since the sequence $\{u^n\}$ is uniformly bounded in $L^{p(x)}\left(0,T,W_0^{\alpha,p(x)}\left(\overline{\Omega}_T\right)\right)$, there exist positive constants M and σ independent of n such that

$$|u^n|_{L^{p(x)}\left(0,T,W_0^{\sigma,p(x)}\left(\overline{\Omega}_T\right)\right)} \leq M,$$

we first show that the limit $u^*$of u^n satisfies the equation of (2) in Ω_T,

we define the operator

$$L^n\upsilon = \frac{\partial^2\upsilon^n}{\partial t^2} + B(\upsilon^n) \equiv \frac{\partial^2\upsilon^n}{\partial t^2} - |\nabla\upsilon|^{p-2}\nabla\upsilon,$$

$$F(x,t) = F\left(x,t,\upsilon^{n-1}\right) - \varphi_i\left(x,0,\upsilon^{n-1}\right)$$

this proves that $\overline{u}$ and $\underline{u}$ are both solutions of (1).

3 The semi-discrete problem of hyperbolic systems

In this section, we discretize the problem (1), we obtain the following problems

$$\begin{cases} u^{n,m+1} + \tau^2\Delta_{p(x)}u^{n,m+1} \\ = \frac{-\tau^2\lambda^{p(x)}\left[\lambda_1 a(x)f(v^{n,m}) + \mu_1 c(x)h(u^{n,m+1})\right]}{2} \\ -\frac{u^{n+1,m+1}+u^{n-1,m+1}}{2} \text{ in } \Omega_T, \\ v^{n,m+1} + \tau^2\Delta_{p(x)}v^{n,m+1} \\ = \frac{-\tau^2\lambda^{p(x)}\left[\lambda_2 b(x)g(v^{n,m+1}) + \mu_2 d(x)\tau(u^{n,m})\right]}{2} \\ -\frac{v^{n+1,m+1}+v^{n-1,m+1}}{2} \text{ in } \Omega_T, \\ u^{n,m+1} = v^{n,m+1} = 0 \quad \text{on } \partial\Omega, \\ u^{0,\cdot} = u_0, v^{0,\cdot} = v_0 \text{ on } \Omega \\ u^{\cdot,m+1} = u_1; v^{\cdot,m+1} = v_1 \text{ on } \Omega \end{cases} \tag{3}$$

Where $N\tau = T$, $0 < \tau < 1$, and for $1 \leq n \leq N$.

Definition 3. *A weak solution to discretized problems (3) is a sequence $(u^{n,m},v^{n,m})_{0\leq n\leq N;0\leq m\leq T}$ such that $u^{0,\cdot} = u_0, v^{0,\cdot} = v_0$ and $u^{\cdot,m+1} = u_1; v^{\cdot,m+1} = v_1$ $(u^{n,m},v^{n,m})$ is defined as a weak solution to the problem*

$$\begin{cases} u^{n,m+1} + \tau^2\Delta_{p(x)}u^{n,m+1} \\ = \frac{-\tau^2\lambda^{p(x)}\left[\lambda_1 a(x)f(v^{n,m}) + \mu_1 c(x)h(u^{n,m+1})\right]}{2} \\ -\frac{u^{n+1,m+1}+u^{n-1,m+1}}{2} \quad in \quad \Omega_T, \\ v^{n,m+1} + \tau^2\Delta_{p(x)}v^{n,m+1} \\ = \frac{-\tau^2\lambda^{p(x)}\left[\lambda_2 b(x)g(v^{n,m+1}) + \mu_2 d(x)\tau(u^{n,m})\right]}{2} \\ -\frac{v^{n+1,m+1}+v^{n-1,m+1}}{2} \quad in \quad \Omega_T, \\ u^{n,m+1} = v^{n,m+1} = 0 \quad on \quad \partial\Omega, \\ u^{0,\cdot} = u_0, v^{0,\cdot} = v_0 \; on \; \Omega, \\ u^{\cdot,m+1} = u_1; v^{\cdot,m+1} = v_1 \; on \; \Omega \end{cases}$$

i.e.

$$\left(u^{n,m},v^{n,m}\right)\in L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right)$$
$$\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)\times L^{q(x)}\left(0,T;W_0^{1,q(x)}(\Omega)\right)$$
$$\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)$$

and for all

$$(\varphi,\psi)\in\left(W_0^{1.p(.)}(\Omega)\times W_0^{1.q(.)}(\Omega)\right),$$

We have, for all

$$\int_\Omega u^{n,m+1}\varphi dx+\tau^2\int_\Omega |\nabla U_n|^{p(x)-2}\nabla U_n.\nabla\varphi dx$$
$$=\int_\Omega\left[\frac{-\tau^2\lambda^{p(x)}\left[\lambda_1 a(x)f(v^{n,m})+\mu_1 c(x)h(u^{n,m+1})\right]}{2} - \frac{u^{n+1,m+1}+u^{n-1,m+1}}{2}\right]\varphi dx, \tag{4}$$

$$\int_\Omega v^{n,m+1}\psi dx+\tau^2\int_\Omega |\nabla v|^{p(x)-2}\nabla v.\nabla\psi dx$$
$$=\int_\Omega\left[\frac{-\tau^2\lambda^{p(x)}\left[\lambda_2 b(x)g(v^{n,m+1})+\mu_2 d(x)\tau(u^{n,m})\right]}{2} - \frac{v^{n+1,m+1}+v^{n-1,m+1}}{2}\right]\times\psi dx. \tag{5}$$

Theorem 3.[6]*Let hypotheses* $(H_1),(H_2),(H_3)$, *with being satisfied, the problem* (3) *has a unique weak solution*

$$\left(u^{n,m},v^{n,m}\right)_{0\le n\le N}\in L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right)$$
$$\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)$$
$$\times L^{q(x)}\left(0,T;W_0^{1,q(x)}(\Omega)\right)\cap L^{\infty}\left(0,T;L^2\left(\overline{\Omega}\right)\right)$$

and for all $n=1,2,3,...,N$.

Lemma 3.[6] *Let* $p(.)$ *be an element of* $L^\infty(\Omega)$, *and let* $u\in W^{1,p(.)}$, *there exists a constant* C_Ω^0 *depending only on* Ω *such that*

$$\int_\Omega |u|^{p(x)}\le C_\Omega^0\int_\Omega |\nabla u|^{p(x)}. \tag{6}$$

Lemma 4.[6] *Nonlinear parabolic problems with Neumann-type boundary conditions and boundary conditions and* L^1-*data) For* $\xi,\eta\in\mathbb{R}^N$ *and* $1<p<\infty$,*we have*

$$\frac{1}{p}|\xi|^p-\frac{1}{p}|\eta|^p\le|\xi|^{p-2}\xi(\xi-\eta). \tag{7}$$

1) Let p,p' be two elements of $C^+(\overline{\Omega})$ such that $\frac{1}{p(.)}+\frac{1}{p'(.)}=1$.

For any $u\in L^{p(.)}(\Omega)$ and $v\in L^{p'(.)}(\Omega)$ we have

$$\left|\int_\Omega u.vdx\right|\le\left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\|u\|_{p(.)}\|v\|_{p'(.)} \tag{8}$$

.

4 Some stability results of hyperbolic systems (1)

In this section, we give some a priori estimates for discrete weak solution $(u^{n,m},v^{n,m})_{0\le n\le N}$ which we use later to derive convergence results for the Euler scheme.

Theorem 4.*Let* $(H_1),(H_2)$, *and* (H_3) *be satisfied. Then, there exists a positive constant* $C(u_0,a,b,c,d)$ *depending on the data but not on* N *such that for all* $n=1,2,3,...,N$, *we have*

$$\|u^{n,m}\|_2^2\le C(u_0,a,b,c,d),$$

*Proof.*Let $k>0$ and $1\le n\le N$, we take $\varphi=\psi=|U_n|^kU_n$ as test function in equality (4) and (5) and we obtain

$$\int_\Omega |u^{n,m}|^{k+2}dx$$
$$+\tau^2\int_\Omega |\nabla u^{n,m}|^{p(x)-2}\nabla u^{n,m}.\nabla(|u^{n,m}|^k u^{n,m})dx=$$
$$\tau^2\left[\frac{\int_\Omega\lambda^{p(x)}[\lambda_1 a(x)f(v^{n,m})+\mu_1 c(x)h(U_n)](|u^{n,m}|^k u^{n,m})dx}{2} + \frac{\int_\Omega u^{n+1,m}+u^{n-1,m}(|U_n|^kU_n)}{2}dx\right] \tag{9}$$

This implies that

$$\|u^{n,m}\|_{k+2}^{k+2}\le\tau^2c_1\|u^{n,m}\|_{k+1}^{k+1} + \|u^{n,m}\|_{k+2}\|u^{n,m}\|_{k+2}^{k+1} \tag{10}$$

If $\|u^{n,m}\|_{k+2}=0$,we get immediately the result

If $\|u^{n,m}\|_{k+2}\ne 0$, inequality (10) becomes $\|u^{n,m}\|_{k+2}\le\tau^2c_1+\|u^{n,m}\|_{k+2}$

$\le Tc_2+\left\|u^{0,0}\right\|_\infty.$

Taking the limit as $k\to\infty$, we deduce the result .

Theorem 5.*Let hypotheses* $(H_1),(H_2)$ *and* (H_3)*be satisfied. Then, there exists a positive constant* $C(u_0,a,b,c,d)$ *depending on the data but not on* N *such that for all* $n=1,2,3,...N$, *we have*

$$\tau^2 \sum_{j=1}^{n} \int_{\Omega} |\nabla v|^{q(x)-2} \nabla v.\nabla u_j dx$$

$$+\tau \sum_{j=1}^{n} \left\| \lambda^{q(x)} \lambda_2 b(x) g(u_j) \right\|_1$$

$$\leq C(u_0, a, b, c, d),$$

*Proof.*Let $k > 0$, we define the following function

$$T_\tau(s) = \begin{cases} s \ \text{ if } |s| \leq k, \\ k\, sign(s) \ \text{ if } |s| > k \end{cases}$$

Where

$$sign(s) = \begin{cases} 1 \ \text{ if } s > 0, \\ 0 \ \text{ if } s = 0, \\ -1 \ \text{ if } s < 0. \end{cases}$$

In equality (4) and (5), we take $\varphi = \psi = T_k(U_i)$ as a test function and dividing this equality by k, taking limits when k goes to 0, we get

$$\|u_j\|_1 + \left\| \lambda^{p(x)} \mu_1 c(x) h(u_j) \right\|_1$$

$$+ \lim_{k\to\infty} \frac{\tau^2}{k} \int_{\{U_j \leq k\}} \left|\nabla u_j\right|^{p(x)} dx \tag{11}$$

$$\leq \tau \left\| \lambda^{p(x)} \lambda_1 a(x) f(v) \right\|_1 + \|u_{j-1}\|_1$$

And respectively,

$$\|u_j\|_1 + \left\| \lambda^{q(x)} \lambda_2 b(x) g(u_j) \right\|_1$$

$$+ \lim_{k\to\infty} \frac{\tau^2}{k} \sum_{j=1}^{n} \int_{\{U_j \leq k\}} |\nabla v|^{q(x)-2} \nabla v.\nabla u_j dx$$

$$\leq \|u_j\|_1 + \left\| \lambda^{q(x)} \lambda_2 b(x) g(u_j) \right\|_1$$

$$+ \left\| |\nabla v|^{q(x)-2} \nabla v \right\|^{q} . \lim_{k\to\infty} \frac{\tau^2}{k} \sum_{j=1}^{n} \int_{\{U_j \leq k\}} |\nabla u|^{q(x)} dx,$$

Which implies

$$\|u_j\|_1 + \left\| \lambda^{q(x)} \lambda_2 b(x) g(u_j) \right\|_1$$

$$+ \left\| |\nabla v|^{q(x)-2} \nabla v \right\|^{q} . \lim_{k\to\infty} \frac{\tau^2}{k} \sum_{j=1}^{n} \int_{\{U_j \leq k\}} |\nabla u_j|^{q(x)} dx$$

$$\leq \tau^2 \left\| \lambda^{q(x)} \mu_2 d(x) \tau(v) \right\|_1 + \|u_{j-1}\|_1 \tag{12}$$

$$\leq \tau^2 \left[\left\| \lambda^{q(x)} \lambda_2 b(x) g(u_j) \right\|_1 + \left\| \lambda^{q(x)} \mu_2 d(x) \tau(v) \right\|_1 \right]$$

4.1 Asymptotic behavior of solutions

In this section, we explore the asymptotic behavior of the solution of system (1) in relation to a weak positive solutions of (2).at $t \to 0$

Definition 4.*A measurable function $u : \Omega_T \to \mathbb{R}$ is a weak solution to hyperbolic systems involving of $(p(x), q(x))-$Laplacien (1) in Ω_T if $u(.,0) = u_0$ in Ω,*

$$u \in C(0,T; L^2(\Omega)) \cap L^p(0,T; H_0^1(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^2(\Omega_T), \nabla u \in \left(L^2(\Omega_T)\right)^N$$

and for all $\varphi \in C^1(\Omega_T)$ and $\psi \in C^1(\Omega_T)$, we have

$$\int_0^T \int_\Omega \frac{\partial^2 u}{\partial t^2} \varphi dxdt$$

$$+ \int_0^T \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dxdt$$

$$+ \int_0^T \int_\Omega (-\lambda^{p(x)} \mu_1 c(x) h(u)) \varphi dxdt$$

$$= \int_0^T \int_\Omega \lambda^{p(x)} \lambda_1 a(x) f(v) \varphi dxdt \tag{13}$$

Lemma 5.

$$\int_0^T \int_\Omega \frac{\partial^2 v}{\partial t^2} \psi dxdt$$

$$+ \int_0^T \int_\Omega |\nabla v|^{q(x)-2} \nabla v \nabla \psi dxdt$$

$$+ \int_0^T \int_\Omega (-\lambda^{q(x)} \lambda_2 b(x) g(u)) \psi dxdt$$

$$= \int_0^T \int_\Omega \lambda^{q(x)} \mu_2 d(x) \sigma(v) \psi dxdt$$

Lemma 6.*Let $\underline{u}, \overline{u}$ be the solutions of 1 with $\underline{u}(x,0) = \varphi_1, \overline{u}(x,0) = \varphi_2$. Then $\underline{u}(x,t)$ is nondercreasing in t , $\overline{u}(x,t)$ is nonincreasing and $\overline{u} > \underline{u}$ for all $t \geq 0, x \in \Omega$*

Theorem 6. *Let hypotheses $(H_1),(H_2)$ and (H_3) be satisfied. and let $u(x,t)$ be the solution of a new class of hyperbolic systems (1) with $\Psi \in S^*$ then*

$$\lim_{t\to\infty} u(x,t) = \begin{cases} \underline{u}_s(x) & if\ \widehat{u}_s \leq \Psi \leq \underline{u}_s \\ \overline{u}_s(x) & if\ \overline{u}_s \leq \Psi \leq \widetilde{u}_s \end{cases}$$

.

Proof. The pair $(\underline{u}_s, \widehat{u}_s)$ and the pair $(\widetilde{u}_s, \overline{u}_s)$ are both sub-super solutions of 1, the maximal and minimal property of $\overline{u}_s$ and $\underline{u}_s$ in S^* ensures that u is the unique:

$\underline{u}_s$ is the unique solution in $[\widehat{u}_s, \underline{u}_s]$ and $\overline{u}_s$ is the unique solution in $[\overline{u}_s, \widetilde{u}_s]$.

5 Conclusion

Our result extends and relates the stationary case, where the authors have already studied the existence of positive solutions for some classes of Laplacian elliptic problems by using one classical method which is sub-supersolution. Here, we use the same methods for proving the existence of results for hyperbolic Laplacian system. This problem is of much interest in physics, since it appears naturally in inflation cosmology and supersymmetric theories, quantum mechanics and nuclear physics . This type of problems has applications in many branches of physics such as nuclear physics, optics and geophysics.

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