On Jain-Beta Linear Operators

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Abstract: Starting from a sequence of linear positive operators introduced by G.C. Jain, we present an integral version of it. Approximation properties and the rate of convergence are investigated. We use the concept of A-statistical convergence. An extension for smooth functions is also given.

Keywords: Linear positive operator, modulus of continuity, A-statistical convergence.

1. Introduction

Set \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) and \( \mathbb{R}_+ = [0, \infty) \). By using the Poisson-type distribution given by

\[
w_\beta(k; \alpha) = \frac{\rho}{k!} (\alpha + k\beta)^k \rho^{-k} e^{-\rho}, \quad k \in \mathbb{N}_0,
\]

for \( 0 < \alpha < \infty \) and \( |\beta| < 1 \), G.C. Jain [7] introduced and studied the following class of positive linear operators

\[
(P_n^{[\beta]} f)(x) = \sum_{k=0}^{\infty} w_\beta(k; nx) f \left( \frac{k}{n} \right), \quad x \geq 0,
\]

where \( \beta \in [0, 1] \) and \( f \in C(\mathbb{R}_+) \), the space of all real-valued continuous functions defined on \( \mathbb{R}_+ \). In the particular case \( \beta = 0 \), \( P_n^{[0]} \), \( n \in \mathbb{N} \), turn into well-known Szász-Mirakjan operators, see [11], [9] (\( P_n^{[0]} \) \( (S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \quad x \geq 0. \)

Due to their properties, the operators \( S_n \) have been intensively studied by many mathematicians. Thus, in our opinion, the class \( (P_n^{[\beta]}) \) should deeper investigate. This paper focuses on an integral variant of the discrete operators defined by (2). The construction is presented in Section 2. The approximation properties of our mixed summation-integral type operators are collected in Section 3.

We mention that a Kantorovich-type extension of \( P_n^{[\beta]} \) was given in [12].

2. The operator \( J_n^{[\beta]} \)

Considering the weight function \( \rho_\lambda : \mathbb{R}_+ \to [1, \infty) \), \( \rho_\lambda(t) = 1 + t^{2+\lambda} (\lambda \geq 0) \), we define the space

\[
C_{\rho_\lambda}(\mathbb{R}_+) = \left\{ f \in C(\mathbb{R}_+) : \frac{f(x)}{\rho_\lambda(x)} \text{ is convergent as } x \to \infty \right\}
\]

endowed with the usual norm \( \| f \|_{\rho_\lambda} = \sup_{x \geq 0} \frac{|f(x)|}{\rho_\lambda(x)} \).

Further on, we introduce a sequence of operators calling it Jain-Beta, as follows

\[
(J_n^{[\beta]} f)(x) = \sum_{k=1}^{\infty} \frac{w_\beta(k; nx)}{B(n+1,k)} \int_0^{\infty} f(t) \frac{t^{k-1}}{(1+t)^{n+k+1}} dt + e^{-nx} f(0),
\]

where \( n \geq 2, f \in C_{\rho_\lambda}(\mathbb{R}_+) \) and \( w_\beta(k; nx) \) is given as in (1). They have Jain and Beta basis functions in summation and integration, respectively. One can see, for any \( f \in C_{\rho_\lambda}(\mathbb{R}_+) \) the integrals from (4) are well-defined. Indeed, if \( f \in C_{\rho_\lambda}(\mathbb{R}_+) \) then a positive constant \( M_f \) exists such that

\[
|f(t)| \leq M_f (1 + t^2).
\]

The convergence of the integrals

\[
\int_0^{\infty} t^{k+1} (1 + t)^{-n-k-1} dt, \quad k \geq 1,
\]

was given in [12].
It is obvious that these operators are linear. Because any function \( f \geq 0 \) implies \( J_n[\beta]f \geq 0 \), they are also positive. For \( \beta = 0 \), the operators \( J_n[\beta] \) reduce to the mixed Szász-Beta operators recently investigated by V. Gupta and M.A. Noor [6].

Let \( e_j, j \in \mathbb{N}_0 \), be the \( j \)-th monomial, \( e_j(t) = t^j \). It is known (see, e.g., [2; Proposition 4.2.5]) that \( \{e_0, e_1, e_2\} \) is a strict Korovkin set in \( C_{\rho_0}(\mathbb{R}_+) \). So, our first concern is to determine values of these test functions. To do this, we recall the following identities established in [7; Eqs. (2.12)-(2.14)]

\[
\begin{align*}
(P_n[\beta]e_0)(x) &= 1, \\
(P_n[\beta]e_1)(x) &= \frac{x}{1-\beta}, \\
(P_n[\beta]e_2)(x) &= \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}, \quad x \geq 0.
\end{align*}
\]

Lemma 1. The operators \( J_n[\beta] \), \( n \geq 2 \), defined by (4) satisfy the following relations

\[
\begin{align*}
J_n[\beta]e_0 &= e_0, \\
J_n[\beta]e_1 &= e_1 - \frac{1}{1-\beta}, \\
J_n[\beta]e_2 &= \frac{n}{(n-1)(1-\beta)^3}(e_1 + \frac{1}{1-\beta}) - \frac{1}{1-\beta}.
\end{align*}
\]

Proof. By simple computation we get

\[
(J_n[\beta]e_0)(x) = \sum_{k=1}^{\infty} w_\beta(k; nx) + e^{-nx} = (P_n[\beta]e_0)(x).
\]

\[
(J_n[\beta]e_1)(x) = \sum_{k=1}^{\infty} w_\beta(k; nx) B(n, k + 1) B(n + 1, k) = \sum_{k=0}^{\infty} w_\beta(k; nx) \frac{k}{n} (P_n[\beta]e_1)(x).
\]

\[
(J_n[\beta]e_2)(x) = \sum_{k=1}^{\infty} w_\beta(k; nx) B(n, k + 2) B(n + 1, k) = \frac{n}{n-1} (P_n[\beta]e_2)(x) + \frac{n}{n-1} (P_n[\beta]e_1)(x).
\]

Taking into account (5) we easily obtain (6) and the proof is completed.

We also introduce the \( s \)-th order central moment of the operator \( J_n[\beta] \), that is \( (J_n[\beta] \varphi_x^s) \), where \( \varphi_x(t) = 1 - x - t \), \( (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \). On the basis of (6), by a straightforward calculation, we obtain

Lemma 2. The first and the second central moment of \( J_n[\beta] \), \( n \geq 2 \), operators are given by

\[
(J_n[\beta] \varphi_x^0)(x) = \frac{1}{1-\beta},
\]

\[
(J_n[\beta] \varphi_x^1)(x) = \frac{\beta}{1-\beta} x^2 + \frac{1}{n-1} \frac{n}{(n-1)(1-\beta)^3} - \frac{1}{1-\beta}.
\]

3. Approximation properties

We establish the rate of convergence of the sequence \( (J_n[\beta]f) \) to \( f \) in terms of the rate of convergence of the test functions. The modulus of continuity \( \omega_{[0,a]}(f; \cdot) \) is also involved, where

\[
\omega_{[0,a]}(f; \delta) = \sup \{|f(x') - f(x'')| : x', x'' \in [0, a], |x' - x''| \leq \delta\}, \quad \delta \geq 0,
\]

is continuous on the interval \([0, a]\).

Theorem 1. Let \( J_n[\beta] \), \( n \geq 2 \), be defined by (4). For any function \( f \in C_{\rho_0}(\mathbb{R}_+) \) one has

\[
|(J_n[\beta]f)(x) - f(x)| \leq 1 + \sqrt{x(x + 1)} \omega_{[0,a]}(f; \sqrt{\delta_{\alpha,\beta}}), \quad x \in [0, a],
\]

where \( \delta_{\alpha,\beta} \) is defined at (9).

Proof. We use the result of Shisha and Mond [10]. Considering the interval \([0, a]\), it says: if \( L \) is a linear positive operator defined on \( C(I), [0, a] \subseteq I \), then for every \( x \in [0, a] \) and \( \delta > 0 \) one has

\[
\frac{|(J_n[\beta]f)(x) - f(x)|}{|(L - I)f(x)|} \leq \frac{|f(|x|)|}{|(L - I)|} + \frac{(Le_0)(x) + \delta^{-1} \sqrt{(Le_0)(x)(Le_0)^2(x)} \omega_{[0,a]}(f; \delta)}{
\}

Knowing that \( J_n[\beta]e_0 = e_0 \) and relation (8), by choosing \( \delta = \sqrt{\delta_{\alpha,\beta}} \) the above inequality leads us to the desired result.

Examining relation (6) and based on famous Korovkin theorem [8], it is clear that \( (J_n[\beta]f)_{n\geq2} \) does not form an approximation process. The next step is to transform it for enjoying of this property. For each \( n \geq 2 \), the constant \( \beta \) will be replaced by a number \( \beta_n \in [0, 1] \). If

\[
\lim_{n \to \infty} \beta_n = 0,
\]
then Lemma 1 ensures $\lim_{n}\langle j^{[\beta_n]}_n e_j(x) = x^j, j = 0, 1, 2, \rangle$
uniformly on any interval compact $K \subset \mathbb{R}_+$. Based on
Korovkin criterion we can state

**Theorem 2.** Let $j^{[\beta_n]}_n, n \geq 2$, be defined as in (4),
where $(\beta_n)_{n \geq 2}$ satisfies (10). For any compact $K \subset \mathbb{R}_+$
and for each $f \in C_{\rho_0}(\mathbb{R}_+)$ one has

$$\lim_{n} j^{[\beta_n]}_n f(x) = f(x), \text{ uniformly in } x \in K.$$ 

Our next concern is the study of statistical convergence of the sequence of operators. For the convenience of the
reader, let recall the concept of this type of convergence.

The density of a set $S \subset \mathbb{N}$ is defined by

$$\delta(S) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_S(k),$$
powered the limit exists, where $\chi_S$ is the characteristic function of $S$. Following [4], a real sequence $x = (x_n)_{n \geq 1}$ is statistically convergent to a real number $L$ if, for every $\varepsilon > 0$,

$$\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0.$$ 

We write $st - \lim x_n = L$. It is known that any convergent
sequence is statistically convergent, but not conversely. Closely related to this notion is A-statistical convergence
where $A = (a_{n,k})$ is an infinite summability matrix. For a given sequence $x = (x_n)_{n \geq 1}$, the A-transform of
x denoted by $Ax = (Ax_n)_{n \geq 1}$ is defined by

$$(Ax)_n = \sum_{k=1}^{n} a_{n,k} x_k, n \in \mathbb{N},$$
powered the series converges for each $n$. Suppose that $A$ is non-negative regular summability matrix, i.e., $a_{n,k} \geq 0$
and the matrix transformation of any convergent sequence preserves its limit.
The sequence $x = (x_n)_{n \geq 1}$ is $A$-statistically convergent
to the real number $L$ if, for every $\varepsilon > 0$, one has

$$\lim_{n \to \infty} \sum_{k \in I(\varepsilon)} a_{n,k} = 0,$$
where $I(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. 

We write $st - \lim x_n = L$, see e.g. [5].

Duman and Orhan [3; Theorem 3] proved the following
weighted Korovkin-type theorem via A-statistical convergence.

**Theorem 3.** Let $A = (a_{n,k})$ be a non-negative regular
summability matrix and let $\rho, \rho'$ weight functions such that

$$\rho(x) \to 0, \quad |x| \to \infty$$

Assume that $(T_n)_{n \geq 1}$ is a sequence of positive linear
operators from $C_\rho(\mathbb{R})$ into $C_{\rho'}(\mathbb{R})$. One has

$$st_A - \lim_{n} \|T_n f - f\|_{\rho'} = 0, \quad f \in C_\rho(\mathbb{R}),$$
if and only if

$$st_A - \lim_{n} \|T_n F_k - F_k\|_{\rho} = 0, \quad k = 0, 1, 2, \quad (12)$$
where $F_k(x) = x^k \rho(x)/(1 + x^2)$.

As regards to our sequence we prove the following result.

**Theorem 4.** Let $A = (a_{n,k})$ be a non-negative regular
summability matrix and $\lambda > 0$ be fixed. Let $j^{[\beta_n]}_n, n \geq 2,$
be defined as in (4), where $(\beta_n)_{n \geq 2}$, $0 \leq \beta_n < 1$, satisfies

$$st_A - \lim \beta_n = 0.$$ 

One has

$$st_A - \lim_{n} \|J^{[\beta_n]}_n f - f\|_{\rho_2} = 0, \quad f \in C_{\rho_0}(\mathbb{R}_+).$$

**Proof.** We use Theorem 3 which is still valid if one replaces
the domain $\mathbb{R}$ by $\mathbb{R}_+$. Also, we choose the weight functions $\rho := \rho_0$ and $\rho' := \rho_2$. Since $\lambda > 0$, relation (11)
is fulfilled and one has $C_{\rho_0}(\mathbb{R}_+) \subset C_{\rho_2}(\mathbb{R}_+)$. The test function are $F_k = e_k, k = 0, 1, 2$. Taking in View
Lemma 1 we have

$$\|J^{[\beta_n]}_n e_k - e_k\|_{\rho_0} = 0,$$

$$\|J^{[\beta_n]}_n e_1 - e_1\|_{\rho_0} = \frac{\beta_n}{1 - \beta_n},$$

$$\|J^{[\beta_n]}_n e_2 - e_2\|_{\rho_0} = \frac{1}{(1 - \beta_n)^2} + \frac{1}{n(1 - \beta_n)^3}.$$ 

Hypothesis (14) and above relations imply

$$st_A - \lim_{n} \|J^{[\beta_n]}_n e_k - e_k\|_{\rho_0} = 0, \quad k = 0, 1, 2.$$ 

Since (13) holds, on the basis of Theorem 3, identity (15)
takes place and this ends the proof.

To increase the rate of convergence we can replace $j^{[\beta_n]}_n$
by its generalization of the $r$-th order, see [1].

Let $f \in C_r(\mathbb{R}_+)$ such that $e_k f^{(s)} \in C_{\rho_0}(\mathbb{R}_+)$ for
$s = 0, 1, \ldots, r$, and let $T_r f(x; \cdot)$ be the $r$-th degree Taylor
polynomial associated to the function $f$ at the point $x \in \mathbb{R}_+$. For $n \geq 2$ and any $x \geq 0$ we define the linear operators

$$(J^{[\beta_n]}_n r f)(x) = J^{[\beta_n]}_n(T_r f; x)$$

$$= \sum_{k=1}^{\infty} w_{\beta_n}(k; n, x) \sum_{s=0}^{r} \frac{1}{s!} \int_{0}^{\infty} f^{(s)}(t) (x - t)^{s-k-1} (1 + t)^{n+k+1} dt$$

$$+ e^{-nx} f(0).$$

Clearly, $J^{[\beta_n]}_n = J^{[\beta_n]}_0, n \geq 2$. These operators keep the
linearity property but loose the positivity.

In what follows, for $\alpha \in (0, 1]$ and $M > 0$, $Lip_M(\alpha)$
stands for the the subset of all Hölder continuous functions
$f$ on $\mathbb{R}_+$, with exponent $\alpha$ and constant $M$, i.e.,

$$|f(x) - f(y)| \leq M|x - y|^\alpha, \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+.$$ 

Applying [1; Theorem 1] we obtain

**Theorem 5.** Let $A$ be a non-negative regular summability
matrix. Let $r \in \mathbb{N}$ be fixed, $\alpha \in (0, 1]$ and $M > 0,$
Let the operators $J_{\beta_n}$ and $J_{\beta_n, r}$, $n \geq 2$, be defined by (4) and (16), respectively.

If $x \geq 0$ and $x^{+\alpha} \in C_{p_0}(\mathbb{R}_+)$ such that

\[ st_A - \lim_{n} \beta_n = 0 \quad \text{and} \quad st_A - \lim_{n} (J_{\beta_n, r} x^{+\alpha})(x) = 0, \]

then

\[ st_A - \lim_{n} |(J_{\beta_n, r} f)(x) - f(x)| = 0 \]

holds for any function $f \in C^r(\mathbb{R}_+) \cap C_{p_0}(\mathbb{R}_+)$ with the properties $e_n f^{(s)} \in C_{p_0}(\mathbb{R}_+), s = 0, 1, \ldots, r$ and $f^{(r)} \in Lip_{M \alpha}$.

In other words, considering that the initial approximation process is A-statistically pointwise convergent, the result says that the property is inherited by the new sequence $J_{\beta_n, r}$ under additional conditions imposed on the smooth signal $f$.

References


Saddika Tarabie is lecturer at Tishrin University of Syria, Faculty of Sciences, and she also teaches at the Arab Academy for Science, Technology and Shipping, Latakia branch. Her area of scientific interest includes approximation theory, complex analysis and integral equations which is proved by the articles and books published in Arabic or in English. In this period Saddika Tarabie completes her PhD dissertation on which she will sustain at Babeș-Bolyai University of Cluj-Napoca, Romania.