Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150510

Solving Systems of Linear Fractional Integro-Differential Equations with Delay using Least Squares Method and Chebyshev Polynomials

Kamal R. Raslan[1], *Khalid K. Ali*[1], *Emad M. H. Mohamed*[1], *Mansour Lotayif*[2] *and Mohamed A. Abd El salam*[1,*]

[1] Department of Mathematics, Faculty of Science, Al azhar University, Nasr-City, 11884, Cairo, Egypt
[2] College of Administrative Sciences, Applied Science University, P.O. Box 5055, East Al-Ekir, Kingdom of Bahrain

Received: 22 Apr. 2021, Revised: 2 Jun. 2021, Accepted: 7 Jul. 2021
Published online: 1 Sep. 2021

Abstract: In this paper, a numerical scheme is introduced for solving a linear system of fractional delay integro-differential equations (FDIDEs). The fractional derivative is considered in the Caputo sense. The spectral least squares method with the aid of the first kind Chebyshev polynomials was introduced to treat the proposed systems. The suggested method reduces this type of system to the solution of a block system of linear algebraic equations. To demonstrate the accuracy and applicability of the presented method some test examples are provided. Numerical results show that this approach is easy to implement and accurate when applied to the proposed system of FDIDEs.

Keywords: Linear system of fractional integro-differential equations with delay, Least squares method, Chebyshev polynomials, Caputo's fractional derivativ.

1 Introduction

Many problems can be modeled by integro-differential equations (IDEs) from various sciences and engineering applications. The inclusion of fractional calculus greatly enriched the consideration of applications of IDEs and made the base of applications much wider than before. Both fractional Fredholm and Volterra types play a pivotal role in the modeling and applications. A particularly rich source is an electric-circuit analysis [1], the control of systems and the activity of interacting inhibitory are all governed by fractional integro-differential equations (FIDEs) [2]. Many applications to floating structures and viscoelastic [3,4] and some computational neuroscience are modeled by FIDEs [5].

Recently, several numerical methods treat systems of FDEs and have been given. The authors in [7,23] are applied the collocation method for solving the following: non-linear fractional Langevin equation involving two fractional orders in different intervals and fractional Fredholm IDEs. Chebyshev collocation method is introduced in [8,9,10,16] for solving multi-terms FDEs, non-linear Volterra, and Fredholm The authors in [11] are applied the variational iteration method for solving FIDE with the non-local boundary conditions. Adomian decomposition method is introduced in [17] to employ FIDEs. References[8,13,14] used the Homotopy perturbation method for solving non-linear Fredholm and systems of linear FIDEs. Taylor series collocation method is introduced in [6] for solving linear FIDEs of Volterra type. In [27] numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method. The appearance of a delay in the argument enriches the meaning and history of the variables. The integro-differential equations that have fractional derivative and their argument being delayed are called fractional delay integro-differential equations (FDIDEs).

In this paper, the least-squares method with aid of Chebyshev polynomials of the first kind is used to approximate systems of FDIDEs [28,29]. The spectral least-squares method has been studied in [24,25,26]. In our previous work [18,19,20,21,22] we introduce some spectral solutions to delay FDEs and FIDEs, and this work is more developable and generalized to systems. The introduced system of fractional delay

* Corresponding author e-mail: mohamed_salam1985@yahoo.com;

integro-differential equations (SFDIDE) of the general form is:

$$\sum_{s=1}^{n} Q_{s_i}(x)y_s^{(\alpha_{s_i})}(x-\tau_{s_i})+\sum_{r=1}^{n} P_{r_i}(x)y_r^{(h_{r_i})}(x)= f_i(x)+\int_a^x k_i(x,t)\left(\sum_{k=1}^{n}\beta_{k_i}y_k(t)\right)dt, \tag{1}$$

with supplementary conditions

$$y_i^{(j)}(\zeta_i)=b_{ij},\quad i=1,\cdots,n,\quad j=0,1,\ldots\lceil\alpha_{s_i}\rceil-1,$$

such that $m-1<\alpha_{s_i}\leq m$, $\alpha_{s_i}\geq 0$, h_{r_i} are integers $\in\{0,1\cdots,m\}$, $m\in\mathbb{N}_0$. Where $y_i^{(\alpha)}$ means $D^\alpha y_i(x)$ and it indicates the αth Caputo's fractional derivative of $y_i(x)$ and we note that: the symbol D^α is used instead of ${}_0^C D_x^\alpha$ for short. Additionaly, $f_i(x)$, $Q_{s_i}(x)$, $P_{r_i}(x)$, $k_i(x,t)$ are given functions, $\beta_{k_i}\in\mathbb{R}$, x,t are real varying in the interval $[a,b]$ and $y_i(x)$ are the unknown functions to be determines.

2 Preliminaries and notations

In this section, we present some necessary definitions and mathematical preliminary theories required for our subsequent development.

2.1 The Caputo's fractional derivative

Definition 1. The Caputo's fractional derivative operator D^v of order v is defined in the following form:

$$D^v g(x)=\frac{1}{\Gamma(m-v)}\int_0^x\frac{g^{(m)}(t)}{(x-t)^{v-m+1}}\,dt,\quad v>0,$$

where, $m-1<v$, $m\in\mathbb{N}_0$ and $x>0$.
Similar to integer-order differentiation, Caputo's fractional derivative operator is a linear operator:

$$D^v\sum_{i=1}^{k}\lambda_i g_i(x)=\lambda_1 D^v g_1(x)+\lambda_2 D^v g_2(x)\cdots+\lambda_k D^v g_k(x) =\sum_{i=1}^{k}\lambda_i D^v g_i(x),$$

where, λ_i are constants.
For the Caputo's derivative we have [12].

$$D^v C=0,\qquad C \text{ is a constant}, \tag{2}$$

$$D^v x^n=\begin{cases}0, & \text{for } n\in\mathbb{N}_0 \text{ and } n<\lceil v\rceil;\\ \frac{\Gamma(n+1)}{\Gamma(n+1-v)}x^{n-v} & \text{for } n\in\mathbb{N}_0 \text{ and } n\geq\lceil v\rceil.\end{cases} \tag{3}$$

We use the ceiling function $\lceil v\rceil$ to denote the smallest integer greater than or equal to v, and $\mathbb{N}_0=\{0,1,2,\cdots\}$. Recall that for $v\in\mathbb{N}_0$, the Caputo's differential operator coincides with the usual differential operator of integer order.

2.2 Chebyshev polynomials of the first-kind

The Chebyshev polynomials $T_n(x)$ of the first kind are orthogonal polynomials of degree n in x defined on the closed interval $[-1,1]$ as:

$$T_n(x)=cos(n\phi),$$

where $cos\phi=x$ and $\phi\in[0,\pi]$. The polynomials $T_n(x)$ be generated by using the following recurrence relation with its initials

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),\ T_0(x)=1,\ T_1(x)=x,\quad n=1,2,\ldots.$$

The Chebyshev polynomials $T_n(x)$ can be expressed in terms of the power x^n in different forms found in [18,19,20,21,22], one of them is

$$T_n(x)=\sum_{k=0}^{\lceil n/2\rceil}c_k^{(n)}x^{n-2k}, \tag{4}$$

where

$$c_k^{(n)}=(-1)^k 2^{n-2k-1}\frac{n}{n-k}\binom{n-k}{k},\quad 2k\leq n.$$

Theorem 1. (Chebyshev truncation theorem) [15].
The error in approximating the well defined function $\phi(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$\phi_m(x)=\sum_{i=0}^{m}b_iT_i(x), \tag{5}$$

then

$$E_T(m)\equiv\left|\phi(x)-\phi_m(x)\right|\leq\sum_{k=m+1}^{\infty}|b_i|, \tag{6}$$

for all $\phi(x)$, all m, and $x\in[-1,1]$.

Corollary 2.1. x and t are the independent variables of (1), $a\leq x,t\leq b$ or they belong to $[a,b]$, and the interval $[a,b]$ is the intersection of the intervals of the different exists delayed arguments and $[-1,1]$, i.e.

$$x,t\in[a,b]=[\tau_{s_i},1+\tau_{s_i}]\bigcap[-1,1],\ and\ \tau_{s_i}<1.$$

3 Numerical scheme

In this section, the least squares method with aid of Chebyshev polynomials is applied to study the numerical solution of the proposed system of fractional integro-differential with delay (1). The method is based on approximating the unknown functions $y_i(x)$ as:

$$y_i(x) = \sum_{j=0}^{m} a_j^i T_j(x), \qquad -1 \leq x \leq 1, \tag{7}$$

where, $T_j(x)$ is the Chebyshev polynomial of the first kind and a_j^i, $i = 1,2,3,\cdots,n$, are constants.
Substituting (7) into (1), we obtain

$$\sum_{s=1}^{n} Q_{s_i}\left[\sum_{j=0}^{m} a_j^i D^{\alpha_{s_i}} T_j(x-\tau_{s_i})\right] + \sum_{r=1}^{n} P_{r_i}\left[\sum_{r=0}^{m} a_r^i D^{h_{r_i}} T_r(x)\right]$$
$$= f_i(x) + \int_a^x k_i(x,t)\left(\sum_{k=1}^{n} \beta_{k_i}\left[\sum_{j=0}^{m} a_j^k T_j(t)\right]\right) dt. \tag{8}$$

Hence, the residual equation is defined as:

$$R_i(x,a_0^i,a_1^i,\cdots,a_m^i) = \sum_{s=1}^{n} Q_{s_i}\left[\sum_{j=0}^{m} a_j^i D^{\alpha_{s_i}} T_j(x-\tau_{s_i})\right] + \sum_{r=1}^{n} P_{r_i}\left[\sum_{r=0}^{m} a_r^i D^{h_{r_i}} T_r(x)\right] - \int_a^x k_i(x,t)\left(\sum_{k=1}^{n} \beta_{ik}\left[\sum_{j=0}^{m} a_j^k T_j(t)\right]\right) dt - f_i(x). \tag{9}$$

Let

$$S_i(a_0^i,a_1^i,\cdots,a_m^i) = \int_a^b \left[R_i(x,a_0^i,a_1^i,\cdots,a_m^i)\right]^2 w(x)\,dx, \tag{10}$$

where, $w(x)$ is the positive weight function defined on the interval $[a,b]$, in this work we take $w(x) = 1$, then (10) obtains as:

$$S_i(a_0^i,a_1^i,\cdots,a_m^i) = \int_a^b \left\{\sum_{s=1}^{n} Q_{s_i}\left[\sum_{j=0}^{m} a_j^i D^{\alpha_{s_i}} T_j(x-\tau_{s_i})\right] + \sum_{r=1}^{n} P_{r_i}\left[\sum_{r=0}^{m} a_r^i D^{h_{r_i}} T_r(x)\right] - \int_a^x k_i(x,t)\left(\sum_{k=1}^{n} \beta_{ik}\left[\sum_{j=0}^{m} a_j^k T_j(t)\right]\right) dt - f_i(x)\right\}^2 dx. \tag{11}$$

So finding the values of a_j^i, $j = 0,\cdots,m$ which minimize S_i is equivalent to finding the best approximation for the solution of the SFDIDE (1). The minimum value of S_i is obtained by setting

$$\frac{\partial S_i}{\partial a_j^i} = 0, \qquad j = 0,1,\cdots,m, \tag{12}$$

$$\int_a^b \left\{\sum_{s=1}^{n} Q_{s_i}\left[\sum_{j=0}^{m} a_j^i D^{\alpha_{s_i}} T_j(x-\tau_{s_i})\right] + \sum_{r=1}^{n} P_{r_i}\left[\sum_{r=0}^{m} a_r^i D^{h_{r_i}} T_r(x)\right] - \int_a^x k_i(x,t)\left(\sum_{k=1}^{n} \beta_{k_i}\left[\sum_{j=0}^{m} a_j^k T_j(t)\right]\right) dt - f_i(x)\right\} \times \left\{\sum_{s=1}^{n} Q_{s_i}\left[\sum_{j=0}^{m} D^{\alpha_{s_i}} T_j(x-\tau_{s_i})\right] + \sum_{r=1}^{n} P_{r_i}\left[\sum_{r=0}^{m} D^{h_{s_r}} T_r(x)\right] - \int_a^x k_i(x,t)\left(\sum_{k=1}^{n} \beta_{k_i}\left[\sum_{j=0}^{m} T_j(t)\right]\right) dt\right\} dx = 0. \tag{13}$$

The above system of equations (13) for $j = 0,\cdots,m$ forms a block system of $n(m+1)\times n(m+1)$ linear equations with $n(m+1)\times n(m+1)$ unknown coefficients a_j^i this system can be formed by using matrices form as follows:

$$\tilde{W}\cdot A = \tilde{F}, \tag{14}$$

where A is the unknown coefficients vector, and

$$\tilde{W} = \begin{pmatrix} W_{00} & W_{02} & \cdots & W_{0n} \\ W_{11} & W_{12} & \cdots & W_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ W_{n1} & W_{n2} & \cdots & W_{nn} \end{pmatrix}, \tag{15}$$

$$\tilde{F} = \begin{pmatrix} F_0 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}, \tag{16}$$

$$W_{ij} = \begin{pmatrix} \int_a^b R_i(x,a_j^i)\gamma_0^i dx & \int_a^b R_i(x,a_j^i)\gamma_0^i dx & \cdots & \int_a^b R_i(x,a_j^i)\gamma_0^i dx \\ \int_a^b R_i(x,a_j^i)\gamma_1^i dx & \int_a^b R_i(x,a_j^i)\gamma_1^i dx & \cdots & \int_a^b R_i(x,a_j^i)\gamma_1^i dx \\ \vdots & \vdots & \vdots & \vdots \\ \int_a^b R_i(x,a_j^i)\gamma_m^i dx & \int_a^b R_i(x,a_j^i)\gamma_m^i dx & \cdots & \int_a^b R_i(x,a_j^i)\gamma_m^i dx \end{pmatrix}, \tag{17}$$

$$F_i = \begin{pmatrix} \int_a^b f_i(x)\gamma_0^i dx \\ \int_a^b f_i(x)\gamma_1^i dx \\ \vdots \\ \int_a^b f_i(x)\gamma_m^i dx \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned} R_i(x,a_j^i) = \int_a^b \Bigg\{ & \sum_{s=1}^{n} Q_{s_i} \left[\sum_{j=0}^{m} a_j^i D^{\alpha_{s_i}} T_j(x-\tau_{s_i}) \right] + \\ & \sum_{r=1}^{n} P_{r_i} \left[\sum_{r=0}^{m} a_r^i D^{h_{r_i}} T_r(x) \right] - \\ & \int_a^x k_i(x,t) \left(\sum_{k=1}^{n} \beta_{ik} \left[\sum_{j=0}^{m} a_j^k T_j(t) \right] \right) dt - f_i(x) \Bigg\}, \end{aligned} \tag{19}$$

$$\begin{aligned} \gamma_j^i = \Bigg\{ & \sum_{s=1}^{n} Q_{s_i} \left[\sum_{j=0}^{m} D^{\alpha_{s_i}} T_j(x-\tau_{s_i}) \right] + \sum_{r=1}^{n} P_{r_i} \left[\sum_{r=0}^{m} D^{h_{r_i}} T_r(x) \right] \\ & - \int_a^x k_i(x,t) \left(\sum_{k=1}^{n} \beta_{ik} \left[\sum_{j=0}^{m} T_j(t) \right] \right) dt \Bigg\}, \\ & j = 0,\cdots,m, \quad i = 1,\cdots,n. \end{aligned} \tag{20}$$

Using the properties of fractional calculus (3), proposition (4) and the binomial expansion theorem, one writes

$$\begin{aligned} D^{\alpha_{s_i}} T_j(x-\tau_{s_i}) &= \sum_{k=0}^{\lceil j/2 \rceil} c_k^{(j)} D^{\alpha_{s_i}} (x-\tau_{s_i})^{j-2k} = \\ &= \sum_{k=0}^{\lceil j/2 \rceil} c_k^{(j)} \sum_{l=0}^{j-2k} \binom{j-2k}{l} (-\tau_{s_i})^{j-2k-l} \frac{\Gamma(l+1)(x)^{l-\alpha_{s_i}}}{\Gamma(l+1-\alpha_{s_i})}, \end{aligned} \tag{21}$$

also,

$$\begin{aligned} D^{h_{r_i}} T_r(x) &= \sum_{k=0}^{\lceil r/2 \rceil} c_k^{(r)} D^{h_{r_i}} (x)^{r-2k} \\ &= \sum_{k=0}^{\lceil r/2 \rceil} c_k^{(r)} \frac{(r-2k)!}{(r-2k-h_{r_i})!} (x)^{r-2k-h_{r_i}}, \end{aligned} \tag{22}$$

where $c_k^{(j)}$ and $c_k^{(r)}$ given in (2.2). By solving the above linear system (14) we obtain the values of the unknown coefficients and consequently the approximate analytic solutions of (1) obtained.

4 Numerical examples

In this section, the Chebyshev polynomials with least squares method have been applied for solving systems of linear fractional delay integro-differential equations with known exact solution to exmamin the proposed method. All results are obtained by using codes ranning on Mathematica 10.0 packge.

Example 1. Consider the following system of linear fractional integro-differential equation with delay

$$\begin{aligned} & D^{\frac{3}{4}} y_1(x-1) + D^{\frac{3}{4}} y_2(x-1) + y_2'(x) = f_1(x) + \\ & \qquad \int_0^x (x+t)[y_1(t)+y_2(t)]\, dt, \\ & D^{\frac{3}{5}} y_2(x-1) + y_1(x) = f_2(x) + \int_0^x \sqrt{x} t^2 [y_1(t)-y_2(t)]\, dt. \end{aligned} \tag{23}$$

Subject to $y_1(0)=0$, $y_2(0)=0$, and

$$\begin{aligned} f_1(x) &= -1 + 2x - \frac{7x^4}{12} + \frac{9x^5}{20} + \frac{4x^{1/4}(-15+8x)}{5\Gamma\left[\frac{1}{4}\right]} - \\ & \qquad \frac{8x^{1/4}(15+4x(-9+4x))}{15\Gamma\left[\frac{1}{4}\right]}, \\ f_2(x) &= x - x^3 - \frac{x^{9/2}}{2} + \frac{x^{11/2}}{5} + \frac{x^{13/2}}{6} + \\ & \qquad \frac{5x^{2/5}(-21+10x)}{14\Gamma\left[\frac{2}{5}\right]}. \end{aligned}$$

The exact solutions of this system $y_1(x) = x - x^3$, $y_2(x) = x^2 - x$. Applying the least squares method with aid of Chebyshev polynomials $T_j(x)$, as mentioned in previous section, $j = 0,\cdots,m$, at $m = 4$, $n = 2$ to the system of the linear fractional delay integro-differential equation (23) and we obtain a system of linear equations with five unknown coefficients a_j^i. The solution obtained using the suggested method as:

$$\begin{aligned} & a_0^1 = -6.0984*10^{-14}, \quad a_1^1 = 0.25, \quad a_2^1 = -1.60035*10^{-13}, \\ & \quad a_3^1 = -0.25, \qquad a_4^1 = -1.0077*10^{-13}, \\ & a_0^2 = 0.5, \qquad a_1^2 = -1, \qquad a_2^2 = 0.5, \\ & \qquad a_3^2 = 1.78626*10^{-13}, \qquad a_4^2 = -9.84045*10^{-15}. \end{aligned}$$

The numerical results are showing in figure.1 It is evident that the overall errors can be made smaller by adding new terms from the series (5). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique.

Example 2. Consider the following system of fractional integro-differential equation [13]

$$\begin{aligned} & D^{\frac{1}{4}} y_1(x-1) + y_2'(x) = f_1(x) + \int_0^x (x+t)[y_1(t)+y_2(t)]\, dt, \\ & D^{\frac{1}{5}} y_2(x-1) + y_1(x) = f_2(x) + \int_0^x (x-t)\,[y_1(t)-y_2(t)]\, dt. \end{aligned} \tag{24}$$

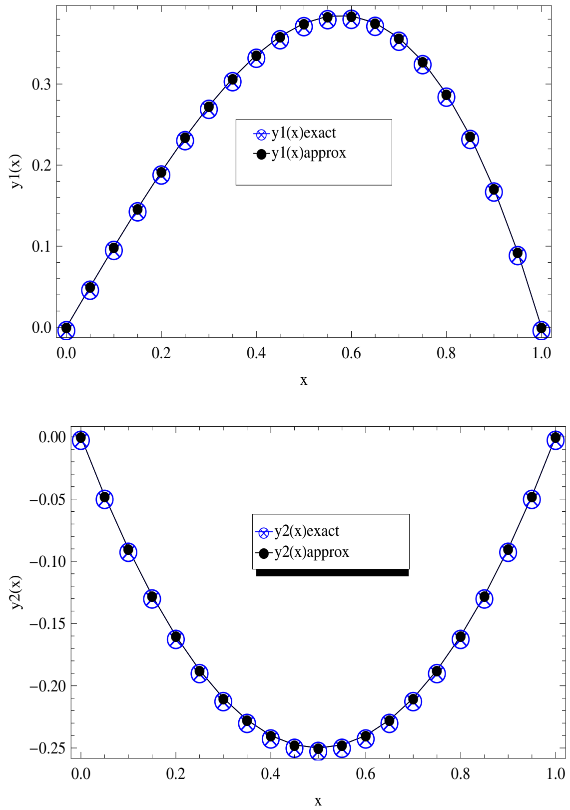

Fig. 1: Results for example (1).

Subject to $y_1(0) = 1$, $y_2(0) = 0$ and

$$f_1(x) = -1 - \frac{3x^2}{2} + \frac{29x^3}{6} + \frac{7x^4}{12} - \frac{11x^6}{30} + \frac{8(7-4x)x^{3/4}}{21\Gamma\left[\frac{3}{4}\right]},$$

$$f_2(x) = 1 - \frac{3x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^6}{30} + \frac{25x^{4/5}(-399+4x(133+5x(-19+5x)))}{1596\Gamma\left[\frac{4}{5}\right]}.$$

with exact solution $y_1(x) = 1 - x^2$, $y_2(x) = x^4 - x$. Similarly, as in example.1 applying the least squares method with Chebyshev polynomials $T_j(x)$, $m = 4$ at $n = 2$ to the fractional integro-differential equation (24) the numerical results are showing figure.2 and we obtain the approximate solution as:

$$y_1 = 0.5 - 3.72906 * 10^{-12}(x) - 0.5(2x^2 - 1) + 8.68425 * 10^{-13} \times (4x^3 - 3x) + 4.45434 * 10^{-13}(1 - 8x^2 + 8x^4),$$

$$y_2 = 0.375 - x + 0.5(-1 + 2x^2) - 8.38486 * 10^{-13} \times (-3x + 4x^3) + 0.125(1 - 8x^2 + 8x^4).$$

The solution obtained using the suggested method is in

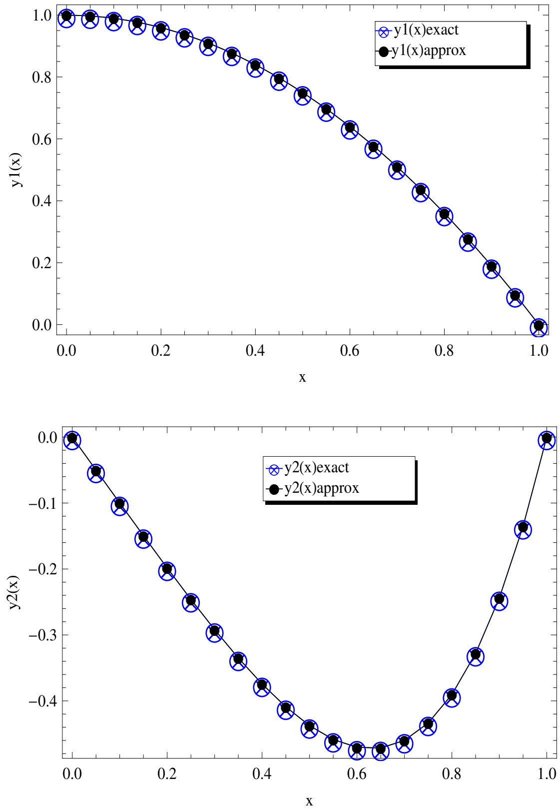

Fig. 2: results for example (2).

excellent agreement with the already exact solution and show that this approach can be solved the problem efectively. It is evident that the overall errors can be made smaller by adding new terms from the series (5). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique.

Example 3. Consider the following system of integro differential equations of fractional order[30]

$$
\begin{aligned}
D^{\frac{3}{4}}y_1(x) &= \frac{1}{\Gamma(1.75)}x^{\frac{3}{4}} - \frac{x^3}{3} + x\cos(x) - \sin(x) + \\
&\qquad \int_0^x [\sin(t)y_1(t) + y_2(t)]\, dt, \\
D^{\frac{1}{2}}y_2(x) &= \frac{2}{\Gamma(2.5)}x^{\frac{3}{2}} + \frac{x^3}{4} + \frac{3}{16}x^4 + \\
&\qquad \int_0^x -\frac{3}{4}\, [y_2(t) + y_3(t)]\, dt, \\
D^{\frac{3}{4}}y_3(x) &= \frac{6}{\Gamma(3.25)}x^{\frac{3}{9}} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \\
&\qquad \int_0^x [y_1(t) + y_2(t) + y_3(t)]\, dt.
\end{aligned} \tag{25}
$$

Subject to $y_1(0) = 0$, $y_2(0) = 0$ and $y_3(0) = 0$ the exact solution of this system is $y_1(x) = x$, $y_2(x) = x^2$ and $y_2(x) = x^3$. Similarly, as pervious in examples.1 and 2 applying the proposed method $m = 3$ at $n = 3$ to the fractional integro-differential equation (25), and we obtain the approximation solution as:

$$
\begin{aligned}
y_1 = 7.88698 * 10^{-10} + x + 2.62372 * 10^{-9}(-1 + 2x^2) + \\
1.02391 * 10^{-10}(-3x + 4x^3),
\end{aligned}
$$

$$
\begin{aligned}
y_2 = 0.5 + 2.61514 * 10^{-10}x + 0.5(-1 + 2x^2) + \\
1.22512 * 10^{-10}(-3x + 4x^3),
\end{aligned}
$$

$$
\begin{aligned}
y_3 = -1.03378 * 10^{-8} + 0.75x + 2.94291 * 10^{-10} \\
\times (-1 + 2x^2) + 0.25(-3x + 4x^3).
\end{aligned}
$$

The numerical results are showing in Figure.3, 4 and the comparision of the absolute errors between the exact solution and the approximate solution with the method in [30]. Tables 1, 2 and 3 show the absolute error between the exact and the approximate solutions comparing with results of [30].

Table 1: The absolute error between the exact solution and the approximate solution for example (3)

x	absolute error y_1 $N = 3$	absolute error y_1 $N = 16$[30]
0.	0	0
0.1	1.1×10^{-9}	1.6×10^{-3}
0.2	4.4×10^{-10}	7.4×10^{-4}
0.3	4.1×10^{-10}	8.1×10^{-4}
0.4	1.3×10^{-9}	1.8×10^{-3}
0.5	2.4×10^{-9}	4.1×10^{-3}

Table 2: The absolute error between the exact solution and the approximate solution for example (3)

x	absolute error y_2 $N = 3$	absolute error y_2 $N = 16$[30]
0.	0	0
0.1	1.2×10^{-8}	2.6×10^{-3}
0.2	1.2×10^{-8}	1.8×10^{-3}
0.3	1.2×10^{-8}	1.1×10^{-3}
0.4	1.1×10^{-8}	1.2×10^{-4}
0.5	1.1×10^{-8}	3.1×10^{-3}

Table 3: The absolute error between the exact solution and the approximate solution for example (3).

x	absolute error y_3 $N = 3$	absolute error y_3 $N = 16$[30]
0.	0	0
0.1	1.06×10^{-8}	6.5×10^{-4}
0.2	1.06×10^{-8}	1.1×10^{-3}
0.3	1.06×10^{-8}	2.0×10^{-3}
0.4	1.06×10^{-8}	8.7×10^{-4}
0.5	1.07×10^{-8}	4.1×10^{-3}

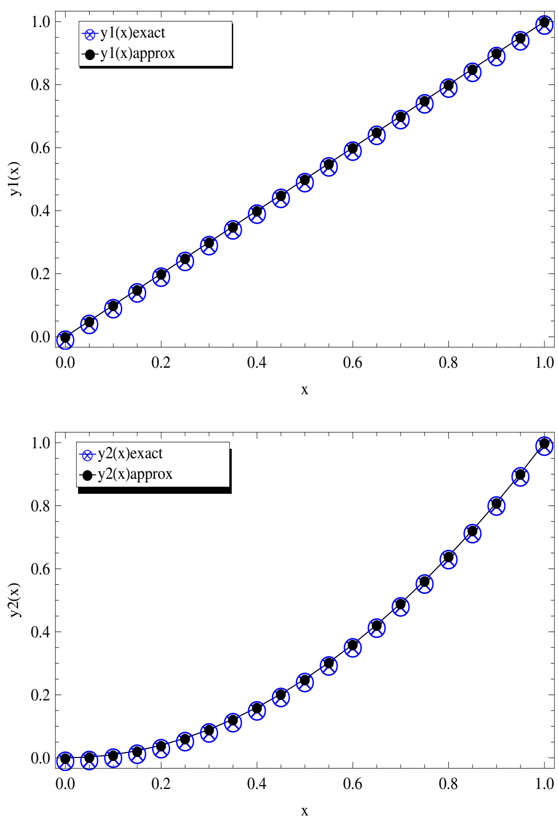

Fig. 3: results for example (3)

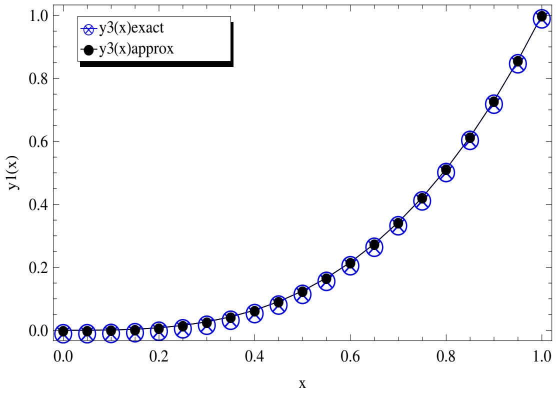

Fig. 4: results for example (3)

5 Conclusion and Remarks

In this article, we introduced an accurate numerical technique for solving a system of linear fractional delay integro-differential equations (FDIDE). An approximate formula for the Caputo's fractional derivative of the first kind Chebyshev polynomials obtained in terms of themselves in matrix forms. The spectral least squares method with the Chebyshev polynomials was introduced to treat the proposed systems. This work is more developable and generalized to systems of FDIDE. The results show that the algorithm converges as the number of terms is increased and the errors are decreased. Some numerical examples are presented to illustrate the theoretical results and compared with the results obtained by other numerical methods.

Acknowledgment

All authors would like to thank editors and reviewers. Thank you very much for good reading of the proposed paper and for useful suggestions and alerts that make the paper looks better.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors carried out the proofs and conceived of the study. All authors read and approved the final manuscript. All authors contributed equally.

References

[1] Doha, E. H., Abdelkawy, M. A., Amin, A. Z. M., & Baleanu, D. (2018). Spectral technique for solving variable order fractional Volterra integro differential equations. Numerical Methods for Partial Differential Equations, 34(5), 1659-1677.

[2] Pouchol, C., Clairambault, J., Lorz, A., & Trelat, E. (2018). Asymptotic analysis and optimal control of an integro-differential system modelling healthy and cancer cells exposed to chemotherapy. Journal de Mathematiques Pures et Appliquees, 116, 268-308.

[3] Younesian, D., Hosseinkhani, A., Askari, H.,& Esmailzadeh, E. (2019). Elastic and viscoelastic foundations: a review on linear and nonlinear vibration modeling and applications. Nonlinear Dynamics, 1-43.

[4] Abd El Salam, M. A., Ramadan, M. A., Nassar, M. A., Agarwal, P., & Chu, Y. M. (2021). Matrix computational collocation approach based on rational Chebyshev functions for nonlinear differential equations. Advances in Difference Equations, 2021(1), 1-17.

[5] Alsaedi, A., Ahmad, B., Aljoudi, S., & Ntouyas, S. K. (2019). A study of a fully coupled two-parameter system of sequential fractional integro-differential equations with nonlocal integro-multipoint boundary conditions. Acta Mathematica Scientia, 39(4), 927-944.

[6] Ahmed, S., & Salh, S. A. H. (2011). Generalized Taylor matrix method for solving linear integro-fractional differential equations of Volterra type. Applied Mathematical Sciences, 5(33-36), 1765-1780.

[7] Bhrawy, A. H., & Alghamdi, M. A. (2012). A shifted Jacobi-Gauss-Lobatto collocation method for solving nonlinear fractional Langevin equation involving two fractional orders in different intervals. Boundary value problems, 2012(1), 1-13.

[8] Bhrawy, A. H., & Alofi, A. S. (2013). The operational matrix of fractional integration for shifted Chebyshev polynomials. Applied Mathematics Letters, 26(1), 25-31.

[9] Doha, E. H., Bhrawy, A. H., & Ezz-Eldien, S. S. (2011). Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. Applied Mathematical Modelling, 35(12), 5662-5672.

[10] Irandoust-Pakchin, S., Kheiri, H., & Abdi-Mazraeh, S. (2013). Chebyshev cardinal functions: An effective tool for solving nonlinear Volterra and Fredholm integro-differential equations of fractional order. Iranian Journal of Science and Technology (Sciences), 37(1), 53-62.

[11] Irandoust-pakchin, S., & Abdi-Mazraeh, S. (2013). Exact solutions for some of the fractional integro-differential equations with the nonlocal boundary conditions by using the modifcation of He's variational iteration method. International Journal of Advanced Mathematical Sciences, 1(3), 139-144.

[12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[13] Saeedi, H., & Samimi, F. (2012). He's homotopy perturbation method for nonlinear ferdholm integrodifferential equations of fractional order. International Journal of Engineering Research and Applications, 2(5), 52-56.
[14] Saeed, R. K., & Sdeq, H. M. (2010). Solving a system of linear fredholm fractional integro-differential equations using homotopy perturbation method. Australian Journal of Basic and Applied Sciences, 4(4), 633-638.
[15] Saleh, M. H., Mohamed, D. S., Ahmed, M. H., & Marjan, M. K. (2015). System of linear fractional integro-differential equations by using Adomian decomposition method. International Journal of Computer Applications, 121(24).
[16] Sweilam, N. H., & Khader, M. (2010). A Chebyshev pseudo-spectral method for solving fractional-order integro-differential equations. The ANZIAM Journal, 51(4), 464-475.
[17] Sweilam, N. H., Nagy, A. M., & El-Sayed, A. A. (2016). On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind. Journal of King Saud University-Science, 28(1), 41-47.
[18] Raslan, K. R., Ali, K. K., & Mohamed, E. M. (2019). Spectral Tau method for solving general fractional order differential equations with linear functional argument. Journal of the Egyptian Mathematical Society, 27(1), 1-16.
[19] Ali, K. K., El Salam, M. A. A., & Mohamed, E. M. (2019). Chebyshev operational matrix for solving fractional order delay-differential equations using spectral collocation method. Arab Journal of Basic and Applied Sciences, 26(1), 342-353.
[20] Mohamed, E. M., Raslan, K. R., Ali, K. K., & Abd El Salam, M. A. (2020). On General Form of Fractional Delay Integro-Differential Equations. Arab Journal of Basic and Applied Sciences, 27(1), 313-323.
[21] Ali, K. K., Abd El Salam, M. A., Mohamed, E. M., Samet, B., Kumar, S., & Osman, M. S. (2020). Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using Chebyshev series. Advances in Difference Equations, 2020(1), 1-23.
[22] Ali, K. K., Abd El salam, M. A., & Mohamed, E. M. (2021). A numerical technique for a general form of nonlinear fractional-order differential equations with the linear functional argument. International Journal of Nonlinear Sciences and Numerical Simulation, 22(1), 83-91.
[23] Yang, Y., Chen, Y., & Huang, Y. (2014). Spectral-collocation method for fractional Fredholm integro-differential equations. Journal of the Korean Mathematical Society, 51(1), 203-224.
[24] Abdulmutallab, D. S. M. (2014). Numerical Solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Chebyshev Polynomial. Hindawi Publishing Corporation, 2014, 5.
[25] Mahdy, A. M. S. (2018). Numerical studies for solving fractional integro-differential equations. Journal of Ocean Engineering and Science, 3(2), 127-132.
[26] Mahdy, A. M. S., Mohamed, E. M. H., & Marai, G. M. A. (2016). Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomials of the third kind method. Theoretical Mathematics & Applications, 6(4), 87-101.
[27] Mahdy, A. M. S., Mohamed, E. M. H., & Marai, G. M. A. (2016). Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomials of the third kind method. Theoretical Mathematics & Applications, 6(4), 87-101.
[28] Khader, M. M., & Sweilam, N. H. (2013). On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method. Applied Mathematical Modelling, 37(24), 9819-9828.
[29] Hesameddini, E., & Rahimi, A. (2013). A new numerical scheme for solving systems of integro-differential equations. Computational Methods for Differential Equations, 1(2), 108-119.
[30] Asgari, M. (2015). Numerical Solution for solving a system of Fractional Integro-differential Equations. IAENG International Journal of Applied Mathematics, 45(2), 85-91.

Kamal R. Raslan received the M.Sc. and Ph.D. degrees from the Faculty of Science, Menoufia University and Al-Azhar University, Egypt, in 1996 and 1999, respectively. He is currently a full Professor of Mathematics with the Faculty of Science, Al-Azhar University, Egypt. He has authored/coauthored over 114 scientific papers in top-ranked International Journals and Conference Proceedings. His research interests include Numerical Analysis, Finite Difference Methods, Finite Element Methods, Approximation Theory, and Computational Mathematics.

Khalid K. Ali MSc in pure Mathematics, Mathematics Department 2015 from Faculty of Science, Al-Azhar University, Cairo, Egypt. Ph.D. in pure Mathematics, Mathematics Department entitled 2018 from Faculty of Science, Al-Azhar University, Cairo, Egypt. He has authored/co-authored over 50 scientific papers in top-ranked International Journals and Conference Proceedings. His research interests include Numerical Analysis, Finite Difference Methods, Finite Element Methods, and Computational Mathematics.

Emad M. H. Mohamed is a PhD student in Mathematics Department, Faculty of Science, Al-Azhar University, Egypt, he resived his Msc degree from the same university in 2018, Pure Mathematics (Numerical Analysis).

Mansour Lotayif acting Dean of Research & Graduate Studies, Applied Science University, Kingdom of Bahrain. Graduated from Cairo university, and earned his PhD from Plymouth University, UK.

Mohamed A. Abd El salam He is an Assistent Professor in Mathematics Department, Faculty of Science, Al-Azhar University, Egypt, he got his Msc and PhD degrees in Pure Mathematics from Al-Azhar University in Egypt, his main research interests are numerical analysis and computational mathematics.