Characterization of Exponential Distribution through Equidistribution Conditions for Consecutive Maxima

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Abstract: A characterization of the exponential distribution based on equidistribution conditions for maxima of random samples with consecutive sizes \( n - 1 \) and \( n \) for an arbitrary and fixed \( n \geq 3 \) is proved. This solves an open problem stated recently in Arnold and Villasenor [3].

Keywords: characterizations, exponential distribution, order statistics, maxima

1 Introduction

Characterizations of the exponential distribution are abundant. Comprehensive surveys can be found in Ahsanullah and Hamedani [1], Arnold and Huang [2], and Johnson, Kotz, and Balakrishnan [5]. Recently, Arnold and Villasenor [3] obtained a series of characterizations based on random sample of size two. They also identified a list of conjectures for possible extensions of their results to larger samples. In this work we confirm that one of these conjectures is true for a sample of any fixed size \( n \geq 2 \). Note that in Yanev and Chakraborty [8] the case of random sample of size three was considered.

Let \( X_1, X_2, \ldots, X_n, n \geq 2 \) be a random sample from an exponentially distributed parent \( X \). It is known that

\[
\max \{X_1, X_2, \ldots, X_{n-1}\} + \frac{1}{n} X_n \overset{d}{=} \max \{X_1, X_2, \ldots, X_n\},
\]

where \( \overset{d}{=} \) denotes equality in distribution. We write \( X \sim \exp(\lambda) \) if the probability density function (pdf) of \( X \) equals \( f_X(x) = \lambda e^{-\lambda x} I(x > 0) \). Our goal is to prove that (1), under analyticity assumptions on the cumulative distribution function (cdf) \( F \) of \( X \), is a sufficient condition for \( X \) to be exponential.

Theorem Let \( X \) be a non-negative continuous random variable with pdf \( f \). If \( f \) is analytic in a neighborhood of zero and (1) holds true, then \( X \sim \exp(\lambda) \) with some \( \lambda > 0 \).

Wesołowski and Ahsanullah [7] and more recently Castaño-Martínez et al. [4] proved characterizations of probability distributions in the context of random translations. The characterization (1) above can be deduced from their results (see Corollary 1 in Wesołowski and Ahsanullah [7] and Corollary 3 in Castaño-Martínez et al. [4]). However, our proof is different from theirs in not referring to uniqueness results for integral equations. The direct approach we follow may also be used in obtaining some more general results, a possibility which we will explore in the future.

2 Preliminaries

Define for all non-negative integers \( n, i \), and any real number \( x \)

\[
H_{n,i}(x) := \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)^i.
\]

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It is known, (e.g., Ruiz [6]) that for all integers \( n \geq 0 \) and all real \( x \)

\[
H_{n,i}(x) = \begin{cases} 
  n! & \text{if } i = n; \\
  0 & \text{if } 0 \leq i \leq n - 1. 
\end{cases}
\]  

(2)

Define \( G_m(x) := F^m(x)f(x) \) for \( m \geq 1 \) and denote by \( g^{(i)}(x) \) for \( i \geq 1 \) the \( i \)th derivative of a function \( g(x) \); \( g^{(0)}(x) := g(x) \).

**Lemma 1** Let \( X \) be a continuous random variable with cdf \( F \) satisfying \( F(0) = 0 \). If for \( 0 \leq r \leq m - 1 \)

\[
f^{(r)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{r-1} f'(0),
\]

(3)

then for \( 0 \leq i \leq 2m \)

\[
G^{(i)}_m(0) = \left[ \frac{f'(0)}{f(0)} \right]^{i-m} f^{m+1}(0)H_{m,i}(m+1).
\]

(4)

**Proof. Case 0 \leq i \leq m - 1.** In this case (2) implies \( H_{m,i}(m+1) = 0 \). On the other hand, in the left-hand side of (4), we have \( G^{(i)}_m(0) = 0 \) because each term in the expansion of \( G^{(i)}_m(0) \) has a factor \( F(0) = 0 \).

**Case i = m.** From (2) it follows that (4) is equivalent to

\[
G^{(m)}_m(0) = m! f^{m+1}(0).
\]

(5)

We shall prove (5) by induction. If \( m = 1 \), then (5) follows from the definition of \( G(x) \) and the assumption \( F(0) = 0 \). Assuming that (5) is true for \( m = k \), we will prove it for \( m = k + 1 \). Since \( G_{k+1}(x) = F(x)G_k(x) \) and \( F(0) = 0 \), we have

\[
G^{(k+1)}_{k+1}(0) = \sum_{j=0}^{k+1} \binom{k+1}{j} F^{(j)}(0)G_k^{(k+1-j)}(0)
\]

\[
= F(0)G_k^{(k+1)}(0) + (k+1) F^{(1)}(0)G_k^{(k)}(0)
\]

\[
= (k+1) f(0)k! f^{k+1}(0)
\]

\[
= (k+1)! f^{k+2}(0),
\]

where we have used that \( G^{(r)}(0) = 0 \) for \( 0 \leq r \leq k - 1 \) and the induction assumption \( G_k^{(k)}(0) = k! f^{k+1}(0) \).

**Case m < i \leq 2m.** Suppose we have proved (4) for \( m = 1, 2, \ldots k \). We want to prove it for \( m = k + 1 \). Observe that

\[
G^{(i)}_{k+1}(0) = \sum_{j=0}^{i} \binom{i}{j} F^{(j)}(0)G_k^{(i-j)}(0).
\]

Since \( G_k^{(j)}(0) = 0 \) for \( 0 \leq r \leq k - 1 \), making use of (3) and the induction assumption, we obtain

\[
G^{(i)}_{k+1}(0) = \sum_{j=1}^{k} \binom{i}{j} f^{(j-1)}(0)G_k^{(i-j)}(0) + \sum_{j=k+1}^{i} \binom{i}{j} f^{(j-1)}(0)G_k^{(i-j)}(0)
\]

\[
= \sum_{j=1}^{k} \binom{i}{j} \left[ \frac{f'(0)}{f(0)} \right]^{j-2} f'(0) \left[ \frac{f'(0)}{f(0)} \right]^{i-j-k} f^{k+1}(0)H_{k,i-j}(k+1)
\]

\[
= \left[ \frac{f'(0)}{f(0)} \right]^{i-k-1} f^{k+2}(0) \sum_{j=1}^{i} \binom{i}{j} H_{k,i-j}(k+1),
\]

(6)
where in the last equality we used that (2) implies $H_{k,j}(k + 1) = 0$ for $j = i + 1, \ldots, k$. Further, we have

$$\sum_{j=1}^{i} \binom{i}{j} H_{i-j}(k+1) = \sum_{r=0}^{k} (-1)^{r} \binom{k}{r} \sum_{j=1}^{i} \binom{i}{j} (k + 1 - r)^{i-j}$$

$$= \sum_{r=0}^{k} (-1)^{r} \binom{k}{r} [(k + 2 - r)^{i} - (k + 1 - r)^{i}]$$

$$= (k + 2)^{i} - \left( \binom{k + 1}{1} (k + 2)^{i} \right) + \left( \binom{k}{1} k^{2} + \binom{k}{2} k \right) + \ldots + (-1)^{k} \left[ \binom{k}{k-1} 2^{i} + 2 \right] + (-1)^{k+1}$$

$$= (k + 1)^{i} - \left( \binom{k + 1}{1} (k + 1)^{i} + \ldots + (-1)^{k} \binom{k + 1}{k} 2^{i} + (-1)^{k+1} \right)$$

$$= \sum_{j=0}^{k} (-1)^{j} \binom{k + 1}{j} (k + 2 - j)^{i} = H_{k+1,i}(k + 2).$$

The lemma’s claim follows by induction, taking into account (6).

The identity below may be of independent interest.

**Lemma 2** For any integers $m \geq 0$ and $k \geq 0$

$$\sum_{j=0}^{m} \frac{(k + 2)^{m-j}}{m-j} H_{k,j}(k + 1) = \sum_{j=0}^{m} \left( \frac{m + 1}{j + 1} \right) H_{k,j}(k + 1).$$

**Proof.** The left-hand side of (7) equals

$$\sum_{j=0}^{m} \frac{(k + 2)^{m-j}}{m-j} \sum_{i=0}^{j} (-1)^{i} \binom{k}{i} (k + 1 - i)^{i} = \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} (k + 2)^{m} \sum_{j=0}^{m} \frac{k + 1 - i}{k + 2}.$$  \hspace{1cm} (8)

For the right-hand side of (7) we obtain

$$\sum_{j=0}^{m} \binom{m+1}{j+1} \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} (k + 1 - i)^{i} = \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} \sum_{j=0}^{m} \frac{(m + 1)}{j + 1} (k + 1 - i)^{j}$$

$$= \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} \frac{1}{k + 1 - i} \sum_{j=0}^{m} \frac{(m + 1)}{j + 1} (k + 1 - i)^{j+1}$$

$$= \frac{1}{k + 1} \sum_{i=0}^{k} (-1)^{i} \binom{k + 1}{i} \sum_{r=1}^{m+1} \binom{m + 1}{r} (k + 1 - i)^{r}$$

$$= \frac{1}{k + 1} \sum_{i=0}^{k} (-1)^{i} \binom{k + 1}{i} \left[ \sum_{r=0}^{m+1} \binom{m + 1}{r} (k + 1 - i)^{r} - 1 \right]$$

$$= \frac{1}{k + 1} \sum_{i=0}^{k} (-1)^{i} \binom{k + 1}{i} (k + 2 - i)^{m+1} - \frac{1}{k + 1} \sum_{i=0}^{k} (-1)^{i} \binom{k + 1}{i} (k + 1 - i)^{m+1}$$

$$= \frac{1}{k + 1} \sum_{r=0}^{k} (-1)^{r} \binom{k + 1}{r} (k + 2 - r)^{m+1},$$
which equals (8). The proof of the lemma is complete.

Next lemma (see also Arnold and Villaseñor [3]) will play a crucial role in the proof of the theorem. In private communications, P. Fitzsimmons pointed out to us that the assumption of analyticity of the density function $f$ is missing in [3].

**Lemma 3** If $F(0) = 0$, the pdf $f$ is analytic in a neighborhood of 0, and

$$f^{(k)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{k-1} f'(0), \quad k = 1, 2, \ldots, \quad (9)$$

then $X \sim \exp\{\lambda\}$ for some $\lambda > 0$.

**Proof.** For the Maclaurin series of $f(x)$, we have for $x > 0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \sum_{k=1}^{\infty} \left[ \frac{f'(0)}{f(0)} \right]^{k-1} f'(0) \frac{x^k}{k!} = f(0) \exp \left\{ \frac{f'(0)}{f(0)} x \right\}. \quad (10)$$

Since $f(x)$ is a pdf, we have $f'(0)/f(0) < 0$. Denoting $\lambda = -f'(0)/f(0) > 0$ and setting the integral of (10) from 0 to $\infty$ to be 1, we obtain $\lambda = f(0)$. Therefore, $f(x) = \lambda e^{-\lambda x}(x > 0)$, i.e., $X \sim \exp\{\lambda\}$.

### 3 Proof of the theorem

Equation (1) can be written as

$$\int_0^x f_{x/n}(y) f_{\max(x_1, \ldots, x_{n-1})}(x-y) dy = n(n-1)f(x) \int_0^x G_{n-2}(y) dy. \quad (11)$$

This is equivalent to

$$\int_0^x nf(ny)(n-1)F^{n-2}(x-y)f(x-y) dy = n(n-1)f(x) \int_0^x G_{n-2}(y) dy,$$

which simplifies to

$$\int_0^x f(ny)G_{n-2}(x-y) dy = f(x) \int_0^x G_{n-2}(y) dy. \quad (11)$$

Differentiating the left-hand side of (11) with respect to $x$, we obtain

$$\frac{d}{dx} \int_0^x f(ny)G_{n-2}(x-y) dy = f(nx)G_{n-2}(0) + \int_0^x f(ny)G'_{n-2}(x-y) dy.$$ 

Differentiating the last equation $2n - 3$ times, we obtain

$$\frac{d^{2n-2}}{dx^{2n-2}} \int_0^x f(ny)G_{n-2}(x-y) dy = \sum_{i=0}^{2n-3} n^{2n-3-i} f^{(2n-3-i)}(x)G^{(i)}_{n-2}(0) + \int_0^x f(ny)G^{(2n-2)}_{n-2}(x-y) dy. \quad (12)$$

On the other hand, applying to the right-hand side of (11) the Leibnitz product rule of differentiation, we have

$$\frac{d^{2n-2}}{dx^{2n-2}} \left[ f(x) \int_0^x G_{n-2}(y) dy \right] = \sum_{i=0}^{2n-3} \binom{2n-2}{i+1} f^{(2n-3-i)}(x)G^{(i)}_{n-2}(0) + f^{(2n-2)}(x) \int_0^x G_{n-2}(y) dy. \quad (13)$$

Therefore, the equation (11), taking into account (12) and (13), becomes

$$\sum_{i=0}^{2n-3} n^{2n-3-i} f^{(2n-3-i)}(x)G^{(i)}_{n-2}(0) + \int_0^x f(ny)G^{(2n-2)}_{n-2}(x-y) dy \quad (14)$$

$$= \sum_{i=0}^{2n-3} \left( \binom{2n-2}{i+1} f^{(2n-3-i)}(x)G^{(i)}_{n-2}(0) + f^{(2n-2)}(x) \int_0^x G_{n-2}(y) dy. \right)$$
Setting \( x = 0 \) and taking into account that \( G_{n-2}^{(i)}(0) = 0 \) for \( 0 \leq i \leq n - 3 \), we obtain that (14) is equivalent to
\[
\sum_{i=n-2}^{2n-4} \frac{n_{2n-3-i}}{i+1} \frac{f(2n-3-i)(0)}{f(0)} G_{n-2}^{(i)}(0) = \sum_{i=n-2}^{2n-4} \frac{2n-2}{i+1} \frac{f(2n-3-i)(0)}{f(0)} G_{n-2}^{(i)}(0).
\]

For \( i = n - 2 \), we have \( f^{(n-1)}(0)G_{n-2}^{(n-2)}(0) = f^{(n-1)}(0)f^{n-1}(0)(n-2)! \). Thus, the equation above can be written as
\[
\left[ n^{n-1} \right] f^{(n-1)}(0)f^{n-1}(0)(n-2)! = \sum_{i=n-1}^{2n-4} \left[ \frac{2n-2}{i+1} - n^{2n-3-i} \right] f^{(2n-3-i)}(0)G_{n-2}^{(i)}(0).
\]

In view of Lemma 3, to complete the proof it suffices to show
\[
f^{(r)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{r-1} f'(0), \quad r = 1, 2, \ldots
\]

Assume (16) for all \( 1 \leq r \leq n - 2 \). We shall prove it for \( r = n - 1 \), i.e.,
\[
f^{(n-1)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{n-2} f'(0), \quad r = 1, 2, \ldots
\]

It follows from Lemma 1 with \( m = n - 2 \) that for \( n - 1 \leq i \leq 2n - 4 \)
\[
f^{(2n-3-i)}(0)G_{n-2}^{(i)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{i-n+2} f^{n-1}(0)H_{n-2,i}(n-1).
\]

Substituting (18) in the right-hand side of (15) we obtain
\[
\left[ n^{n-1} \right] f^{(n-1)}(0)(n-2)! = \left[ \frac{f'(0)}{f(0)} \right]^{n-2} f'(0) \sum_{i=n-1}^{2n-4} \left[ \frac{2n-2}{i+1} - n^{2n-3-i} \right] H_{n-2,i}(n-1).
\]

To establish (18) we need to prove
\[
\left[ n^{n-1} \right] = \sum_{i=n-1}^{2n-4} \left[ \frac{2n-2}{i+1} - n^{2n-3-i} \right] H_{n-2,i}(n-1)
\]
or equivalently
\[
\sum_{i=n-2}^{2n-4} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=n-2}^{2n-4} \frac{2n-2}{i+1} H_{n-2,i}(n-1).
\]

Since (2) implies \( H_{n-2,i}(n-1) = 0 \) for \( 0 \leq i \leq n - 3 \) and for \( i = 2n - 3 \) we have \( n^{2n-3-i} = \left( \frac{2n-2}{i+1} \right) = 1 \), we obtain that (19) is equivalent to
\[
\sum_{i=0}^{2n-3} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=0}^{2n-3} \frac{2n-2}{i+1} H_{n-2,i}(n-1),
\]

which follows from Lemma 3 with \( m = 2n - 3 \). This completes the induction argument and thus proves (16). Referring to (16) and Lemma 2 we complete the proof of the theorem.

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References