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Rank and Signed-Rank Tests for Random Coefficient Regression Model

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Abstract: In this paper, we propose nonparametric locally and asymptotically optimal tests for the problem of detecting randomness in the coefficient of a linear regression model (in the Le Cam and Hájek sense). That is, the problem of testing the null hypothesis of a Standard Linear Regression (SLR) model against the alternative of a Random Coefficient Regression (RCR) model. A Local Asymptotic Normality (LAN) property, which allows for constructing locally asymptotically optimal tests, is therefore established for RCR models in the vicinity of SLR ones. Rank and signed-rank based versions of the optimal parametric tests are provided. These tests are optimal, most powerful and valid under a wide class of densities. A Monte-Carlo study confirms the performance of the proposed tests.

Keywords: Local asymptotic normality, optimal test, rank test, signed rank test, Random Coefficient Regression Model.

1 Introduction and notations

Regression analysis has been investigated by many researchers and mainly used to explore the relationship between dependent and independent variables. Random coefficient and varying parameter regression models are widely applied in many areas such as social sciences, ecology, finance and econometrics (see e.g. [20] and [14]). Besides, it was demonstrated in many applications that these models are superior to ordinary linear regression ones.

Earlier contributions to the study of these models are due to Raj and Ullah [18] and Nicholls and Pagan [15]. Consequently, there is a growing theoretical interest in these models. One such model is the Random Coefficient Regression *(RCR model)* which is considered in this paper and defined by:

$$Y_i = \mu + (\beta + \xi_i)X_i + \varepsilon_i, \qquad i = 1, \dots, n, \tag{1}$$

where,

- ▷ Y_i is the observed response for individual i, X_i is a non-stochastic exogenous regressor, and μ and β are given regression parameters;
- ▷ $(\varepsilon_i)_{1\le i\le n}$ is a zero-mean $i.i.d.$ sequence of error terms with variance σ^2 and density $\varepsilon \mapsto f(\varepsilon) := (1/\sigma)f_1(\varepsilon/\sigma)$;
- ▷ $(\xi_i)_{1\le i\le n}$ is an $i.i.d.$ sequence of random coefficients $(0, \sigma_\xi^2)$ with density $\xi \mapsto h(\xi) := (1/\sigma_\xi)h_1(\xi/\sigma_\xi)$;
- ▷ ξ_i and ε_i are independent for all $i = 1, \dots, n$.

In this model, the regression slope varies randomly around its mean, β, according to a distribution whose standard deviation is σ_ξ.

Note that, if $\sigma_\xi^2 = 0$, the RCR model reduces to the SLR one:

$$Y_i = \mu + \beta X_i + \varepsilon_i, \qquad i = 1, \dots, n. \tag{2}$$

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It will be useful to have an available test of the null hypothesis that the fixed parameter model is appropriate against the alternative of stochastic parameter. Ramanathan and Rajarshi [19] obtained a signed-rank-based (ranks of the squares of residuals) version test by considering that ξ_i and ε_i have a symmetric logistic distribution.

In this paper, the main technical tool is *Le Cam's asymptotic theory* of statistical experiments and the properties of *locally asymptotically normal (LAN)* families (see, for details, [12], [22, chapter 6 – 9] and [11]). This powerful method has been used quite successfully in various inference problems (see, e.g., [4], [9], [3] and [13]). *The logarithm of the likelihood ratio* in a *LAN family* with parameter θ admits local approximation of the form $\tau\Delta_\theta^{(n)} - \frac{1}{2}\tau^2\gamma_\theta$, where the random variable $\Delta_\theta^{(n)}$, called a *central sequence*, is asymptotically normal $\mathscr{N}(\tau\gamma_\theta, \gamma_\theta)$ under sequences of parameter values of the form $\theta + n^{-1/2}\tau$ (*local alternatives*). Let $\phi(\Delta)$ be an optimal test in the *Gaussian shift model* describing a hypothetical observation Δ with distribution in the family $\{\mathscr{N}(\tau\gamma_\theta, \gamma_\theta) | \tau \in \mathbb{R}\}$ (γ_θ specified). Then, the sequence $\phi(\Delta_\theta)$ is a sequence of *locally asymptotically optimal tests* for the original problem (optimality is based on the local convergence of risk functions to the risk functions of Gaussian shift experiments). Furthermore, this procedure allows to provide a class of rank and signed-rank tests which remain valid under arbitrary densities, without any moment assumption (see, e.g., [8], [10] and [6]). Several arguments justify the use of rank tests: The vector of ranks (with the vector of signs for the symmetric case) is a *maximal invariant* with respect to the group of *order-preserving transformations* of residuals for a broad class of densities; the rank (signed-rank) tests produced are more *robust* regarding some outliers than their parametric counterparts and rank (signed-rank) based procedures are *asymptotically powerful* compared to parametric ones.

The asymptotic behavior of these rank and signed-rank tests is studied under the null and sequences of local alternatives. It is shown that the studied tests are most powerful and present a uniformly good power behavior, which is demonstrated by a Monte-Carlo simulations.

The paper is organized as follows. In subsection 2.1, the notation and main assumptions are provided. Subsection 2.2 establishes the LAN property. A rank and signed-rank version of the LAN result are established in section 4 based on the optimal procedure version obtained in section 3. The van der Waerden and Wilcoxon versions of the rank and the signed-rank test statistics are described in subsections 4.1 and 4.2. Finally, section 5 provides some simulation results of the various proposed tests. The conclusion and perspectives appear in section 6.

2 Local Asymptotic Normality

2.1 Notation and main assumptions

The null hypothesis, we are interested in, is the traditional standard linear regression (SLR) dependence (2) in which the hypothesis $\mathbb{P}(\xi_i = 0) = 1$ is equivalent to assume that $\sigma_\xi^2 = 0$. In this case we denote by $\mathbb{P}_{0;f_1}^{(n)} =: \mathbb{P}_{f_1}^{(n)}$ the probability distribution under the null. Under the alternative, $\mathbb{P}_{\sigma_\xi^2;f_1,h_1}^{(n)}$ is the probability distribution of the observation $\mathbf{Y}^{(\mathbf{n})} := \left(Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}\right)'$ generated by (1). Let's consider the class of general standardized densities:

$$\mathscr{F}_0 := \left\{ f_1 : \int_{-1}^{1} f_1(z)dz = 0.5 = \int_{-\infty}^{0} f_1(z)dz \right\}$$

and the class of symmetric standardized densities:

$$\mathscr{F}_0^+ := \left\{ f_1 : f_1(-z) = f_1(z) \text{ and } \int_{-1}^{1} f_1(z)dz = 0.5 = \int_{-\infty}^{0} f_1(z)dz \right\}.$$

Note that, under $\mathscr{F}_0$ and $\mathscr{F}_0^+$ the median and median absolute deviation are respectively 0 and σ. This standardization which, contrary to the usual one based on the mean and the standard deviation, avoids all moment assumptions, plays the role of an identification constraint and has no impact on subsequent results.

The main technical tool in our derivation of optimal tests is the local asymptotic normality (LAN), with respect to σ_ξ^2, of the families of distributions

$$\mathscr{P}_{f_1,h_1}^{(n)} := \left\{ \mathbb{P}_{\sigma_\xi^2;f_1,h_1}^{(n)} : \sigma_\xi^2 \geq 0 \right\}$$

at $\sigma_{\xi}^2 = 0$[1]. Establishing the LAN requires some technical assumptions, which are about the density f_1, the asymptotic behavior of the regressor and the density h_1.

Assumption (A)

(A.1) $f_1 \in \mathscr{F}_0$ (resp. $f_1 \in \mathscr{F}_0^+$ for the symmetric distributions);
(A.2) $f_1(z) > 0$ for all $z \in \mathbb{R}$;
(A.3) $z \mapsto f_1(z)$ is $\mathscr{C}^2$ on $\mathbb{R}$, with second derivative $\ddot{f}_1$ and letting $\psi_{f_1} := \ddot{f}_1/f_1$, assume that $\mathscr{I}_{\psi}(f_1) := \int_{\mathbb{R}} \psi_{f_1}^2(z) f_1(z)dz$ is finite.

Denote by $\mathscr{F}_A$ (resp. $\mathscr{F}_A^+$ for the symmetric distributions) the set of all densities satisfying **Assumption (A)**. For instance,

▷ the logistic distribution, with standardized density is given as

$$f_1 = \ell_1 := \sqrt{b}\exp(-\sqrt{b}z)/(1+\exp(-\sqrt{b}z))^2, \tag{3}$$

with $b = (\ln 3)^2$ and $\mathscr{I}_{\psi}(\ell_1) = b^2/5$;

▷ the Gaussian distribution, with standardized density takes the form

$$f_1 = \phi_1 := \sqrt{a/2\pi}\exp(-az^2/2), \tag{4}$$

with $a \approx 0.4549$ and $\mathscr{I}_{\psi}(\phi_1) = 2a^2$

▷ and the Student-t_ν distribution (with $\nu > 0$ degrees of freedom), with standardized density is formulated as

$$f_1 = f_{t_\nu} := \frac{\sqrt{a_\nu}\left(\frac{\nu}{a_\nu x^2+\nu}\right)^{\frac{\nu+1}{2}}}{\sqrt{\nu}\mathscr{B}\left(\frac{\nu}{2},\frac{1}{2}\right)}, \tag{5}$$

the normalizing constant $a_\nu > 0$ is such that $f_{t_\nu} \in \mathscr{F}_0^+$. $\mathscr{I}_{\psi}(f_{t_\nu}) = 2a_\nu^2 \frac{(\nu+1)(\nu+2)(\nu(\nu+5)+10)}{\nu(\nu+3)(\nu+5)(\nu+7)}$.

Assumption (B)

Suppose that the classical Noether condition hold ([16, p.501]):

$$\lim_{n\to\infty} \frac{\max_{1\le i\le n}\left(X_i - \overline{X}^{(n)}\right)^2}{\sum_{i=1}^n \left(X_i - \overline{X}^{(n)}\right)^2} = 0,$$

where $\overline{X}^{(n)} := n^{-1}\sum_{i=1}^n X_i$.

This condition originates from Noether (1949). It essentially keeps one of the constants from dominating the others and allows to give the asymptotic behavior of a test statistic.

Assumption (C)

(C.1) $\int_{\mathbb{R}} \xi h_1(\xi)d\xi = 0$ and $\int_{\mathbb{R}} \xi^2 h_1(\xi)d\xi = 1$;
(C.2) Denote by $\mathscr{I}_{\psi\psi}^x(f_1;y)$ the Fisher information associated to σ_ξ, such as

$$\begin{aligned}\mathscr{I}_{\psi\psi}^x(f_1;y) &:= \begin{cases} \frac{1}{y^2}\int_{z=-\infty}^{\infty} \frac{\left[\int_{w=0}^{y}\int \ddot{f}_1(z-xvw)x^2v^2h_1(v)dvdw\right]^2}{\int f_1(z-xyv)h_1(v)dv}dz & \text{if } y>0 \\ x^4\mathscr{I}_{\psi}(f_1) & \text{if } y=0 \end{cases} \\ &= \begin{cases} \frac{1}{y^2}\int_{z=-\infty}^{\infty}\left[\frac{\int \dot{f}_1(z-xyv)xvh_1(v)dv}{\left[\int f_1(z-xyv)h_1(v)dv\right]^{1/2}}\right]^2 dz & \text{if } y>0 \\ x^4\mathscr{I}_{\psi}(f_1) & \text{if } y=0 \end{cases}.\end{aligned} \tag{6}$$

The function $y \mapsto \mathscr{I}_{\psi\psi}^x(f_1;y)$ is continuous from the right at $y=0$, for all x.

Note that, assumption (C.2) is an assumption which involves the couple of densities (f_1,h_1).

Let $\mathscr{F}_{C|f_1} := \{h_1 | h_1$ and (f_1,h_1) satisfy Assumptions (C.1) and (C.2) , respectively$\}$ (resp. $\mathscr{F}_{C|f_1}^+$ for $f_1 \in \mathscr{F}_0^+$).

[1] This is a one-sided test and takes the following form: $\begin{cases}\mathscr{H}_0: \sigma_\xi^2 = 0 \\ \mathscr{H}_1: \sigma_\xi^2 > 0\end{cases}$.

2.2 LAN

In the following, we establish the *local asymptotic normality (LAN)* result (on which optimal test will be based (section 3)) with respect to σ_ξ^2 for a fixed density f_1. Let's consider a sequence of local alternatives of the form[2] $\left(0 + n^{-1/2} K^{(n)} \tau\right)$, where $K^{(n)} = \left(\sum_{i=1}^n X_i^4\right)^{-1/2}$ and $\tau \in \mathbb{R}^+$. Define the standardized residuals as: $Z_i := \sigma^{-1}(Y_i - \mu - \beta X_i)$, for $i = 1, \ldots, n$ and note that these residuals coincide with $\dfrac{\varepsilon_i}{\sigma}$ under the null hypothesis $\mathbb{P}_{f_1}^{(n)}$. We then have the following proposition.

Proposition 1(LAN). *Let Assumptions (B) and (C) hold. Fix $f_1 \in \mathscr{F}_A$ and $h_1 \in \mathscr{F}_{C|f_1}$. Then, as $n \to \infty$,*

(i) *the family $\mathscr{P}_{f_1,h_1}^{(n)}$ is LAN at 0, with central sequence*

$$\Delta_{f_1}^{(n)} := \frac{1}{2\sigma^2 \sqrt{n}} \sum_{i=1}^n \psi_{f_1}(Z_i) K^{(n)} X_i^2 \tag{7}$$

and variance

$$\gamma_{f_1} := \frac{1}{4n\sigma^4} \mathscr{I}_\psi(f_1), \tag{8}$$

(ii) *for any $\tau \in \mathbb{R}^+$, we have, under $\mathbb{P}_{f_1}^{(n)}$,*

$$\begin{aligned} \Lambda_{n^{-1/2}K^{(n)}\tau/0;f_1,h_1}^{(n)} :=& \log\left(\frac{d\mathbb{P}_{n^{-1/2}K^{(n)}\tau;f_1,h_1}^{(n)}}{d\mathbb{P}_{f_1}^{(n)}}\right) \\ =& \tau \Delta_{f_1}^{(n)} - \frac{1}{2}\tau^2 \gamma_{f_1} + o_{\mathbb{P}}(1), \end{aligned}$$

(iii) $\Delta_{f_1}^{(n)} \xrightarrow{\mathscr{L}} \mathscr{N}(0, \gamma_{f_1})$, *under* $\mathbb{P}_{f_1}^{(n)}$.

Proof. (See appendix.)

Remark. The expressions ((7), (8)) of the central sequence $\Delta_{f_1}^{(n)}$ and the variance γ_{f_1} do not depend on the density h_1 of the random criterion ξ. It has no influence and will not appear in the test statistics.

3 Optimal parametric test

For specified $f_1 \in \mathscr{F}_A$, we consider the null hypothesis

$$\mathscr{H}_0^{(n)} := \bigcup_{f_1 \in \mathscr{F}_A} \mathbb{P}_{f_1}^{(n)}$$

of non-randomness coefficient in model (1) (i.e. $\sigma_\xi^2 = 0$), against alternatives of the form

$$\bigcup_{f_1 \in \mathscr{F}_A} \bigcup_{\sigma_\xi^2 > 0} \bigcup_{h_1 \in \mathscr{F}_{C|f_1}} \left\{ \mathbb{P}_{\sigma_\xi^2; f_1, h_1}^{(n)} \right\}.$$

Recall that, based on Le Cam's third lemma, one can give the distribution under the alternative hypothesis (see, e.g., [22, chapter 6]).

[2] This is equivalent to test: $\begin{cases} \mathscr{H}_0 : \mathbb{P}_{f_1}^{(n)} & : \ \tau = 0 \\ \mathscr{H}_1 : \mathbb{P}_{\sigma_\xi^2; f_1, h_1}^{(n)} & : \ \tau > 0 \end{cases}.$

Definition 1(Le Cam's third lemma). *Let $S^{(n)}$ be a measurable statistic and $\Lambda^{(n)}$ is a version of* $\log \frac{d\mathbb{Q}^{(n)}}{d\mathbb{P}^{(n)}}$ *($\mathbb{P}^{(n)}$ and $\mathbb{Q}^{(n)}$ are respectively, the null and the local alternative hypothesis). Suppose that under $\mathbb{P}^{(n)}$, as $n \to \infty$,*

$$\begin{pmatrix} S^{(n)} \\ \Lambda^{(n)} \end{pmatrix} \xrightarrow{\mathscr{L}} \mathscr{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \right), \text{ with } \mu_2 = -\sigma_2^2/2. \tag{9}$$

Then,

(i) $\mathbb{P}^{(n)}$ and $\mathbb{Q}^{(n)}$ are mutually contiguous;
(ii) under $\mathbb{Q}^{(n)}$, $S^{(n)} \xrightarrow{\mathscr{L}} \mathscr{N}\left(\mu_1 + \sigma_{12}, \sigma_1^2\right)$.

Applying definition 1 yields

$$\begin{pmatrix} \Delta_{f_1}^{(n)} \\ \Lambda_{n^{-1/2}K^{(n)}\tau/0;f_1,h_1}^{(n)} \end{pmatrix} \xrightarrow{\mathscr{L}} \mathscr{N}\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\tau^2\gamma_{f_1} \end{pmatrix}, \begin{pmatrix} \gamma_{f_1} & \tau\gamma_{f_1} \\ \tau\gamma_{f_1} & \tau^2\gamma_{f_1} \end{pmatrix} \right), \text{ under } \mathbb{P}_{f_1}^{(n)}. \tag{10}$$

Therefore

$$\Delta_{f_1}^{(n)} \xrightarrow{\mathscr{L}} \mathscr{N}\left(\tau\gamma_{f_1}, \gamma_{f_1}\right), \text{ under } \mathbb{P}_{n^{-1/2}K^{(n)}\tau;f_1,h_1}^{(n)}. \tag{11}$$

The LAN structure and the convergence of local experiments to the Gaussian shift experiment $\Delta \sim \mathscr{N}(\gamma\tau, \gamma)$, imply that locally optimal inference on σ_ξ^2 should be based on $\Delta_{f_1}^{(n)}$, hence on $T_{f_1}^{(n)}$ with

$$\begin{aligned} T_{f_1}^{(n)} &:= \left(\gamma_{f_1}\right)^{-1/2} \Delta_{f_1}^{(n)} \\ &= \frac{K^{(n)}}{\sqrt{\mathscr{I}_\psi(f_1)}} \sum_{i=1}^{n} \psi_{f_1}(Z_i) X_i^2. \end{aligned} \tag{12}$$

Thus, we have the following result.

Proposition 2. *Let Assumptions (B) and (C) hold. Fix $f_1 \in \mathscr{F}_A$ and $h_1 \in \mathscr{F}_{C|f_1}$. Then,*

(i) $T_{f_1}^{(n)}$ is asymptotically normal, with mean zero under $\mathbb{P}_{f_1}^{(n)}$, mean $\gamma_{f_1}^{1/2}\tau$ under $\mathbb{P}_{n^{-1/2}K^{(n)}\tau;f_1,h_1}^{(n)}$ and variance one under both;
(ii) the sequence of tests rejecting the null hypothesis $\mathscr{H}_0^{(n)}$ (with standardized density f_1) whenever[3]

$$T_{f_1}^{(n)} > z_{1-\alpha},$$

is locally asymptotically most powerful at asymptotic level α, against local alternatives hypothesis of the form $\bigcup_{f_1 \in \mathscr{F}_A} \bigcup_{\sigma_\xi^2 > 0} \bigcup_{h_1 \in \mathscr{F}_{C|f_1}} \left\{ \mathbb{P}_{\sigma_\xi^2;f_1,h_1}^{(n)} \right\}$.

4 Optimal rank and signed-rank tests

The null hypothesis $\mathscr{H}_0^{(n)}$ of nullity of randomness criterion indeed is generated by the group $\left(\mathscr{G}_0^{(n)}, \circ\right)$ of all transformations $\mathscr{G}_l$ of $\mathbb{R}^n$ such that $\mathscr{G}_l(Y_1, \ldots, Y_n) := (l(Y_1), \ldots, l(Y_n))$, where $\lim_{y \to \pm\infty} l(y) = \pm\infty$ and $l(y)$ is continuous and monotone increasing.

[3] $z_{1-\alpha}$ is the $(1-\alpha)-$standard normal quantile.

4.1 Optimal rank test

A maximal invariant for the group $\mathscr{G}_l$ is known to be the vector $(R_1^{(n)}, R_2^{(n)}, \ldots, R_n^{(n)})'$, where $R_i^{(n)}$ denotes the rank of residual Z_i among $Z_1, Z_2, \ldots, Z_n$.

General results on semi-parametric efficiency (see, for details, [10]) indicate that, in such context, the expectation of the efficient central sequence $\Delta_{f_1}^{(n)}$ conditional on those ranks yields to a version of the semi-parametrically efficient central sequence. The rank based version of the efficient central sequence $\Delta_{f_1}^{(n)}$ is given as:

$$\underset{\sim}{\Delta}_{f_1}^{(n)} := \frac{K^{(n)}}{2\sigma^2\sqrt{n}} \sum_{i=1}^{n} X_i^2 \left\{ \psi_{f_1}\left(F_1^{-1}\left(\frac{R_i^{(n)}}{n+1}\right)\right) - \overline{\psi}_{f_1}^{(n)} \right\}, \tag{13}$$

with $\overline{\psi}_{f_1}^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \psi_{f_1}\left(F_1^{-1}\left(\frac{i}{n+1}\right)\right)$.

Let

$$\begin{aligned} s^{2\,(n)}_{f_1} &:= \overline{\psi^2}_{f_1}^{(n)} - \left(\overline{\psi}_{f_1}^{(n)}\right)^2 \\ &= \frac{n-1}{n^2} \sum_{i=1}^{n} \psi_{f_1}^2\left(F_1^{-1}\left(\frac{i}{n+1}\right)\right) \\ &\quad - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} \psi_{f_1}\left(F_1^{-1}\left(\frac{i}{n+1}\right)\right)\psi_{f_1}\left(F_1^{-1}\left(\frac{j}{n+1}\right)\right), \end{aligned} \tag{14}$$

with $\overline{\psi^2}_{f_1}^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \psi_{f_1}^2\left(F_1^{-1}\left(\frac{i}{n+1}\right)\right)$. The variance of $\underset{\sim}{\Delta}_{f_1}^{(n)}$, is

$$\mathbb{V}ar(\underset{\sim}{\Delta}_{f_1}^{(n)}) = \frac{{K^{(n)}}^2 \sum_{i=1}^{n}\left(X_i^2 - \overline{X^2}\right)^2}{4(n-1)\sigma^4} s^{2\,(n)}_{f_1}. \tag{15}$$

The proof of the next proposition is based on Hájek's projection theorem, followed from Hallin et al. [7] and reinforced by Hallin and Werker [10, proposition 3.1]; this result allows us to give the distribution under $\mathbb{P}_{g_1}^{(n)}$ such that $g_1 \in \mathscr{F}_0$.

Proposition 3. *Let assumption (B) hold, for all* $f_1 \in \mathscr{F}_A$ *and* $g_1 \in \mathscr{F}_0$*, we have, under* $\mathbb{P}_{g_1}^{(n)}$*, as* $n \to \infty$[4],

$$\underset{\sim}{\Delta}_{f_1}^{(n)} = \frac{K^{(n)}}{2\sigma^2\sqrt{n}} \sum_{i=1}^{n}\left(X_i^2 - \overline{X^2}\right)\psi_{f_1}\left(F_1^{-1}(G_1(Z_i))\right) + o_{\mathbb{P}}(1). \tag{16}$$

Then, from Le Cam's third lemma, we have

$$\begin{pmatrix} \frac{K^{(n)}}{2\sigma^2\sqrt{n}}\sum_{i=1}^{n}\left(X_i^2 - \overline{X^2}\right)\psi_{f_1}\left(F_1^{-1}(G_1(Z_i))\right) \\ \tau\Delta_{g_1}^{(n)} - \frac{1}{2}\tau^2\gamma_{g_1} \end{pmatrix} \xrightarrow{\mathscr{L}} \mathscr{N}\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\tau^2\gamma_{g_1}\end{pmatrix}, \begin{pmatrix} \gamma_{f_1}^{*(n)} & \tau\underset{\sim}{\gamma}_{f_1,g_1}^{(n)} \\ \tau\underset{\sim}{\gamma}_{f_1,g_1}^{(n)} & \tau^2\gamma_{g_1}\end{pmatrix}\right), \text{ under } \mathbb{P}_{g_1}^{(n)}. \tag{17}$$

Therefore

$$\underset{\sim}{\Delta}_{f_1}^{(n)} \xrightarrow{\mathscr{L}} \mathscr{N}\left(\tau\underset{\sim}{\gamma}_{f_1,g_1}^{(n)}, \gamma_{f_1}^{*(n)}\right), \text{ under } \mathbb{P}^{(n)}_{n^{-1/2}K^{(n)}\tau; g_1, h_1}, \tag{18}$$

with

$$\underset{\sim}{\gamma}_{f_1,g_1}^{(n)} := \frac{{K^{(n)}}^2\sum_{i=1}^{n}\left(X_i^2 - \overline{X^2}\right)^2}{4n\sigma^4}\mathscr{I}_{\psi}(f_1,g_1), \qquad \mathscr{I}_{\psi}(f_1,g_1) := \int_0^1 \psi_{f_1}\left(F_1^{-1}(u)\right)\psi_{g_1}\left(G_1^{-1}(u)\right)\, du \tag{19}$$

[4] G_1 is the distribution function associated with g_1

and

$$\gamma_{f_1}^{*(n)} := \frac{K^{(n)^2} \sum_{i=1}^{n} \left(X_i^2 - \overline{X^2}\right)^2}{4n\sigma^4} \mathscr{I}_{\psi}(f_1). \tag{20}$$

From proposition 3 and 17, as $n \to \infty$, we have: $\mathbb{V}ar(\underset{\sim}{\Delta}_{f_1}^{(n)}) = \gamma_{f_1}^{*(n)} + o(1)$.

Local asymptotic optimality at density f_1 is achieved by the test based on $\underset{\sim}{T}_{f_1}^{(n)}$, where

$$\underset{\sim}{T}_{f_1}^{(n)} := \underset{\sim}{\Delta}_{f_1}^{(n)} / \sqrt{\mathbb{V}ar(\underset{\sim}{\Delta}_{f_1}^{(n)})} = \left[\frac{1-\frac{1}{n}}{s_{f_1}^{2(n)} \sum_{i=1}^{n} \left(X_i^2 - \overline{X^2}\right)^2}\right]^{1/2} \sum_{i=1}^{n} X_i^2 \left\{\psi_{f_1}\left(F_1^{-1}\left(\frac{R_i^{(n)}}{n+1}\right)\right) - \overline{\psi}_{f_1}^{(n)}\right\}. \tag{21}$$

More precisely, we have the following proposition.

Proposition 4. *Let Assumptions (B) and (C) hold and fix $f_1 \in \mathscr{F}_A$. Then,*

(i) *for any $g_1 \in \mathscr{F}_A$, $\underset{\sim}{T}_{f_1}^{(n)}$ is asymptotically normal, with mean zero under $\mathbb{P}_{g_1}^{(n)}$, mean $\tau\underset{\sim}{\gamma}_{f_1,g_1}^{(n)}/\gamma_{f_1}^{*(n)}$ under $\mathbb{P}_{n^{-1/2}K^{(n)}\tau;g_1,h_1}^{(n)}$ $(h_1 \in \mathscr{F}_{C|g_1})$ and variance one under both;*

(ii) *the sequence of tests rejecting the null hypothesis whenever*

$$\underset{\sim}{T}_{f_1}^{(n)} > z_{1-\alpha},$$

is locally asymptotically most powerful, at asymptotic level α, against local alternatives.

The two most important particular cases for the test statistic presented in proposition 4, are the *van der Waerden (normal scores)* and the *Wilcoxon (logistic scores)* test statistics, which are respectively optimal at normal and logistic distributions.

• ***The van der Waerden test statistic*** is defined in a Gaussian case ($f_1 = \phi_1$), with $\psi_{f_1}\left(F_1^{-1}(x)\right) = a\left[\left(\Phi^{-1}(x)\right)^2 - 1\right]$ (Φ is the standard normal distribution function) and the test statistic becomes

$$\underset{\sim}{T}_{vdW}^{(n)} = \left[\frac{1-\frac{1}{n}}{s_{vdW}^{2(n)} \sum_{i=1}^{n} \left(X_i^2 - \overline{X^2}\right)^2}\right]^{1/2} \sum_{i=1}^{n} X_i^2 \left\{\left(\Phi^{-1}\left(\frac{R_i^{(n)}}{n+1}\right)\right)^2 - \overline{\psi}_{vdW}^{(n)}\right\}, \tag{22}$$

with $\overline{\psi}_{vdW}^{(n)} := \frac{1}{n}\sum_{i=1}^{n}\left(\Phi^{-1}\left(\frac{i}{n+1}\right)\right)^2$ and

$$\begin{aligned} s_{vdW}^{2(n)} := & \frac{n-1}{n^2}\sum_{i=1}^{n}\left[\left(\Phi^{-1}\left(\frac{i}{n+1}\right)\right)^2 - 1\right]^2 \\ & - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n}\left[\left(\Phi^{-1}\left(\frac{i}{n+1}\right)\right)^2 - 1\right]\left[\left(\Phi^{-1}\left(\frac{j}{n+1}\right)\right)^2 - 1\right]. \end{aligned}$$

• ***The Wilcoxon test statistic*** is defined in a logistic case ($f_1 = \ell_1$) with $\psi_{f_1}\left(F_1^{-1}(x)\right) = b\left(6x^2 - 6x + 1\right)$ and the test statistic is writing under the following form

$$\underset{\sim}{T}_{W}^{(n)} = \left[\frac{1-\frac{1}{n}}{s_{W}^{2(n)} \sum_{i=1}^{n} \left(X_i^2 - \overline{X^2}\right)^2}\right]^{1/2} \sum_{i=1}^{n} X_i^2 \left\{6\left(\frac{R_i^{(n)}}{n+1}\right)^2 - 6\left(\frac{R_i^{(n)}}{n+1}\right) - \overline{\psi}_{W}^{(n)}\right\}, \tag{23}$$

with $\overline{\psi}_{W}^{(n)} = -\frac{n+2}{n+1}$ and $s_{W}^{2(n)} = \frac{(n-2)(n-1)(n+2)}{5(n+1)^3}$.

4.2 Optimal signed-rank test

In this subsection, we focus on symmetric densities and we assume that $f_1 \in \mathscr{F}_0^+$. A maximal invariant for the group $\mathscr{G}_l$ is known to be the vector of signs $(s_1, s_2, \ldots, s_n)'$, along with the vector of ranks $(R_{+,1}^{(n)}, R_{+,2}^{(n)}, ..., R_{+,n}^{(n)})'$, where s_i is the sign of Z_i and $R_{+,i}^{(n)}$ the rank of $|Z_i|$ among $|Z_1|, |Z_2|, \ldots, |Z_n|$. Knowing that ψ_{f_1} is *even*, then the sign have no impact on the signed-rank based version of the efficient central sequence $\Delta_{f_1}^{(n)}$. We have then

$$\underset{\sim}{\Delta}_{f_1}^{+(n)} := \frac{K^{(n)}}{2\sigma^2\sqrt{n}} \sum_{i=1}^{n} X_i^2 \left\{ \psi_{f_1}\left(F_{1,+}^{-1}\left(\frac{R_{+,i}^{(n)}}{n+1}\right)\right) - \overline{\psi}_{f_1}^{+(n)} \right\}, \tag{24}$$

with $\overline{\psi}_{f_1}^{+(n)} := \frac{1}{n}\sum_{i=1}^{n} \psi_{f_1}\left(F_{1,+}^{-1}\left(\frac{i}{n+1}\right)\right)$. Let

$$\begin{aligned} s_{f_1}^{2+(n)} &:= \overline{\psi^2}_{f_1}^{+(n)} - \left(\overline{\psi}_{f_1}^{+(n)}\right)^2 \\ &= \frac{n-1}{n^2}\sum_{i=1}^{n} \psi_{f_1}^2\left(F_{1,+}^{-1}\left(\frac{i}{n+1}\right)\right) \\ &\quad - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} \psi_{f_1}\left(F_{1,+}^{-1}\left(\frac{i}{n+1}\right)\right)\psi_{f_1}\left(F_{1,+}^{-1}\left(\frac{j}{n+1}\right)\right), \end{aligned} \tag{25}$$

with $\overline{\psi^2}_{f_1}^{+(n)} := \frac{1}{n}\sum_{i=1}^{n} \psi_{f_1}^2\left(F_{1,+}^{-1}\left(\frac{i}{n+1}\right)\right)$ and $F_{1,+}$ is such that:

- ▷ F_1 is a cumulative distribution function *(cdf.)* of the random variable Z, $F_{1,+} : x \mapsto 2F_1(x) - 1$ is a *cdf.* of $|Z|$;
- ▷ $F_{1,+}^{-1}(v) = F_1^{-1}\left(\frac{v+1}{2}\right)$ for all $v \in [0,1]$.

The proof of the next proposition is an immediate consequence of proposition 3, knowing that $\psi_{f_1}\left(F_{1,+}^{-1}(G_{1,+}(Z_i))\right) = \psi_{f_1}\left(F_1^{-1}(G_1(Z_i))\right)$.

Proposition 5. *Let assumption (B) hold, for all* $f_1 \in \mathscr{F}_A^+$ *and* $g_1 \in \mathscr{F}_0^+$*, we have, under* $\mathbb{P}_{g_1}^{(n)}$*, as* $n \to \infty$*,*

$$\underset{\sim}{\Delta}_{f_1}^{+(n)} = \frac{K^{(n)}}{2\sigma^2\sqrt{n}} \sum_{i=1}^{n} \left(X_i^2 - \overline{X^2}\right) \psi_{f_1}\left(F_{1,+}^{-1}(G_{1,+}(Z_i))\right) + o_{\mathbb{P}}(1). \tag{26}$$

In this case, local asymptotic optimality at density f_1 is achieved by the test based on $\underset{\sim}{T}_{f_1}^{+(n)}$, where

$$\underset{\sim}{T}_{f_1}^{+(n)} := \left[\frac{1-\frac{1}{n}}{s_{f_1}^{2+(n)}\sum_{i=1}^{n}\left(X_i^2 - \overline{X^2}\right)^2}\right]^{1/2} \sum_{i=1}^{n} X_i^2 \left\{ \psi_{f_1}\left(F_{1,+}^{-1}\left(\frac{R_{+,i}^{(n)}}{n+1}\right)\right) - \overline{\psi}_{f_1}^{+(n)} \right\}. \tag{27}$$

Then, the result given in the next proposition is followed from proposition 5 and the Le Cam's third lemma.

Proposition 6. *Let Assumptions (B) and (C) hold and fix* $f_1 \in \mathscr{F}_A^+$*. Then,*

(i) for any $g_1 \in \mathscr{F}_A^+$*,* $\underset{\sim}{T}_{f_1}^{+(n)}$ *is asymptotically normal, with mean zero under* $\mathbb{P}_{g_1}^{(n)}$*, mean* $\tau\underset{\sim}{\gamma}_{f_1,g_1}^{(n)}/\gamma_{f_1}^{*(n)}$ *under* $\mathbb{P}_{n^{-1/2}K^{(n)}\tau;g_1,h_1}^{(n)}(h_1 \in \mathscr{F}_{C|g_1}^+)$ *and variance one under both;*
(ii) the sequence of tests rejecting the null hypothesis whenever

$$\underset{\sim}{T}_{f_1}^{+(n)} > z_{1-\alpha},$$

is locally asymptotically most powerful, at asymptotic level α*, against local alternatives.*

• ***The signed van der Waerden test statistic (normal scores)*** : given for $f_1 = \phi_1$, where $\psi_{f_1}\left(F_{1,+}^{-1}(x)\right) = a\left[\left(\Phi^{-1}\left(\frac{x+1}{2}\right)\right)^2 - 1\right]$ and

$$\underset{\sim}{T}_{vdW}^{+(n)} = \left[\frac{1-\frac{1}{n}}{s_{vdW}^{2+(n)}\sum_{i=1}^{n}\left(X_i^2-\overline{X^2}\right)^2}\right]^{1/2}\sum_{i=1}^{n}X_i^2\left\{\left(\Phi^{-1}\left(\frac{1}{2}+\frac{R_{+,i}^{(n)}}{2(n+1)}\right)\right)^2-\overline{\psi}_{vdW}^{+(n)}\right\}, \tag{28}$$

with $\overline{\psi}_{vdW}^{+(n)} = \frac{1}{n}\sum_{i=1}^{n}\left(\Phi^{-1}\left(\frac{1}{2}+\frac{i}{2(n+1)}\right)\right)^2$ and

$$s_{vdW}^{2+(n)} := \frac{n-1}{n^2}\sum_{i=1}^{n}\left[\left(\Phi^{-1}\left(\frac{1}{2}+\frac{i}{2(n+1)}\right)\right)^2-1\right]^2 - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left[\left(\Phi^{-1}\left(\frac{1}{2}+\frac{i}{2(n+1)}\right)\right)^2-1\right]\left[\left(\Phi^{-1}\left(\frac{1}{2}+\frac{j}{2(n+1)}\right)\right)^2-1\right].$$

• ***The signed Wilcoxon test statistic (logistic scores)*** : given for $f_1 = \ell_1$, where $\psi_{f_1}\left(F_{1,+}^{-1}(x)\right) = \frac{b}{2}\left(3x^2-1\right)$ and the test statistic is writing under the form

$$\underset{\sim}{T}_{W}^{+(n)} = 3\left[\frac{1-\frac{1}{n}}{s_{W}^{2+(n)}\sum_{i=1}^{n}\left(X_i^2-\overline{X^2}\right)^2}\right]^{1/2}\sum_{i=1}^{n}X_i^2\left\{\left(\frac{R_{+,i}^{(n)}}{n+1}\right)^2-\overline{\psi}_{W}^{+(n)}\right\}, \tag{29}$$

with $\overline{\psi}_{W}^{+(n)} = \frac{2n+1}{6(n+1)}$ and $s_W^{2+(n)} = \frac{(n-1)(2n+1)(8n+11)}{20(n+1)^3}$.

Remark

Due to the expressions of $s_{vdW}^{(n)}$ and $s_W^{(n)}$ for the rank tests (respectively to $s_{vdW}^{+(n)}$ and $s_W^{+(n)}$ for the signed-rank tests), the scale factors a for van der Waerden and b for Wilcoxon are simplified in the final expressions of the rank (respectively signed-rank) test statistics. This confirms that the choice of the median of absolute deviation as a scale parameter in the definition of $\mathscr{F}_0$ (respectively $\mathscr{F}_0^+$) has no impact on the results.

5 Simulation

The purpose of this section is to evaluate the performance of the proposed tests, in propositions 4 and 6, at asymptotic level α. Let's consider the model

$$Y_i = \mu + \beta X_i + \xi_i X_i + \varepsilon_i, \qquad i = 1,\ldots,n = 100, \tag{30}$$

where,

(a) $\mu = 1$ and $\beta = 10$;
(b) the X_i's are $i.i.d.$ uniform $(0,10)$;
(c) the ξ_i's are $i.i.d.$ Gaussian with mean zero and standard deviation $\sigma_\xi = 0$ (**for null hypothesis**), $= 0.1,\ 0.2,\ 0.3,\ 0.4$ or 0.5 (**for increasing alternatives**). When asymmetric densities are used, the non null values of σ_ξ considered are $0.05,\ 0.1,\ 0.15,\ 0.2$ and 0.25;
(d) the ε_i's are $i.i.d.$ with a symmetric density – Gaussian (ϕ_1), logistic (ℓ_1), Student with $\nu = 1$, 3 and 5 degrees of freedom (t_ν) – or with an asymmetric density – the skew normal $(s\mathcal{N})$ or skew Student t_5 (st_5) densities[5] (both with skewness parameter value $\delta = 10$).

[5] for example, the skew-normal distribution with shape parameter $\delta \neq 0$ is defined as $f_{s\mathcal{N}}(z) := 2\phi_1(z)\Phi(\delta z)$ where Φ stands for the standard normal distribution function. See, for details, [2].

We generated $N = 2500$ independent samples of size $n = 100$ from (30).

For symmetric densities, table 1 and 2 show the rejection frequencies for the van der Waerden (T_{vdW^+}), the Wilcoxon (T_{W^+}), the Student ($T_{t_\nu^+}$, with $\nu = 1,\ 3,\ 5$) and the Ramanathan and Rajarshi [19] (T_{RR}) tests. Furthermore, when the densities are asymmetric, we present in table 3 the rejection frequencies, for the van der Waerden (T_{vdW}), the Wilcoxon (T_W), the Student (T_{t_ν}, with $\nu = 1,\ 3,\ 5$) and the Ramanathan and Rajarshi [19] (T_{RR}) tests. The six tests are performed at asymptotic levels $\alpha = 1\%$, 5% and 10%.

Table 1: Rejection frequencies (out of 2500 replications), at asymptotic levels $\alpha = 1\%$, 5% and 10%, for $\sigma_\xi = 0$ (null hypothesis), 0.1, 0.2, 0.3, 0.4, 0.5 (alternative hypotheses), with error density g_1 that is Gaussian (ϕ_1) and logistic (ℓ_1), of the van der Waerden test (T_{vdW^+}), the Wilcoxon test (T_{W^+}), the Student $T_{t_\nu^+}$ tests (t_ν-score with $\nu = 1,3,5$) and Ramanathan and Rajarshi test (T_{RR}), for $n = 100$.

g_1	*Test*	α	σ_ξ					
			0	0.1	0.2	0.3	0.4	0.5
ϕ_1	T_{vdW^+}	1%	0.0092	0.0652	0.4440	0.8184	0.9460	0.9808
		5%	0.0468	0.2260	0.6928	0.9440	0.9900	0.9996
		10%	0.0920	0.3480	0.7968	0.9764	0.9980	0.9988
	T_{W^+}	1%	0.0096	0.0568	0.3628	0.7412	0.9100	0.9688
		5%	0.0492	0.1876	0.6132	0.9072	0.9756	0.9940
		10%	0.0960	0.2988	0.7308	0.9572	0.9912	0.9968
	$T_{t_1^+}$	1%	0.0084	0.0124	0.0248	0.0656	0.1016	0.1516
		5%	0.0516	0.0672	0.1164	0.1992	0.2992	0.4044
		10%	0.1016	0.1408	0.2096	0.3308	0.4568	0.5460
	$T_{t_3^+}$	1%	0.0084	0.0456	0.3120	0.6844	0.8884	0.9592
		5%	0.0452	0.1652	0.5772	0.8932	0.9712	0.9920
		10%	0.1008	0.2800	0.7016	0.9472	0.9924	0.9980
	$T_{t_5^+}$	1%	0.0104	0.0532	0.3108	0.6872	0.8824	0.9560
		5%	0.0456	0.1740	0.5588	0.8748	0.9632	0.9908
		10%	0.0932	0.2784	0.6848	0.9392	0.9836	0.9956
	T_{RR}	1%	0.0104	0.0560	0.3588	0.7384	0.9072	0.9680
		5%	0.0488	0.1852	0.6072	0.9052	0.9748	0.9936
		10%	0.0988	0.2948	0.7276	0.9572	0.9912	0.9964
ℓ_1	T_{vdW^+}	1%	0.0096	0.0504	0.3396	0.7264	0.9108	0.9700
		5%	0.0496	0.1560	0.5860	0.8912	0.9804	0.9928
		10%	0.0936	0.2556	0.7112	0.9412	0.9920	0.9968
	T_{W^+}	1%	0.0096	0.0476	0.3112	0.6816	0.8904	0.9640
		5%	0.0492	0.1568	0.5812	0.8768	0.9704	0.9928
		10%	0.0956	0.2620	0.7172	0.9372	0.9872	0.9972
	$T_{t_1^+}$	1%	0.0064	0.0168	0.0364	0.0648	0.1224	0.1612
		5%	0.0444	0.0840	0.1588	0.2428	0.3340	0.4136
		10%	0.0940	0.1708	0.2640	0.3748	0.4904	0.5832
	$T_{t_3^+}$	1%	0.0088	0.0496	0.2836	0.6436	0.8544	0.9428
		5%	0.0456	0.1596	0.5420	0.8516	0.9608	0.9900
		10%	0.0968	0.2552	0.6848	0.9188	0.9836	0.9952
	$T_{t_5^+}$	1%	0.0104	0.0460	0.2796	0.6296	0.8516	0.9460
		5%	0.0456	0.1540	0.5524	0.8504	0.9600	0.9896
		10%	0.0932	0.2524	0.6900	0.9216	0.9836	0.9952
	T_{RR}	1%	0.0096	0.0460	0.3064	0.6780	0.8888	0.9632
		5%	0.0488	0.1548	0.5792	0.8752	0.9696	0.9924
		10%	0.0968	0.2584	0.7144	0.9368	0.9872	0.9972

Table 2: Rejection frequencies (out of 2500 replications), at asymptotic levels $\alpha = 1\%$, 5% and 10%, for $\sigma_\xi = 0$ (null hypothesis), 0.1, 0.2, 0.3, 0.4, 0.5 (alternative hypotheses), with error density g_1 that is Student (t_ν, with $\nu = 1$, 3 and 5) of the van der Waerden test (T_{vdW^+}), the Wilcoxon test (T_{W^+}), the Student $T_{t_\nu^+}$ tests (t_ν-score with $\nu = 1,3,5$) and the T_{RR} one, for $n = 100$.

g_1	*Test*	α	σ_ξ					
			0	0.1	0.2	0.3	0.4	0.5
t_1	T_{vdW^+}	1%	0.0100	0.0236	0.0712	0.1480	0.2512	0.3700
		5%	0.0444	0.0904	0.2012	0.3524	0.5096	0.6364
		10%	0.0888	0.1680	0.3180	0.4836	0.6500	0.7708
	T_{W^+}	1%	0.0084	0.0376	0.1576	0.3472	0.5384	0.7028
		5%	0.0480	0.1388	0.3700	0.6072	0.7800	0.8852
		10%	0.0904	0.2252	0.5036	0.7464	0.8688	0.9448
	$T_{t_1^+}$	1%	0.0072	0.0484	0.1616	0.2756	0.3100	0.3924
		5%	0.0432	0.1744	0.3928	0.5412	0.6040	0.6840
		10%	0.0936	0.2812	0.5408	0.6852	0.7464	0.8000
	$T_{t_3^+}$	1%	0.0084	0.0396	0.1888	0.3748	0.5600	0.6860
		5%	0.0504	0.1452	0.4204	0.6528	0.8112	0.8824
		10%	0.1048	0.2464	0.5668	0.7792	0.8956	0.9372
	$T_{t_5^+}$	1%	0.0088	0.0408	0.1796	0.3936	0.5940	0.7436
		5%	0.0484	0.1456	0.4040	0.6584	0.8164	0.9116
		10%	0.0936	0.2316	0.5420	0.7920	0.8984	0.9584
	T_{RR}	1%	0.00800	0.0356	0.1532	0.3432	0.5328	0.6960
		5%	0.0472	0.1384	0.3644	0.6036	0.7772	0.8824
		10%	0.0900	0.2248	0.4996	0.7456	0.8688	0.9436
t_3	T_{vdW^+}	1%	0.0104	0.0328	0.2060	0.5288	0.7576	0.9080
		5%	0.0500	0.1300	0.4512	0.7552	0.9192	0.9764
		10%	0.0996	0.2112	0.5764	0.8464	0.9600	0.9900
	T_{W^+}	1%	0.0112	0.0416	0.2596	0.5976	0.8176	0.9304
		5%	0.0488	0.1512	0.5268	0.8148	0.9448	0.9832
		10%	0.0968	0.2524	0.6588	0.8900	0.9740	0.9920
	$T_{t_1^+}$	1%	0.0084	0.0236	0.0508	0.1044	0.1548	0.1924
		5%	0.0492	0.0992	0.1896	0.2948	0.3868	0.4656
		10%	0.1048	0.1840	0.3180	0.4440	0.5376	0.6080
	$T_{t_3^+}$	1%	0.0104	0.0436	0.2464	0.5672	0.7968	0.9116
		5%	0.0440	0.1464	0.5060	0.7976	0.9360	0.9784
		10%	0.0936	0.2560	0.6456	0.8804	0.9680	0.9908
	$T_{t_5^+}$	1%	0.0116	0.0424	0.2640	0.5772	0.8060	0.9144
		5%	0.0472	0.1568	0.5260	0.8148	0.9380	0.9796
		10%	0.0968	0.2628	0.6616	0.8864	0.9676	0.9904
	T_{RR}	1%	0.0108	0.0400	0.2552	0.5924	0.8152	0.9284
		5%	0.0472	0.1480	0.5212	0.8144	0.9440	0.9832
		10%	0.0948	0.2488	0.6572	0.8888	0.9736	0.9920
t_5	T_{vdW^+}	1%	0.0084	0.0476	0.2868	0.6612	0.8692	0.9524
		5%	0.0492	0.1544	0.5436	0.8640	0.9668	0.9908
		10%	0.0968	0.2544	0.6824	0.9204	0.9856	0.9968
	T_{W^+}	1%	0.0084	0.0428	0.3000	0.6524	0.8612	0.9524
		5%	0.0480	0.1740	0.5628	0.8604	0.9616	0.9896
		10%	0.1016	0.2832	0.7040	0.9228	0.9840	0.9964
	$T_{t_1^+}$	1%	0.0088	0.0200	0.0444	0.0736	0.1288	0.1840
		5%	0.0500	0.0776	0.1676	0.2616	0.3492	0.4480
		10%	0.0988	0.1556	0.2896	0.3972	0.4976	0.5968
	$T_{t_3^+}$	1%	0.0104	0.0476	0.2520	0.6212	0.8356	0.9188
		5%	0.0576	0.1520	0.5032	0.8400	0.9504	0.9844
		10%	0.0960	0.2604	0.6520	0.9140	0.9736	0.9916
	$T_{t_5^+}$	1%	0.0084	0.0424	0.2832	0.6216	0.8364	0.9324
		5%	0.0460	0.1716	0.5460	0.8384	0.9524	0.9860
		10%	0.1028	0.2796	0.6848	0.9120	0.9812	0.9956
	T_{RR}	1%	0.0084	0.0412	0.2948	0.6484	0.8600	0.9512
		5%	0.0444	0.1720	0.5588	0.8584	0.9612	0.9896
		10%	0.1012	0.2800	0.7016	0.9224	0.9836	0.9952

Table 3: Rejection frequencies (out of 2500 replications), at asymptotic levels $\alpha = 1\%$, 5% and 10%, for $\sigma_\xi = 0$ (null hypothesis), $0.05,\ 0.1,\ 0.15,\ 0.2,\ 0.25$ (alternative hypotheses), with error density g_1 that is skew-normal density ($f_{s\mathcal{N}}$) and skew-Student density with 5 $d.f$ (st_5), of the van der Waerden test (T_{vdW}), the Wilcoxon test (T_W), the Student T_{t_ν} tests (t_ν-score with $\nu = 1,3,5$) and the Ramanathan and Rajarshi test (T_{RR}), for $n = 100$.

g_1	Test	α	σ_ξ					
			0	0.05	0.1	0.15	0.2	0.25
$s\mathcal{N}$	T_{vdW}	1%	0.0084	0.0696	0.3180	0.6840	0.8588	0.9468
		5%	0.0468	0.2032	0.5804	0.8720	0.9624	0.9900
		10%	0.0956	0.3280	0.7100	0.9332	0.9824	0.9964
	T_W	1%	0.0064	0.0420	0.2420	0.5896	0.8072	0.9264
		5%	0.0436	0.1392	0.4808	0.8132	0.9348	0.9804
		10%	0.0872	0.2440	0.6152	0.8936	0.9672	0.9912
	T_{t_1}	1%	0.0076	0.0068	0.0116	0.0336	0.0696	0.1160
		5%	0.0444	0.0400	0.0692	0.1476	0.2248	0.3148
		10%	0.0976	0.0916	0.1320	0.2656	0.3620	0.4552
	T_{t_3}	1%	0.0092	0.0368	0.1980	0.5208	0.7800	0.9012
		5%	0.0508	0.1336	0.4348	0.7680	0.9228	0.9748
		10%	0.1052	0.2180	0.5668	0.8600	0.9668	0.9904
	T_{t_5}	1%	0.0072	0.0324	0.2144	0.5372	0.7696	0.9044
		5%	0.0420	0.1208	0.4420	0.7876	0.9168	0.9768
		10%	0.0860	0.2212	0.5756	0.8756	0.9548	0.9876
	T_{RR}	1%	0.0088	0.0140	0.0488	0.1536	0.3568	0.5708
		5%	0.0500	0.0728	0.1600	0.3776	0.6116	0.8108
		10%	0.1008	0.1364	0.2540	0.5088	0.7428	0.8916
st_5	T_{vdW}	1%	0.0088	0.0680	0.3444	0.6664	0.8580	0.9456
		5%	0.0484	0.2088	0.6092	0.8660	0.9596	0.9924
		10%	0.1024	0.3208	0.7268	0.9248	0.9820	0.9964
	T_W	1%	0.0100	0.0388	0.2688	0.5768	0.8000	0.9192
		5%	0.0500	0.1392	0.4900	0.8056	0.9380	0.9848
		10%	0.1040	0.2308	0.6344	0.8844	0.9692	0.9928
	T_{t_1}	1%	0.0088	0.0068	0.0172	0.0360	0.0748	0.1148
		5%	0.0520	0.0392	0.0800	0.1388	0.2380	0.3224
		10%	0.0988	0.0888	0.1456	0.2420	0.3752	0.4740
	T_{t_3}	1%	0.0096	0.1692	0.1656	0.4248	0.6688	0.8316
		5%	0.0500	0.3980	0.3856	0.6792	0.8596	0.9432
		10%	0.1028	0.5260	0.5224	0.7952	0.9180	0.9732
	T_{t_5}	1%	0.0104	0.0344	0.2304	0.5256	0.7652	0.9020
		5%	0.0484	0.1192	0.4508	0.7700	0.9216	0.9772
		10%	0.1036	0.2072	0.5916	0.8600	0.9604	0.9900
	T_{RR}	1%	0.0120	0.0144	0.0484	0.1640	0.3604	0.5864
		5%	0.0552	0.0760	0.1632	0.3740	0.6132	0.8072
		10%	0.1024	0.1464	0.2712	0.5092	0.7332	0.8812

6 Conclusion and perspectives

The approach used in this paper allows detecting a randomness criterion in the regression model. It is shown that the distribution of the random criterion (h_1) has no influence on the test statistics (which justifies the only choice of the density h_1 as Gaussian in the simulation section).

It is clearly seen that all the considered tests here are extremely conservatives. It is explained by the fact that the considered tests do not get the nominal rejection frequencies under the null and their powers are increasing with respect to σ_ξ under the alternatives. The power of the Student t_ν-score rank and signed-rank tests are increasing as much as ν increases. The simulation results show that the Wilxoxon signed-rank test given in (29) is equivalent to the Ramanathan and Rajatchi test (table 1 and 2). It also appears from the skew-normal and the skew-Student simulations (table 3) that asymmetry significantly improves the superiority of rank tests over Ramanathan and Rajarshi procedure.

The developed work shows its power in the case of non symmetric distribution and it could be extended to the case of unknown regression parameters. Future investigations will be devoted to this goal and also to a large class of stochastic parameter regression models.

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A Appendix (Proof of proposition 1)

The proof relies on Swensen's conditions 1.2 to 1.7 of [21, lemma 1]. More precisely, the only delicate one is the condition 1.2. This condition is a direct consequence of the *quadratic mean differentiability*, at $\sigma_\xi = 0$ of

$$g^{1/2}_{\sigma_\xi}(\xi) := \left\{ \int_{\mathbb{R}} f\left(\varepsilon - \sigma_\xi \xi x\right) h(\xi) d\xi \right\}^{1/2}$$

However, this quadratic mean differentiability is somewhat non standard, as it involves the second-order derivatives $\ddot{f}$ of the density f. As in Akharif and Hallin [1], the proof is decomposed into the following three parts.

(i) $y^2 \mapsto g(y) = \int_{\mathbb{R}} f(\varepsilon - xyv)h(v)\, dv$ is absolutely continuous in a right-neighborhood of $y = 0$, with a.e. derivative

$$\frac{1}{2y} \int_{w=0}^{y} \int_{\mathbb{R}} \ddot{f}(\varepsilon - xwv)x^2v^2h(v)\, dv\, dw. \tag{31}$$

Indeed, from the absolutely continuity of f and $\dot{f}$, and Fubini's theorem, we obtain

$$\begin{aligned} g(y) - g(0) &= \int_{\mathbb{R}} \left[f(\varepsilon - xyv) - f(\varepsilon)\right] h(v)\, dv \\ &= - \int_{\mathbb{R}} \int_{a=0}^{y} \dot{f}(\varepsilon - xav)xv\, da\, h(v)\, dv \\ &= \int_{\mathbb{R}} \int_{a=0}^{y} \int_{w=0}^{a} \ddot{f}(\varepsilon - xwv)x^2v^2\, dw\, da\, h(v)\, dv \end{aligned}$$

$$g(y) - g(0) = \frac{1}{2} \int_{b=0}^{y^2} b^{-\frac{1}{2}} \int_{w=0}^{b^{\frac{1}{2}}} \int_{v=-\infty}^{+\infty} \ddot{f}(\varepsilon - xwv)x^2v^2\, h(v)\, dv\, dw\, db. \tag{32}$$

The value of the a.e. derivative in (31) follows for each $y > 0$. At $y = 0$, the right derivative is defined as the limit, as $y \to 0$, of $\left[g(y) - g(0)\right]/y^2$, (32) yields $\frac{0}{0}$, but by applying L'Hospital's rule, it leads to $\frac{1}{2}\ddot{f}(\varepsilon)x^2 \int_{\mathbb{R}} v^2 h(v)\, dv = \frac{1}{2}\ddot{f}(\varepsilon)x^2$.

(ii) It follows that $y^2 \mapsto s_{\varepsilon,x}(y) := [g(y)]^{1/2}$ is absolutely continuous in a neighborhood of $y = 0$, with a.e. derivative

$$\dot{s}_{\varepsilon,x}(y) = \frac{1}{4y} \int_{w=0}^{y} \frac{\int_{\mathbb{R}} \ddot{f}(\varepsilon - xwv)x^2v^2\, h(v)\, dv}{\left[\int_{\mathbb{R}} f(\varepsilon - xyv)\, h(v)\, dv\right]^{\frac{1}{2}}} dw. \tag{33}$$

L'Hospital's rule at $y = 0$ yields $\dot{s}_{\varepsilon,x}(0) = \frac{1}{4} f^{-\frac{1}{2}}(\varepsilon)\ddot{f}(\varepsilon)x^2$. Hence, for all ε,

$$\lim_{y\to 0} \left[s_{\varepsilon,x}(y) - s_{\varepsilon,x}(0)\right]/y^2 = \dot{s}_{\varepsilon,x}(0). \tag{34}$$

(iii) The partial quadratic mean differentiability property to be proved takes the form

$$\lim_{y\to 0} \int_{\mathbb{R}} \left\{ \frac{1}{y^2}\left[s_{\varepsilon,x}(y) - s_{\varepsilon,x}(0)\right] - \dot{s}_{\varepsilon,x}(0) \right\}^2 d\varepsilon = 0. \tag{35}$$

From (ii) above,

$$\begin{aligned} \left\{ \frac{1}{y^2}\left[s_{\varepsilon,x}(y) - s_{\varepsilon,x}(0)\right] \right\}^2 &= \left(\frac{1}{y^2}\right)^2 \left(\int_{\lambda=0}^{y^2} \dot{s}_{\varepsilon,x}(\sqrt{\lambda})\, d\lambda \right)^2 \\ &\leq \frac{1}{y^2} \int_{\lambda=0}^{y^2} \left(\dot{s}_{\varepsilon,x}(\sqrt{\lambda})\right)^2\, d\lambda, \end{aligned} \tag{36}$$

for all ε. Fubini's theorem and (33) yields

$$\begin{aligned}\int_{\mathbb{R}}\left\{\frac{1}{y^2}\left[s_{\varepsilon,x}(y)-s_{\varepsilon,x}(0)\right]\right\}^2 d\varepsilon \leq & \frac{1}{y^2}\int_{\lambda=0}^{y^2}\int_{\mathbb{R}}\left(\dot{s}_{\varepsilon,x}(\sqrt{\lambda})\right)^2 d\varepsilon\, d\lambda \\ = & \frac{1}{16y^2}\int_{\lambda=0}^{y^2}\mathscr{I}^x_{\psi\psi}(f;\sqrt{\lambda})\, d\lambda,\end{aligned} \tag{37}$$

with $\mathscr{I}^x_{\psi\psi}$ defined in (6) and from the continuity assumption in (C.2) , this latter quantity converges, as $y \to 0$, to $\mathscr{I}^x_{\psi\psi}(f;0)/16 = \int_{\mathbb{R}}(\dot{s}_{\varepsilon,x}(0))^2\, d\varepsilon$. Which, together with (37), entails that

$$\limsup_{y\longrightarrow 0}\int_{\mathbb{R}}\left\{\frac{1}{y^2}\left[s_{\varepsilon,x}(y)-s_{\varepsilon,x}(0)\right]\right\}^2 d\varepsilon \leq \int_{\mathbb{R}}(\dot{s}_{\varepsilon,x}(0))^2\, d\varepsilon. \tag{38}$$

In view of Theorem V.I.3 of Hájek and Šidák [5] [also in Hájek et al. [6]], (34) and (38) jointly imply (35). This completes the proof.

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