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The Asymptotic Estimations of the Eigen-values and Eigen-functions for the Fourth Order Boundary Value Problem with Smooth Coefficients

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Abstract: In this paper, we have an Eigen-value problem generated by a fourth order differential equation and suitable boundary conditions which contain a spectral parameter. We obtain asymptotic expressions for the fourth linearly independent solutions, and we also found new asymptotic formulas for the Eigen-values and Eigen-functions of this boundary value problem.

Keywords: Eigen-value Problem; Eigen-value; Eigen-function; Spectral Parameter; Asymptotic Formula.

1 Introduction

In this work we have a fourth order linear differential operator which is generated by the differential equation and the boundary conditions of the form:

$$l(y) = y^{(4)}(x) + q(x)y(x) = \lambda^4 y(x), \quad x \in [0,a] \tag{1}$$

$$U_j(y) = \begin{cases} y^{(j)}(0) = 0 & j = 0,1 \\ \sum_{i=1}^{4} (iw_j)^{i-1} y^{(4-i)}(a,\lambda) = 0 & j = 2,3 \end{cases} \tag{2}$$

Where λ is the spectral parameter and $q(x)$ is an arbitrary complex-valued function such that $q(x) \in C^2[0,a]$ and also satisfies,$q^{'}(0) = q^{'}(a) = 0, \int_0^a q(x)dx = 0$ and $q(a) \neq 0$. The spectral properties of Eigen-values and Eigen-functions of a differential equations was investigated by many authors such as, G. D. Birkhoff[1], V. M. Kurbanov[2], H. Menken [3], K. H. F. Jwamer[4, 5] and G. A. Auginov [6], V. A. Chernyatin,[7] and so forth. We can notes that [8] studies the differential equation of order "$2n$" $y^{(2n)}(x) + q(x)y(x) = \lambda^{2n}\rho(x)y(x), x \in [0,a]$ and considered, $\rho(x) \neq 1$, then they got the asymptotes formulas only for the Eigen-values. The aim of this work is to find a new expression for the fourth linearly independent solutions and asymptotic formulas for the Eigen-values and Eigen-functions of (1) and (2) with a new accurate, but before doing this we need some auxiliary results as we proved in section 2 .

2 Auxiliary Results

If $\lambda = \sigma + i\tau$, then the complex plane can be divided into 8 sectors as we see in [9], so that for each sector T_k and $\overline{T_k}, k = 0:3$, different roots of 1 can be arranged as:

$$Re(i\lambda w_0^{'}) \leq Re(i\lambda w_1^{'}) \leq Re(i\lambda w_2^{'}) \leq Re(i\lambda w_3^{'}) \tag{3}$$

Where $w_j^{'}$ is one of w_j and w_j is the root of unity of degree 4, which can be listed as:

$$w_0 = 1, w_1 = i, w_2 = -1, w_3 = -i \tag{4}$$

The numbering depend of arranging $w_j^{'}$ so that satisfying equation (3). We introduce the sectors T_k and $\overline{T_k}, k = 0:3$ of the complex plain. The numbering of the sectors depending on $w_j^{'}$ such that satisfy (3) as we see in the figure 1. So the sectors are:

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$$T_0 : 0 \le arg(\lambda) \le \frac{\pi}{4}, \qquad T_1 : \frac{\pi}{4} \le arg(\lambda) \le \frac{\pi}{2}$$
$$T_3 : \frac{\pi}{2} \le arg(\lambda) \le \frac{3\pi}{4} \qquad T_2 : \pi \le arg(\lambda) \le \frac{5\pi}{4}$$
$$\bar{T_0} : \frac{7\pi}{4} \le arg(\lambda) \le 2\pi \qquad \bar{T_1} : \frac{3\pi}{2} \le arg(\lambda) \le \frac{7\pi}{4}$$
$$\bar{T_3} : \frac{5\pi}{4} \le arg(\lambda) \le \frac{3\pi}{2} \qquad \bar{T_2} : \frac{3\pi}{4} \le arg(\lambda) \le \pi$$

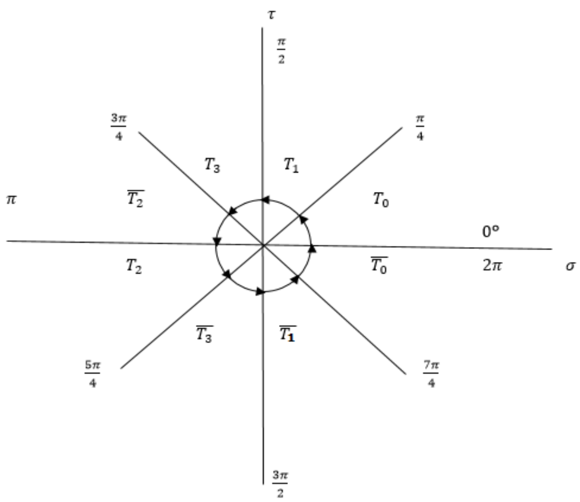

Fig. 1: The positions of the sectors

If λ located in some fixed sector T_k or $\bar{T_k}$ we denote all possible amount sums as:

$$S_1 = Re(i\lambda(w_2^{'} + w_3^{'})) \qquad S_2 = Re(i\lambda(w_1^{'} + w_3^{'}))$$
$$S_3 = Re(i\lambda(w_0^{'} + w_3^{'})) \qquad S_4 = Re(i\lambda(w_1^{'} + w_2^{'}))$$
$$S_5 = Re(i\lambda(w_0^{'} + w_2^{'})) \qquad S_6 = Re(i\lambda(w_0^{'} + w_1^{'}))$$

Lemma 1. The following relation hold for S_j and $w_2^{'}, w_1^{'}$:

$$S_1 \ge S_2, \quad And \quad S_1 \ge S_j + |\lambda| sin(\frac{\pi}{4}), \qquad j = 3:6.$$
$$w_2^{'} = -w_k, w_1^{'} = w_k, \qquad if \quad \lambda \in T_k.$$
$$w_2^{'} = \bar{w_k}, w_1^{'} = -\bar{w_k}, \qquad if \quad \lambda \in \bar{T_k}. \quad for \quad k = 0:3.$$

Proof. We prove the Theorem for every sectors, first if $\lambda \in T_0$ and $\lambda = \sigma + i\tau$, then $0 \le arg(\lambda) \le \frac{\pi}{4}, \sigma \ge 0, \tau \ge 0$ and $\sigma \ge \tau$. To arrange $w_j^{'}$ according to w_j such that satisfy (3), that is

$$Re(i\lambda w_0^{'}) \le Re(i\lambda w_1^{'}) \le Re(i\lambda w_2^{'}) \le Re(i\lambda w_3^{'})$$

And since, $Re(i\lambda w_j^{'}) = -Im(\lambda w_j^{'})$, then $Im(\lambda w_0^{'}) \ge Im(\lambda w_1^{'}) \ge Im(\lambda w_2^{'}) \ge Im(\lambda w_3^{'})$ By using the above inequalities for σ and τ we get that:

$$w_0^{'} = i, w_1^{'} = 1, w_2^{'} = -1, w_3^{'} = -i$$

Now,
$S_2 = S_1 - 2Im(\lambda) = S_1 - 2|\lambda| \sin(arg(\lambda))$. Thus, $S_2 = S_1 - 2|\lambda| \sin(arg(\lambda))$. And since, $0 \le arg(\lambda) \le \frac{\pi}{4}$, and $\sin(x) \ge 0, for x \in [0, \pi]$. so $\sin(arg(\lambda)) \ge 0$ and hence $S_2 \le S_1$.
For S_3
$S_3 = S_1 - |\lambda|[\sin(arg(i\lambda)) + \sin(arg(\lambda))] = S_1 - |\lambda|[\sin(\alpha + \frac{\pi}{2}) + \sin(\alpha)]$,where $\alpha = arg(\lambda)$ So $S_3 = S_1 - |\lambda|[\sin(\alpha + \frac{\pi}{2}) + \sin(\alpha)]$.
To evaluate $\sin(\alpha + \frac{\pi}{2}) + \sin(\alpha)$,
for $0 \le \alpha \le \frac{\pi}{4}$, we get : $\sin(\pi - \frac{\pi}{4}) \le \sin(\alpha + \frac{\pi}{2}) \le \sin(\frac{\pi}{2})$.
So

$$\sin(\frac{\pi}{4}) \le \sin(\alpha + \frac{\pi}{2}) \le \sin(\frac{\pi}{2}) \tag{5}$$

And since $0 \le \alpha \le \frac{\pi}{4}$, then

$$0 \le \sin(\alpha) \le \sin(\frac{\pi}{4}) \tag{6}$$

Then from (5) and (6) we get: $\sin[(\alpha + \frac{\pi}{2}) + \sin(\alpha)] \ge \sin(\frac{\pi}{4})$ So $S_3 = S_1 - |\lambda|[\sin(\alpha + \frac{\pi}{2}) + \sin(\alpha)] \le S_1 - |\lambda| \sin(\frac{\pi}{4})$.
Thus $S_3 \le S_1 - |\lambda| \sin(\frac{\pi}{4})$
For S_4
$S_4 = S_1 - |\lambda|[\sin(\alpha + \frac{\pi}{2}) + \sin(\alpha)]$,where $\alpha = arg(\lambda) \le S_1 - |\lambda| \sin(\frac{\pi}{4})$ as in the above case So $S_4 \le S_1 - |\lambda| \sin(\frac{\pi}{4})$.
For S_5
$S_5 = S_1 - 2|\lambda| \sin(\alpha + \frac{\pi}{2}) \le S_1 - 2|\lambda| \sin(\frac{\pi}{4}) \le S_1 - 2|\lambda| \sin(\frac{\pi}{4})$ as in (6).
So $S_5 \le S_1 - |\lambda| \sin(\frac{\pi}{4})$.
For S_6
$S_6 = S_1 - 2|\lambda|[\sin(arg(i\lambda)) + \sin(arg(\lambda))] \le S_1 - 2|\lambda| \sin(\frac{\pi}{4}) \le S_1 - |\lambda| \sin(\frac{\pi}{4})$ as in the above case ,
So $S_6 \le S_1 - |\lambda| \sin(\frac{\pi}{4})$. If $\lambda \in T_0$, then $S_1 \ge S_2$ and $S_1 \ge S_j - |\lambda| \sin(\frac{\pi}{4}), j = 3 : 6$ and $w_2^{'} = -1 = -w_0, w_1^{'} = 1 = w_0$.
We can use a similar arguments as above for all other sectors to get the result of the lemma.

3 Expressions of Fundamental Solutions

In this section we find a new asymptotic expression for the fundamental solutions of (1).

Theorem 1. If we have the differential equation (1), where $q(x) \in C^{(n-5)}[0,a]$, then for $\lambda \in T_{k'}$ or $\overline{T_{k'}}, k' = 0:3$ and $w_k, k = 0:3$ are fourth root of unity we can find four linearly independent solutions which it is and their derivatives can be expressed as:

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k)^s e^{i\lambda w_k x}\Bigg[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3} + \frac{A_{4sk}(x)}{\lambda^4} + \frac{A_{5sk}(x)}{\lambda^5} + \frac{A_{6sk}(x)}{\lambda^6} + \dots + \frac{A_{nsk}(x)}{\lambda^n} + O(\frac{1}{\lambda^{n+1}})\Bigg]$$

where

$$A_{1sk} = A_{1k}(x), \qquad A_{2sk} = A_{2k}(x), \qquad A_{3sk} = A_{3k}(x)$$

$$A_{4sk} = A_{4k}(x) - \binom{s}{1} i w_k^3 A_3 k'(x),$$

$$A_{5sk} = A_{5k}(x) - \binom{s}{1} i w_k^3 A_{4k}'(x) - \binom{s}{2} w_k^2 A_{3k}''(x),$$

$$A_{6sk} = A_{6k}(x) - \binom{s}{1} i w_k^3 A_{5k}'(x) - \binom{s}{2} w_k^2 A_{4k}''(x) + \binom{s}{3} i w_k A_{3k}'''(x).$$

And so on for $n \geq 7$ we have:

$$A_{nsk} = A_{n,k}(x) - \binom{s}{1} i w_k^3 A_{n-1,k}'(x) - \binom{s}{2} w_k^2 A_{n-2,k}''(x) + \binom{s}{3} i w_k A_{n-3,k}'''(x) + \binom{s}{4} A_{n-4,k}^{(4)}(x).$$

And

$$A_{0k}(x) = 1, \qquad A_{1k}(x) = 0, \qquad A_{2k}(x) = 0,$$

$$A_{3k}(x) = -\frac{iw_k}{4}\int_0^x q(t)A_{0k}(t)dt,$$

$$A_{4k}(x) = -\frac{iw_k}{4}\int_0^x (-6w_k^2 A_{3k}''(t) + q(t)A_{1k}(t))dt,$$

$$A_{5k}(x) = -\frac{iw_k}{4}\int_0^x \Big(-6w_k^2 A_{4k}''(t) + 4iw_k A_{3k}'''(t) + q(t)A_{2k}(t)\Big)dt,$$

And for integer , $n \geq 6$:

$$A_{nk}(x) = -\frac{iw_k}{4}\int_0^x \Big(-6w_k^2 A_{n-1,k}''(t) + 4iw_k A_{n-2,k}'''(t) + A_{n-3,k}^{(4)}(t) + q(t)A_{n-3,k}(t)\Big)dt.$$

Proof. As we see in [1] the solution of the differential equation can be written in a power series of the form

$$y_k(x,\lambda) = e^{\lambda \int_0^x \phi_k(t)dt}\sum_{j=0}^{\infty}\frac{A_j(x)}{\lambda^j}$$

, where $\phi_k(x) = iw_k \sqrt[4]{p(x)}$, but in our problem $p(x) = 1$, so we can write

$$y_k(x,\lambda) = e^{i\lambda w_k x}\sum_{j=0}^{\infty}\frac{A_j(x)}{\lambda^j}$$

We want to find $y_k', y_k'', y_k''', y_k^{(4)}$ and putting in the differential equation (1).
Now

$$y_k(x,\lambda) = e^{i\lambda w_k x}[A_0(x) + \frac{A_1(x)}{\lambda} + \dots + \frac{A_n(x)}{\lambda^n} + O(\frac{1}{\lambda^{n+1}})] \tag{7}$$

$$y_k'(x,\lambda) = i\lambda w_k e^{i\lambda w_k x} * \Bigg[A_0(x) + \frac{1}{\lambda}(A_1(x) - iw_k^3 A_0'(x)) + \frac{1}{\lambda^2}(A_2(x) - iw_k^3 A_1'(x)) + \frac{1}{\lambda^3}(A_3(x) - iw_k^3 A_2'(x)) + \frac{1}{\lambda^4}(A_4(x) - iw_k^3 A_3'(x)) + \dots + \frac{1}{\lambda^n}(A_n(x) - iw_k^3 A_{n-1}'(x)) + O(\frac{1}{\lambda^{n+1}})\Bigg] \tag{8}$$

$$y_k''(x,\lambda) = (i\lambda w_k)^2 e^{i\lambda w_k x} * \Bigg[A_0(x) + \frac{1}{\lambda}(A_1(x) - 2iw_k^3 A_0'(x)) + \frac{1}{\lambda^2}(A_2(x) - 2iw_k^3 A_1'(x) - w_k^2 A_0''(x)) + \frac{1}{\lambda^3}(A_3(x) - 2iw_k^3 A_2'(x) - w_k^2 A_1''(x)) + \frac{1}{\lambda^4}(A_4(x) - 2iw_k^3 A_3'(x) - w_k^2 A_2''(x)) + \frac{1}{\lambda^5}(A_5(x) - 2iw_k^3 A_4'(x) - w_k^2 A_3''(x)) + \dots + \frac{1}{\lambda^n}(A_n(x) - 2iw_k^3 A_{n-1}'(x) - w_k^2 A_{n-2}''(x)) + O(\frac{1}{\lambda^{n+1}})\Bigg] \tag{9}$$

$$y_k'''(x,\lambda) = (i\lambda w_k)^3 e^{i\lambda w_k x} * \Bigg[A_0(x) + \frac{1}{\lambda}(A_1(x) - 3iw_k^3 A_0'(x)) + \frac{1}{\lambda^2}(A_2(x) - 3iw_k^3 A_1'(x) - 3w_k^2 A_0''(x)) + \frac{1}{\lambda^3}(A_3(x) - 3iw_k^3 A_2'(x) - 3w_k^2 A_1''(x) + iw_k A_0'''(x)) + \frac{1}{\lambda^4}(A_4(x) - 3iw_k^3 A_3'(x) - 3w_k^2 A_2''(x) + iw_k A_1'''(x)) + \frac{1}{\lambda^5}(A_5(x) - 3iw_k^3 A_4'(x) - 3w_k^2 A_3''(x) + iw_k A_2'''(x)) + \frac{1}{\lambda^6}(A_6(x) - 3iw_k^3 A_5'(x) - 3w_k^2 A_4''(x) + iw_k A_3'''(x)) + \dots + \frac{1}{\lambda^n}(A_n(x) - 3iw_k^3 A_{n-1}'(x) - 3w_k^2 A_{n-2}''(x) + iw_k A_{n-3}'''(x)) + O(\frac{1}{\lambda^{n+1}})\Bigg] \tag{10}$$

$$y_k^{(4)}(x,\lambda) = \lambda^4 e^{i\lambda w_k x}\Big[A_0(x) + \frac{1}{\lambda}(A_1(x) - 4iw_k^3 A_0'(x)) + \frac{1}{\lambda^2}(A_2(x) - 4iw_k^3 A_1'(x) - 6w_k^2 A_0''(x)) + \frac{1}{\lambda^3}(A_3(x) - 4iw_k^3 A_2'(x) - 6w_k^2 A_1''(x) + 4iw_k A_0'''(x)) + \frac{1}{\lambda^4}(A_4(x) - 4iw_k^3 A_3'(x) - 6w_k^2 A_2''(x) + 4iw_k A_1'''(x) + A_0^{(4)}(x)) + \frac{1}{\lambda^5}(A_5(x) - 4iw_k^3 A_4'(x) - 6w_k^2 A_3''(x) + 4iw_k A_2'''(x) + A_1^{(4)}(x)) + \frac{1}{\lambda^6}(A_6(x) - 4iw_k^3 A_5'(x) - 6w_k^2 A_4''(x) + 4iw_k A_3'''(x) + A_2^{(4)}(x)) + \ldots + \frac{1}{\lambda^n}(A_n(x) - 4iw_k^3 A_{n-1}'(x) - 6w_k^2 A_{n-2}''(x) + 4iw_k A_{n-3}'''(x) + A_{n-4}^{(4)}(x)) + O(\frac{1}{\lambda^{n+1}})\Big] \quad (11)$$

Putting $y_k, y_k^{(4)}$ in (1) , then we get:

$$\lambda^4 e^{i\lambda w_k x}\Big[\frac{1}{\lambda}(-4iw_k^3 A_0'(x)) + \frac{1}{\lambda^2}(-4iw_k^3 A_1'(x) - 6w_k^2 A_0''(x)) + \frac{1}{\lambda^3}(-4iw_k^3 A_2'(x) - 6w_k^2 A_1''(x) + 4iw_k A_0'''(x)) + \frac{1}{\lambda^4}(-4iw_k^3 A_3'(x) - 6w_k^2 A_2''(x) + 4iw_k A_1'''(x) + A_0^{(4)}(x) + q(x)A_0(x)) + \frac{1}{\lambda^5}(-4iw_k^3 A_4'(x) - 6w_k^2 A_3''(x) + 4iw_k A_2'''(x) + A_1^{(4)}(x) + q(x)A_1(x)) + \frac{1}{\lambda^6}(-4iw_k^3 A_5'(x) - 6w_k^2 A_4''(x) + 4iw_k A_3'''(x) + A_2^{(4)}(x) + q(x)A_2(x)) + \ldots + \frac{1}{\lambda^n}(-4iw_k^3 A_{n-1}'(x) - 6w_k^2 A_{n-2}''(x) + 4iw_k A_{n-3}'''(x) + A_{n-4}^{(4)}(x) + q(x)A_{n-4}(x)) + O(\frac{1}{\lambda^{n+1}})\Big] = 0$$

By equating the coefficients of the same power of $\frac{1}{\lambda}$,then We get the following relation:

$$A_{0,k}(x) = 1, \qquad A_{1,k}(x) = 0, \qquad A_{2,k}(x) = 0,$$

$$A_{3,k}(x) = -\frac{iw_k}{4}\int_0^x (q(t)A_{0,k}(t))dt,$$

$$A_{4,k}(x) = -\frac{iw_k}{4}\int_0^x (-6w_k^2 A_{3,k}''(t) + q(t)A_{1,k}(t))dt,$$

$$A_{5,k}(x) = -\frac{iw_k}{4}\int_0^x (-6w_k^2 A_{4,k}''(t) + 4iw_k A_{3,k}'''(t) + q(t)A_{2,k}(t))dt,$$

$$A_{6,k}(x) = -\frac{iw_k}{4}\int_0^x (-6w_k^2 A_{5,k}''(t) + 4iw_k A_{4,k}'''(t) + A_{3,k}^{(4)}(t) + q(t)A_{3,k}(t))dt,$$

$$A_{7,k}(x) = -\frac{iw_k}{4}\int_0^x (-6w_k^2 A_{6,k}''(t) + 4iw_k A_{5,k}'''(t) + A_{4,k}^{(4)}(t) + q(t)A_{4,k}(t))dt,$$

And hence for integer $n \geq 6$ we get that:

$$A_{n,k}(x) = -\frac{iw_k}{4}\int_0^x (-6w_k^2 A_{n-1,k}''(t) + 4iw_k A_{n-2,k}'''(t) + A_{n-3,k}^{(4)}(t) + q(t)A_{n-3,k}(t))dt.$$

By using the above recursion relations for A_{ik}.derivatives of the solution of the differential equation (1) have the following forms:

$$y_k'(x,\lambda) = i\lambda w_k e^{i\lambda w_k x} * \Big[A_{0,k}(x) + \frac{1}{\lambda}(A_{1,k}(x) - iw_k^3 A_{0,k}'(x)) + \frac{1}{\lambda^2}(A_{2,k}(x) - iw_k^3 A_{1,k}'(x)) + \frac{1}{\lambda^3}(A_{3,k}(x) - iw_k^3 A_{2,k}'(x)) + \frac{1}{\lambda^4}(A_{4,k}(x) - iw_k^3 A_{3,k}'(x)) + \ldots + \frac{1}{\lambda^n}(A_{n,k}(x) - iw_k^3 A_{n-1,k}'(x)) + O(\frac{1}{\lambda^{n+1}})\Big]$$

$$y_k''(x,\lambda) = (i\lambda w_k)^2 e^{i\lambda w_k x} * \Big[A_{0,k}(x) + \frac{1}{\lambda}(A_{1,k}(x) - 2iw_k^3 A_{0,k}'(x)) + \frac{1}{\lambda^2}(A_{2,k}(x) - 2iw_k^3 A_{1,k}'(x) - w_k^2 A_{0,k}''(x)) + \frac{1}{\lambda^3}(A_{3,k}(x) - 2iw_k^3 A_{2,k}'(x) - w_k^2 A_{1,k}''(x)) + \frac{1}{\lambda^4}(A_{4,k}(x) - 2iw_k^3 A_{3,k}'(x) - w_k^2 A_{2,k}''(x)) + \frac{1}{\lambda^5}(A_{5,k}(x) - 2iw_k^3 A_{4,k}'(x) - w_k^2 A_{3,k}''(x)) + \ldots + \frac{1}{\lambda^n}(A_{n,k}(x) - 2iw_k^3 A_{n-1,k}'(x) - w_k^2 A_{n-2,k}''(x)) + O(\frac{1}{\lambda^{n+1}})\Big]$$

$$
\begin{aligned}
y_k^{'''}(x,\lambda) &= (i\lambda w_k)^3 e^{i\lambda w_k x}\Big[A_{0,k}(x) \\
&+\frac{1}{\lambda}(A_{1,k}(x)-3iw_k^3A_{0,k}^{'}(x)) \\
&+\frac{1}{\lambda^2}(A_{2,k}(x)-3iw_k^3A_{1,k}^{'}(x)-3w_k^2A_{0,k}^{''}(x)) \\
&+\frac{1}{\lambda^3}(A_{3,k}(x)-3iw_k^3A_{2,k}^{'}(x)-3w_k^2A_{1,k}^{''}(x) \\
&+iw_kA_{0,k}^{'''}(x))+\frac{1}{\lambda^4}(A_{4,k}(x)-3iw_k^3A_{3,k}^{'}(x) \\
&-3w_k^2A_{2,k}^{''}(x)+iw_kA_{1,k}^{'''}(x))+\frac{1}{\lambda^5}(A_{5,k}(x) \\
&-3iw_k^3A_{4,k}^{'}(x)-3w_k^2A_{3,k}^{''}(x)+iw_kA_{2,k}^{'''}(x)) \\
&+\frac{1}{\lambda^6}(A_{6,k}(x)-3iw_k^3A_{5,k}^{'}(x)-3w_k^2A_{4,k}^{''}(x) \\
&+iw_kA_{3,k}^{'''}(x))+\ldots+\frac{1}{\lambda^n}(A_{n,k}(x)-3iw_k^3A_{n-1,k}^{'}(x) \\
&-3w_k^2A_{n-2,k}^{''}(x)+iw_kA_{n-3,k}^{'''}(x))+O(\frac{1}{\lambda^{n+1}})\Big]
\end{aligned}
$$

$$
\begin{aligned}
y_k^{(4)}(x,\lambda) &= \lambda^4 e^{i\lambda w_k x}\Big[A_{0,k}(x)+\frac{1}{\lambda}(A_{1,k}(x)-4iw_k^3A_{0,k}^{'}(x)) \\
&+\frac{1}{\lambda^2}(A_{2,k}(x)-4iw_k^3A_{1,k}^{'}(x)-6w_k^2A_{0,k}^{''}(x)) \\
&+\frac{1}{\lambda^3}(A_{3,k}(x)-4iw_k^3A_{2,k}^{'}(x)-6w_k^2A_{1,k}^{''}(x) \\
&+4iw_kA_{0,k}^{'''}(x))+\frac{1}{\lambda^4}(A_{4,k}(x)-4iw_k^3A_{3,k}^{'}(x) \\
&-6w_k^2A_{2,k}^{''}(x)+4iw_kA_{1,k}^{'''}(x)+A_{0,k}^{(4)}(x)) \\
&+\frac{1}{\lambda^5}(A_{5,k}(x)-4iw_k^3A_{4,k}^{'}(x)-6w_k^2A_{3,k}^{''}(x) \\
&+4iw_kA_{2,k}^{'''}(x)+A_{1,k}^{(4)}(x))+\frac{1}{\lambda^6}(A_{6,k}(x) \\
&-4iw_k^3A_{5,k}^{'}(x)-6w_k^2A_{4,k}^{''}(x)+4iw_kA_{3,k}^{'''}(x) \\
&+A_{2,k}^{(4)}(x))+\ldots+\frac{1}{\lambda^n}(A_{n,k}(x)-4iw_k^3A_{n-1,k}^{'}(x) \\
&-6w_k^2A_{n-2,k}^{''}(x)+4iw_kA_{n-3,k}^{'''}(x)+A_{n-4,k}^{(4)}(x)) \\
&+O(\frac{1}{\lambda^{n+1}})\Big]
\end{aligned}
$$

Since, $\binom{n}{k}=\frac{n!}{(n-k)!k!}$, then we get the result of the theorem.

4 Asymptotic behavior of the Eigen-values

In this section we try to find the Eigen-values of the problem (1)-(2)

Theorem 2. Consider the boundary value problem(1)-(2) , where $q(x)$ is smooth function, for which satisfy the conditions $q^{'}(a)=0, q^{'}(0)=0, \int_0^a q(x)dx=0$, and $q(a)\neq 0$, then for $\lambda\in T_{k'}$ or $\lambda\in \bar{T}_{k'}, k'=0:3$ then asymptotic of eigenvalues of the problem for sufficiently large $|m|$,has the following forms:

$$
\begin{aligned}
\hat{\lambda}_{0,m} = & (\frac{1}{2a})^4\Big((2m\pi+i)^4+112a^4q(a)[\frac{8}{m\pi}-\frac{12i}{(m\pi)^2} \\
& -\frac{6}{(m\pi)^3}+\frac{i}{(m\pi)^4}]\Big)+O(\frac{1}{m^6}), \qquad \lambda\in T_0,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\lambda}_{\bar{0},m} &= (\frac{1}{2a})^4\Big((2im\pi-33)^4 \\
&+4a^4(32iq(a)+24q(0))[-\frac{8i}{m\pi}+12\frac{33}{(m\pi)^2} \\
&+6i\frac{(33)^2}{(m\pi)^3}-\frac{(33)^3}{(m\pi)^4}]\Big)+O(\frac{1}{m^6}), \quad \lambda\in \bar{T}_0, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\lambda}_{1,m} &= (\frac{1}{2a})^4\Big((2im\pi-1)^4+32a^4iq(a)[\frac{1}{im\pi}-\frac{3}{2}\frac{1}{(mi\pi)^2} \\
&+\frac{3}{4}\frac{1}{(mi\pi)^3}-\frac{1}{2}\frac{1}{(mi?)^4}]\Big)+O(\frac{1}{m^6}), \quad \lambda\in T_1, \quad (13)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\lambda}_{\bar{1},m} &= (\frac{1}{2a})^4\Big((2im\pi-1)^4 \\
&-4(1+2i)a^4q(a)[\frac{(2)^3}{mi\pi}-\frac{3(2)^2}{(mi\pi)^2}+\frac{3(2)}{(mi\pi)^3} \\
&-\frac{1}{(mi\pi)^4}]\Big)+O(\frac{1}{m^6}), \quad \lambda\in \bar{T}_1, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\lambda}_{2,m} &= (\frac{1}{2a})^4\Big((2m\pi+(i-8))^4 \\
&+4a^4(2q(a)+3q(0))[\frac{8}{m\pi}+(i-8)\frac{12}{(m\pi)^2} \\
&+(i-8)^2\frac{6}{(m\pi)^3}+(i-8)^3\frac{1}{(m\pi)^4}]\Big) \\
&+O(\frac{1}{m^6}), \lambda\in T_2, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\lambda}_{\bar{2},m} &= (\frac{1}{2a})^4\Big((2im\pi+1)^4-(2+i)4a^4q(a)[-\frac{8i}{m\pi} \\
&-\frac{6}{(m\pi)^2}+\frac{6i}{(m\pi)^3}+\frac{1}{(m\pi)^4}]\Big) \\
&+O(\frac{1}{m^6}), \qquad \lambda\in \bar{T}_2, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\lambda}_{3,m} &= (\frac{1}{2a})^4\Big((2im\pi+33)^4 \\
&+4a^4(i32q(a)-24q(0))[\frac{8}{im\pi}+33\frac{12}{(im\pi)^2} \\
&+(33)^2\frac{6}{(im\pi)^3}+(33)^3\frac{1}{(im\pi)^4}]\Big) \\
&+O(\frac{1}{m^6}), \qquad \lambda\in T_3, \qquad (17)
\end{aligned}
$$

$$\hat{\lambda}_{\bar{3},m} = (\frac{1}{2a})^4\Big((2im\pi+(2+8i))^4 - i(2q(a)+3q(0))4a^4[\frac{8}{im\pi}+(2+8i)\frac{12}{(im\pi)^2} + (2+8i)^2\frac{6}{(im\pi)^3}+(2+8i)^3\frac{1}{(im\pi)^4}]\Big) + O(\frac{1}{m^6}), \qquad \lambda\in\overline{T}_3. \tag{18}$$

for $m = N, N+1, N+2, \dots$ Where N is a large integer.

Proof. If we choose five terms of $y_k^{(s)}(x,\lambda)$ in Theorem 1 then:

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k')^s e^{i\lambda w_k' x}\Big[A_{0sk}(x)+\frac{A_{1sk}(x)}{\lambda}+\frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3}+\frac{A_{4sk}(x)}{\lambda^4}+\frac{A_{5sk}(x)}{\lambda^5}+O(\frac{1}{\lambda^6})\Big] \tag{19}$$

For, $s = 1,2,3, k = 0,1,2,3$. We have:

$$A_{0sk} = 1, \quad A_{1sk} = A_{2sk} = 0,$$

$$A_{3sk} = -\frac{iw_k'}{4}\int_0^x q(t)dt,$$

$$A_{40k} = \frac{3}{8}\Big[q(x)-q(0)\Big],$$

$$A_{41k} = \Big[\frac{1}{8}q(x)-\frac{3}{8}q(0)\Big],$$

$$A_{42k} = \Big[-\frac{1}{8}q(x)-\frac{3}{8}q(0)\Big],$$

$$A_{43k} = \Big[-\frac{3}{8}q(x)-\frac{3}{8}q(0)\Big],$$

$$A_{50k} = \frac{5i(w_k')^3}{16}\Big[q'(x)-q'(0)\Big],$$

$$A_{51k} = \Big[-\frac{i}{16}(w_k')^3q'(x)-\frac{5i}{16}(w_k')^3q'(0)\Big],$$

$$A_{52k} = \Big[-\frac{3i}{16}(w_k')^3(q'(x))-\frac{5i}{16}(w_k')^3q'(0)\Big],$$

$$A_{53k} = \Big[-\frac{i}{16}(w_k')^3(q'(x))-\frac{5i}{16}(w_k')^3q'(0)\Big].$$

Now, to find the boundary conditions $U_j(y_k)$ for $k, j = 0,1,2,3$. Where $U_0(y) = y(0) = 0, U_1(y) = y'(0) = 0, U_j(y) = \sum_{l=1}^4 (iw_j\lambda)^{l-1}y^{(4-l)}(a,\lambda) = 0$ $j = 2,3$, where, $w_k = \sqrt[4]{1} = e^{\frac{2\pi k}{4}i} = e^{\frac{\pi k}{2}i}, k = 0,1,2,3.$and $q(x)$is smooth function. we know that $w_0 = -w_2 = 1$, $w_1 = -w_3 = i$. w_j' are the w_j which numbering so that satisfy (3) We can easily find out the form of each boundary conditions up to order six in each sectors:

$$U_0(y_k) = \Big[1+O(\frac{1}{\lambda^6})\Big] \tag{20}$$

$$U_1(y_k) = i\lambda w_k'\Big[1-\frac{1}{4}\frac{q(0)}{\lambda^4}-\frac{i(w_k')^3\frac{3}{8}q'(0)+\frac{1}{4}q(0)}{\lambda^5} + O(\frac{1}{\lambda^6})\Big] \tag{21}$$

$$\begin{aligned} U_j(y_k) = &-i\lambda^3 e^{i\lambda w_k a}\Big[(w_k')^3\Big(1-\frac{iw_k'}{4}\frac{\int_0^a q(t)dt}{\lambda^3} + \frac{[-\frac{3}{8}q(a)-\frac{3}{8}q(0)]}{\lambda^4} + \frac{[-\frac{i}{16}(w_k')^3(q'(a))-\frac{5i}{16}(w_k')^3q'(0)]}{\lambda^5}+O(\frac{1}{\lambda^6})\Big) \\ &+(w_j)(w_k')^2\Big(1+\frac{-\frac{iw_k'}{4}\int_0^a q(t)dt}{\lambda^3} + \frac{[-\frac{1}{8}q(a)-\frac{3}{8}q(0)]}{\lambda^4} + \frac{[-\frac{3i}{16}(w_k')^3(q'(a))-\frac{5i}{16}(w_k')^3q'(0)]}{\lambda^5} \\ &+O(\frac{1}{\lambda^6})\Big)+(w_j)^2(w_k')\Big(1+\frac{-iw_k'}{4}\frac{\int_0^a q(t)dt}{\lambda^3} + \frac{[\frac{1}{8}q(a)-\frac{3}{8}q(0)]}{\lambda^4} + \frac{[-\frac{i}{16}(w_k')^3q'(a)-\frac{5i}{16}(w_k')^3q'(0)]}{\lambda^5}+O(\frac{1}{\lambda^6})\Big) \\ &+(w_j)^3\Big(1-\frac{iw_k'}{4}\frac{\int_0^a q(t)dt}{\lambda^3}+\frac{\frac{3}{8}[q(a)-q(0)]}{\lambda^4} + \frac{5i(w_k')^3}{16}\frac{[q'(a)-q'(0)]}{\lambda^5}+O(\frac{1}{\lambda^6})\Big)\Big] \end{aligned} \tag{22}$$

For $j = 2,3$.
If $\lambda\in T_0$, then
$w_0' = i, w_1' = w_0 = 1, w_2' = -w_0 = -1, w_3' = -i.$
$q'(a) = 0$ and $q'(0) = 0$ and $\int_0^a q(x)dx = 0$, then after along computation from equations (21 and 22) we get:

$$U_0(y_k) = A, \quad U_1(y_0) = -\lambda B, \quad U_1(y_1) = i\lambda B, \quad U_1(y_2) = -i\lambda B, \quad U_1(y_3) = \lambda B \tag{23}$$

$$\begin{aligned} &U_2(y_0) = (i+1)\lambda^3 e^{i\lambda w_0' a}C, \qquad U_2(y_1) = i\lambda^3 e^{i\lambda w_1' a}C, \\ &U_2(y_2) = i\lambda^3 e^{i\lambda w_2' a}D, U_2(y_3) = (i-1)\lambda^3 e^{i\lambda w_3' a}C, \\ &U_3(y_0) = \lambda^3 e^{iw_0' a}C, \qquad U_3(y_1) = (i+1)\lambda^3 e^{i\lambda w_1' a}C, \\ &U_3(y_2) = (1-i)\lambda^3 e^{i\lambda w_2' a}C, \quad U_3(y_3) = \lambda^3 e^{i\lambda w_3' a}D. \end{aligned} \tag{24}$$

Where,

$$A = \Big[1+O(\frac{1}{\lambda^6})\Big], B = \Big[1-\frac{1}{4}\frac{q(0)}{\lambda^4}-\frac{1}{4}\frac{q(0)}{\lambda^5}+O(\frac{1}{\lambda^6})\Big],$$

$$C = \Big[-\frac{1}{2}\frac{q(a)}{\lambda^4}+O(\frac{1}{\lambda^6})\Big], D = \Big[4-\frac{3}{2}\frac{q(0)}{\lambda^4}+O(\frac{1}{\lambda^6})\Big]$$

we form the determinant $\Delta(\lambda) = det[U_j(y_k)]$as was proved in [9],the eigenvalues of the problem (1)-(2) are the zeros

of $\Delta(\lambda)$.
So we will find $\Delta(\lambda)$, in T_0: Since $\Delta(\lambda) = det[U_j(y_k)]$, for $k, j = 0:3$,then

$$\Delta(\lambda) = \begin{vmatrix} U_0(y_0) & U_0(y_1) & U_0(y_2) & U_0(y_3) \\ U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix}$$

Substituting the expressions of (23)-(24) in $\Delta(\lambda)$ Then by Laplace expansion theorem for determinant as we see in [9], [10] and Lemma 1 we can reduce $\Delta(\lambda)$ to

$$\Delta(\lambda) = \lambda^7 ABe^{i\lambda w_3' a}\Big\{(1+i)\begin{vmatrix} iD & (i-1)C \\ (1-i)C & D \end{vmatrix} e^{i\lambda w_2' a} + (1-i)\begin{vmatrix} iC & (i-1)C \\ (1+i)C & D \end{vmatrix} e^{i\lambda w_1' a} + O(e^{-|\lambda|\sin\frac{\pi}{4}})\Big\} \quad (25)$$

Calculating equation (25)leads the following form for $\Delta(\lambda)$

$$\Delta(\lambda) = -(1-i)\lambda^7 ABe^{i\lambda w_3' a}e^{i\lambda w_1' a}\Big\{[DD-2CC]e^{i\lambda w_2' a}e^{i\lambda w_1' a} + [iCD+2CC] + O(e^{-|\lambda|\sin\frac{\pi}{4}})\Big\} \quad (26)$$

From (26) it is clear that $\Delta(\lambda) = 0$ for sufficiently large $|\lambda|$ if and only if

$$[DD-2CC]e^{i\lambda w_2' a}e^{i\lambda w_1' a} + [iCD+2CC] = 0 \quad (27)$$

We can easily obtain that
$[DD-2CC]^{-1} = -14 + 12\frac{q(0)}{\lambda^4} + O(\frac{1}{\lambda^6})$.
And
$[iCD+2CC] = 2i\frac{q(a)}{\lambda^4} + O(\frac{1}{\lambda^6})$.
So from (27) we find out:

$$e^{i\lambda(w_2'-w_1')a} + [DD-2CC]^{-1}[iCD+2CC] = 0$$

Then,

$$e^{i\lambda(w_2'-w_1')a} = [28i\frac{q(a)}{\lambda^4} + O(\frac{1}{\lambda^6})]$$

And since in T_0 we have $w_2' = -w_0, w_1' = w_0$, then

$$e^{-2i\lambda a)} - 1 = [-1 + 28i\frac{q(a)}{\lambda^4} + O(\frac{1}{\lambda^6})]$$

Then according to [3],[13], [14] and [11] by using Rouche's theorem we can solve it and we get:

$$\lambda_{0,m} = -\frac{1}{2ai}[2m\pi - 1 + 28i\frac{q(a)}{(\frac{1}{a}m\pi)^4} + O(\frac{1}{m^6})],$$

For, $m = N, N+1, N+2, \ldots$ Where N is a large integer. And we know that the eigen-values of the problem are $\hat{\lambda}_{0,m} = (\lambda_{0,m})^4$,thus we obtained (12). By the same way we can find the eigen-values in all other sectors.

5 Asymptotic Formulas for the Eigen-Functions

In this section we find an expression for the eigen functions of the boundary value problem (1)-(2) in each sectors T_k and $\bar{T_k}$ that we defined in section 2.

Theorem 3. Asymptotic behavior of the Eigen-function for the boundary value problem corresponding to $\lambda_{j,m}, \lambda_{\bar{j},m}$, for $j = 0:3$ has the form:

$$y_{k,m}(x,\lambda_{k,m}) = ie^{\bar{w_k}\frac{1}{a}m\pi x} + e^{-i\bar{w_k}\frac{1}{a}m\pi x} + O(\frac{1}{m^3}), \qquad \lambda \in T_k. \quad (28)$$

$$y_{\bar{k},m}(x,\lambda_{k,m}) = ie^{-w_k\frac{1}{a}m\pi x} + e^{iw_k\frac{1}{a}m\pi x} + O(\frac{1}{m^3}), \qquad \lambda \in \bar{T_k}. \quad (29)$$

$k, j = 0:3$, for $m = N, N+1, N+2, \ldots$

Where N is a large integer.

Proof. If we choose three terms of $y_k^{(s)}(x,\lambda)$ in Theorem 1 then:

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k')^s e^{i\lambda w_k' x)} \Big[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + O(\frac{1}{\lambda^3})\Big]. \quad (30)$$

For $s = 0,1,2,3, k = 0,1,2,3$, We have:

$$A_{0sk} = 1, A_{1sk} = 0, A_{2sk} = 0,$$

And to finding the boundary conditions $U_j(y_k)$ for $k = 0,1,2,3, j = 1,2,3$ up to order $O(\frac{1}{\lambda^3})$ and $q(x)$ satisfies $\int_0^a q(t)dt = 0$, and If $\lambda \in T_0$, then

$$w_0' = i, w_1' = w_0 = 1, w_2' = -w_0 = -1, w_3' = -i.$$

Now we can easily find

$$\begin{aligned} &U_1(y_0) = -\lambda[1+O(\frac{1}{\lambda^3})], U_1(y_1) = i\lambda[1+O(\frac{1}{\lambda^3})], \\ &U_1(y_2) = -i\lambda[1+O(\frac{1}{\lambda^3})], U_1(y_3) = \lambda[1+O(\frac{1}{\lambda^3})], \\ &U_2(y_0) = 0, \quad U_2(y_1) = 0, \\ &U_2(y_2) = 4i\lambda^3 e^{-i\lambda a}[1+O(\frac{1}{\lambda^3})], U_2(y_3) = 0, \\ &U_3(y_0) = 0, \quad U_3(y_1) = 0, \quad U_3(y_2) = 0, \\ &U_3(y_3) = 4\lambda^3 e^{\lambda a}[1+O(\frac{1}{\lambda^3})]. \end{aligned} \quad (31)$$

According to [12] , [14] we can write the Eigen-function in T_0 as follows:

$$y_{0,m}(x,\lambda) = \frac{1}{16i\lambda^7} e^{(i-1)\lambda a} * \begin{vmatrix} y_0(x,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) \\ U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix}. \quad (32)$$

Substituting (31) in (32), we get

$$y_{0,m}(x,\lambda) = \frac{1}{16i\lambda^7} e^{(i-1)\lambda a} 16i\lambda^7 e^{-i\lambda a} e^{\lambda a} * \left\{ \begin{vmatrix} e^{-\lambda x} & e^{i\lambda x} & e^{-i\lambda x} & e^{\lambda x} \\ -1 & i & -i & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + O(\frac{1}{\lambda^3}) \right\}.$$

Calculating this determinant leads the following form for the eigen function

$$y_{0,m}(x,\lambda) = \left\{ \begin{vmatrix} e^{-\lambda x} & e^{i\lambda x} \\ -1 & i \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + O(\frac{1}{\lambda^3}) \right\}.$$

so we get that

$$y_{0,m}(x,\lambda) = \left[ie^{-\lambda x} + e^{i\lambda x} + O(\frac{1}{\lambda^3}) \right].$$

Since in $T_0, \lambda = \lambda_{0,m}$,
where $\lambda_{0,m} = -\frac{1}{2a}\left[2m\pi + i + 28\frac{q(a)}{(\frac{1}{a}m\pi)^4}\right] + O(\frac{1}{m^3})$.
so

$$y_{0,m}(x,\lambda_{0,m}) = [ie^{x\frac{1}{a}m\pi} + e^{-ix(\frac{1}{a}m\pi)} + O(\frac{1}{m^3})].$$

Then by Lemma 1 we get :

$$y_{0,m}(x,\lambda_{0,m}) = ie^{\bar{w}_k \frac{1}{a} m\pi x} + e^{-i\bar{w}_k \frac{1}{a} m\pi x} + O(\frac{1}{m^3}), \quad \lambda \in T_0.$$

for $m = N, N+1, N+2, \ldots$ Where N is a large integer.
By the same way we can find the Eigen functions in all other sectors.

6 Conclusion

As a conclusion we find the asymptotic expression for the fundamental solution of the differential equation (1), and also we find the asymptotic formulas for the Eigen values and Eigen functions of the boundary value problem (1)-(2) under a certain conditions.

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