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An Elementary Proof of the Vanishing of Formal Local Cohomology Modules

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Abstract: In this paper, we presented a proof of a theorem of vanishing for the formal local cohomology module. Also is put its connection with local cohomology modules.

Keywords: Inverse limit, Direct limit, Local cohomology, Formal local cohomology, Commutative algebra theory

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity. The theory of local cohomology if has developed so much of six decades after of its introduction by Grothendieck. The local cohomology theory of Grothendieck has proved to be an important tool in commutative algebra and algebraic geometry. There exists a relation between local cohomology, given by [2], and formal local cohomology, given by [6]. Not so much is known about these modules. The motivation of this work is precisely the formal local cohomology of P. Schenzel. Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. For an R-module M, let $\mathrm{H}^i_{\mathfrak{a}}(M)$, for $0 \leq i \in \mathbb{Z}$, denote the local cohomology module of M with respect to $\mathfrak{a}$ (cf. [2] for the basic definitions). There are the following integers related to these local cohomology modules, such as

$$\mathrm{grade}\,(\mathfrak{a},M) = \inf\left\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\right\} \text{ and,}$$

$$\mathrm{cd}\,(\mathfrak{a},M) = \sup\left\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\right\},$$

called the grade, respectively the cohomological dimension, of M with respect to $\mathfrak{a}$. In general we have the bounds:

$$\mathrm{height}_M(\mathfrak{a}) \leq \mathrm{cd}\,(\mathfrak{a},M) \leq \dim(M).$$

In the case of $\mathfrak{m}$ the maximal ideal it follows that

$$\mathrm{grade}\,(\mathfrak{m},M) = \mathrm{depth}\,(M) \text{ and,}$$

$$\mathrm{cd}\,(\mathfrak{m},M) = \dim(M).$$

From the local cohomology we obtain the formal local cohomology, and the purpose here is a systematic study of the formal local cohomology modules. Here we consider the asymptotic behavior of the family of local cohomology modules given by $\left\{\mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right)\right\}_{n\in\mathbb{N}}$ for an integer $i \in \mathbb{Z}$. By the natural homomorphisms these families form a projective system. Their projective limit given by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right)$$

is called the i-th formal local cohomology module of M with respect to $\mathfrak{a}$, or with respect to the ideal $\mathfrak{m}$.

The purpose of this note is to establish a kind of vanishing theorem for the formal local cohomology module. We observe that the formal local cohomology module is used for to characterize some values. For example, we have the result: “Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. Then

$$\dim\left(\frac{M}{\mathfrak{a}M}\right) = \sup\left\{i \in \mathbb{Z} \mid \varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{M}{\mathfrak{a}^n M}\right) \neq 0\right\},$$

for a finitely generated R-module M.”

For see properties of commutative algebra and homological algebra we refer the reader to the books [3] and [5].

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2 Some preliminaries and the definition of formal local cohomology module

Let I be an ideal of R, and let M be an R-module. In [3, Definition 3.5.2], the local cohomology functors, denoted by $\mathrm{H}_I^i(\bullet)$, are the right derived functors of $\Gamma_I(\bullet)$ (see definition of $\Gamma_I(\bullet)$ in [3, section 3.5]). In other words, if $\mathbf{I}^\bullet$ is an injective resolution of the R-module M, then $\mathrm{H}_I^i(M) \cong \mathrm{H}^i(\Gamma_I(\mathbf{I}_M^\bullet))$ for all $0 \leq i \in \mathbb{Z}$, where $\mathbf{I}_M^\bullet$ denotes the deleted injective resolution of M (for more details about injective resolution, see [5]).

The following two remarks are important for the theory of the local cohomology, which is presented in this paper.

Remark([3][Remark 3.5.3(a)**])** Let M be an R-module. Then we have that $\mathrm{H}_I^0(M) \cong \Gamma_I(M)$; and moreover, we have that $\mathrm{H}_I^i(M) = 0$, for all $i < 0$.

Remark([3][Remark 3.5.3(c)**])** For any R-module M and for all $i \geq 0$, we have that

$$\mathrm{H}_I^i(M) \cong \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}_R^i\left(R/I^t, M\right).$$

The following definition will be used in the next section.

Definition If $\mathfrak{p}$ is a prime ideal of R, then the *coheight* of $\mathfrak{p}$, written $\mathrm{coht}(\mathfrak{p})$, is defined to be the Krull dimension of the ring $R/\mathfrak{p}$.

Now, for $\mathfrak{a}$ other ideal of R, consider the family of local cohomology modules given by $\left\{\mathrm{H}_I^i(M/\mathfrak{a}^nM)\right\}_{n\in\mathbb{N}}$. According to [6], for every $n \in \mathbb{N}$, there exists a natural homomorphism:

$$\phi_{n+1,n} : \mathrm{H}_I^i\left(M/\mathfrak{a}^{n+1}M\right) \to \mathrm{H}_I^i(M/\mathfrak{a}^nM).$$

Now, for $(R,\mathfrak{m})$ a local ring these families, for $I = \mathfrak{m}$, form an inverse system of R-modules. Their inverse limit that is given by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}_\mathfrak{m}^i(M/\mathfrak{a}^nM)$$

is called, according to [6], the i-th formal local cohomology module of M with respect to the ideal $\mathfrak{a}$ (or with respect to the ideal $\mathfrak{m}$), and will be denoted by $\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^i(M)$, for all $0 \leq i \in \mathbb{Z}$.

3 A connection between the formal local cohomology module and Krull dimension

Our result of vanishing is the theorem following.

Theorem Let $(R,\mathfrak{m})$ be a local Noetherian ring and suppose that M is an R-module of finite Krull dimension and non-zero. And moreover, suppose that the R-module $M/\mathfrak{a}^nM$ also have finite Krull dimension, we say that $\dim(M/\mathfrak{a}^nM) = s$, for all $n \in \mathbb{N}$. Then we have that,

$$\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^i(M) = 0 \text{ whenever } i > s.$$

*Proof.*Let's to prove firstly that $\mathrm{H}_\mathfrak{m}^i(M/\mathfrak{a}^nM) = 0$, whenever $i > s$ and for all $n \in \mathbb{N}$. This will be by induction on s. Suppose, firstly, that $s = 0$. Then, since any R-module is a direct limit of its finitely generated submodules (see [4, 4.9 of Chapter 8]), it follows that it is sufficient to prove the result when $M/\mathfrak{a}^nM$ is a non-zero, finitely generated R-module, for all $n \in \mathbb{N}$.

In fact, note that $\tilde{M} = M/\mathfrak{a}^nM$, for all $n \in \mathbb{N}$, is a non-zero R-module (and will be finitely generated due to the exposed above), since if we have $\tilde{M} = 0$ then $M = \mathfrak{a}^nM$ implying, by the Nakayama's lemma, that $M = 0$; contradiction, since, by hypothesis, we have that $M \neq 0$.

By [7, Corollary 4 I – 19], there exists a finite chain of submodules of $\tilde{M}$:

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_r = \tilde{M},$$

such that for each $i = 1, \ldots, r$, we have $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Note that, each of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \mathrm{Supp}_R\left(\tilde{M}\right)$, so, since $\dim_R\left(\tilde{M}\right) = 0$, each of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ must be maximal.

Now, by inductive argument we have that it is sufficient to prove that $\mathrm{H}_\mathfrak{m}^i(R/\mathfrak{m}) = 0$ whenever $i > 0$. Consider a minimal injective resolution for $R/\mathfrak{m}$:

$$0 \to R/\mathfrak{m} \to E^0(R/\mathfrak{m}) \to E^1(R/\mathfrak{m}) \to \ldots \to E^m(R/\mathfrak{m}) \to \ldots.$$

By [1, Lemma 2.7], if $\mathfrak{p}$ is a prime ideal of R such that $\mathfrak{p} \subset \mathfrak{m}$ properly, then $\mu^i(\mathfrak{p}, R/\mathfrak{m}) = 0$, for all $i \geq 0$, while $\mu^i(\mathfrak{m}, R/\mathfrak{m})$ is finite for all $i \geq 0$. Thus, for each $i \geq 0$, we have $E^i(R/\mathfrak{m}) \cong \bigoplus_i \mu^i(\mathfrak{m}, R/\mathfrak{m}) E(R/\mathfrak{m})$.

Thus, as every element of $E(R/\mathfrak{m})$ is annihilated by some power of $\mathfrak{m}$, it follows that $\Gamma_\mathfrak{m}\left(E^i(R/\mathfrak{m})\right) = E^i(R/\mathfrak{m})$ for each $i \geq 0$. Hence, we have that $\mathrm{H}_\mathfrak{m}^i(R/\mathfrak{m}) = 0$ for all $i > 0$. Therefore, $\mathrm{H}_\mathfrak{m}^i(M/\mathfrak{a}^nM) = 0$, for all $n \in \mathbb{N}$ and for all $i > 0$. Then, it follows that:

$$\mathfrak{F}_{\mathfrak{a},\mathfrak{m}}^i(M) := \varprojlim_{n\in\mathbb{N}} \mathrm{H}_\mathfrak{m}^i(M/\mathfrak{a}^nM) = 0,$$

for all $i > 0$. This concludes the proof for the case $s = 0$.

Now, suppose that $s = \dim(M/\mathfrak{a}^nM) > 0$, for all $n \in \mathbb{N}$, with s finite.

Suppose, inductively, that $k > 0$ and that whenever N is an R-module with $\dim_R(N) = s < k$, then $\mathrm{H}_\mathfrak{m}^i(N) = 0$ for all $i > s$. Suppose that $\tilde{M} = M/\mathfrak{a}^nM$ is such that its Krull dimension is k. Since any submodule of $\tilde{M}$ will have Krull dimension $\leq k$ and any R-module is a direct limit of its finitely generated submodules, it follows that it is sufficient to assume that $\tilde{M}$ is an non-zero R-module and finitely generated. By [7, Corollary 4 I – 19], there exists a finite chain of submodules of $\tilde{M}$:

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_r = \tilde{M},$$

such that, for each $i = 1, \ldots, r$, we have $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Note that we have

$\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \mathrm{Supp}_R(\tilde{M})$, and so it follows that it is sufficient to prove that, if $\mathfrak{p}$ is a prime ideal of R of coheight k (see Definition 2), then $\mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p}) = 0$ for all $i > k$. (The inductive hypothesis is used here, of course).

Let $K = (R/\mathfrak{p})_{\mathfrak{p}}$ regarded as an R-module in the natural way. Let then

$$\alpha : R/\mathfrak{p} \to (R/\mathfrak{p})_{\mathfrak{p}},$$

be the natural R-homomorphism: α is injective. Let $H = \mathrm{Coker}(\alpha)$, so that

$$0 \to R/\mathfrak{p} \xrightarrow{\alpha} K \to H \to 0,$$

is an exact sequence.

Note that $\mathrm{H}^i_{\mathfrak{m}}(K) = 0$, for all $i > 0$. Indeed, there exists $x \in \mathfrak{m} \setminus \mathfrak{p}$, and the multiplication by x on K provides an automorphism of K. Hence, if $i \geq 0$, since $\mathrm{H}^i_{\mathfrak{m}}(\bullet)$ is an R-linear functor, multiplication by x provides an automorphism of $\mathrm{H}^i_{\mathfrak{m}}(K)$. But if $z \in \mathrm{H}^i_{\mathfrak{m}}(K)$, then (from the construction of $\mathrm{H}^i_{\mathfrak{m}}(K)$ by means of applying the functor $\Gamma_{\mathfrak{m}}(\bullet)$ to an injective resolution of K) it follows that there exists an integer $j > 0$ such that $\mathfrak{m}^j z = 0$. Thus, $x^j z = 0$, and so $z = 0$. Therefore, we have that $\mathrm{H}^i_{\mathfrak{m}}(K) = 0$ for all $i \geq 0$.

Therefore, we conclude that, the formal local cohomology module

$$\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M) = 0,$$

for all $i > \dim_R(M/\mathfrak{a}^n M)$, for all $n \in \mathbb{N}$, as required.

4 Conclusion

According to what was elaborated in the work, we observed that we can to find applications for the formal local cohomology module, such as the result of vanishing that was presented in this paper. For future works, we intend to elaborate results that determine under which assumptions the formal local cohomology modules are Artinian and Noetherian modules.

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