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Different Method Estimations of the Three Parameters of Exponentiated Gompertz Distribution

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Abstract: This article discusses the maximum likelihood, least squares, weighted least squares, and percentiles estimates of three parameters of an exponentiated Gompertz distribution based on complete sample. By using the mean square error through Monte Carlo simulation, this study compares the performances of estimates. Real data set is used as an example of the methods of estimations for the three parameters of exponentiated Gompertz distribution.

Keywords: Exponentiated Gompertz distribution; maximum likelihood estimators; Least squares estimators; weighted least squares estimators; percentiles estimators; mean square error; Monte Carlo simulation.

1 Introduction

In analyzing lifetime data, one often uses the exponential, Gompertz, Weibull, and generalized exponential (GE) distributions. One interesting aim of statistics is to search for distributions with certain properties that facilitate descriptions of some devices' lifetimes. Properties of the exponentiated Weibull (EW) family are discussed in [1,2,3,4]. Classical estimators of parameters of the EW distribution are discussed in [5,6,7,8]. Different estimators of the parameters of the exponentiated gamma distribution are considered in [9] and their performances are compared through Monte Carlo simulations. The different methods used for estimating the parameters of the exponentiated Pareto distribution are discussed in [10,11].

The exponentiated Gompertz (EGpz) distribution which may have bathtub shaped HF has generalized many well-known distributions including the traditional Gompertz distribution. This distribution function is

$$G(t) = \left(1 - e^{-\alpha\left(e^{\beta t}-1\right)}\right)^{\theta}. \tag{1}$$

Let $T \sim EGpz(\theta, \alpha, \beta)$ to denote the random variable T which follows an exponentiated Gompertz distribution with three parameters: β (scale parameter) and θ and α (shape parameters). Therefore, the probability density function is

$$g(t) = \theta\alpha\beta e^{\beta t} e^{-\alpha\left(e^{\beta t}-1\right)} \left(1 - e^{-\alpha\left(e^{\beta t}-1\right)}\right)^{\theta-1}. \tag{2}$$

Different properties of this distribution have been discussed by [12,13] when $\alpha = \delta/\beta$. In addition, [14] has discussed the maximum likelihood estimator and other methods estimator of the shape parameter θ of $EGpz$ distribution. In addition, [15] derived the Bayes estimates of shape parameter θ of EGpz distribution. Moreover, [16] discussed goodness-of-fit tests for the three parameters $EGpz$ distribution based on different types of samples, including complete and type II censored sampling.

The main aims of this paper are to study the different estimators of the unknown three-parameter of an EGpz distribution and to examine how the different estimators behave for different sample sizes. In addition, the maximum likelihood (ML) estimator is compared with percentiles (PC), least squares (LS), and weighted least squares (WLS) estimators based on complete sample.

The remaining sections are organized as follows. In Section 2, the ML estimator is discussed. Sections 3 to 5 present other methods. In Section 6, the performances of

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the estimates are compared by using the mean squared errors (MSEs) through Monte Carlo (MC) simulation study based on different sample sizes. Concluding remarks are presented in Section 7. Finally, Section 8 uses a real data set as an example to find the different estimators for the three parameters of $EGpz$ distribution.

2 Maximum Likelihood Estimators

Maximum likelihood (ML) estimation is a very popular technique used for estimating the parameters of continuous distributions. In this section the ML estimators of $EGpz(\theta,\alpha,\beta)$ are considered. Let T_1, T_2, $\ldots$, T_m is a random sample from $EGpz(\theta,\alpha,\beta)$, when the θ, α and β are unknown, then the likelihood function, $\ell(\theta,\alpha,\beta|\underline{t})$, is

$$\ell(\theta,\alpha,\beta|\underline{t}) = (\theta\alpha\beta)^m e^{\left\{-\alpha\sum_{i=1}^{m}\left(e^{\beta t_i}-1\right)\right\}} \times e^{\exp\left\{\beta\sum_{i=1}^{m}t_i\right\}} \prod_{i=1}^{m}\left(1-e^{-\alpha\left(e^{\beta t_i}-1\right)}\right)^{\theta-1}, \tag{3}$$

and the log-likelihood function, $L(\theta,\alpha,\beta|\underline{t})$, is

$$L(\theta,\alpha,\beta|\underline{t}) = m\log(\theta\alpha\beta) + \beta\sum_{i=1}^{m}t_i - \alpha\sum_{i=1}^{m}\left(e^{\beta t_i}-1\right) + (\theta-1)\sum_{i=1}^{m}\log\left(1-e^{-\alpha\left(e^{\beta t_i}-1\right)}\right). \tag{4}$$

The ML estimators of parameters θ, α, and β, says $\hat{\theta}_{\mathrm{ML}}$,$\hat{\alpha}_{\mathrm{ML}}$, and $\hat{\beta}_{\mathrm{ML}}$, can be found by setting the first partialy derivatives of the-loglikelihood to zero with respect to θ,α, and β respectily. The next non-linear equations can be solved by using iterative procedure:

$$\frac{m}{\theta} + \sum_{i=1}^{m}\log\left(1-e^{-\alpha\left(e^{\beta t_i}-1\right)}\right) = 0, \tag{5}$$

$$\frac{m}{\alpha} + \sum_{i=1}^{m}\left(e^{\beta t_i}-1\right)\left[(\theta-1)\frac{e^{-\alpha\left(e^{\beta t_i}-1\right)}}{1-e^{-\alpha\left(e^{\beta t_i}-1\right)}} - 1\right] = 0, \tag{6}$$

and

$$\frac{m}{\beta} + \sum_{i=1}^{m}t_i + \sum_{i=1}^{m}t_i e^{\beta t_i}\left[(\theta-1)\frac{e^{-\alpha\left(e^{\beta t_i}-1\right)}}{1-e^{-\alpha\left(e^{\beta t_i}-1\right)}} - 1\right] = 0. \tag{7}$$

The equations (5), (6), and (7) are nonlinear and do not have closed form; therefore, a numerical technique is required.

3 Least Squares Estimators

The method of least squares is often used to generate estimators and other statistics in regression analysis, as proposed by [17]. A random sample of size m is $T_1, \ldots, T_m$ from a distribution function $F(.)$ and $T_{(1)} < T_{(2)}, \ldots < T_{(m)}$ denotes the order statistics of the observed sample. This technique uses the distribution of $F(T_{(k)})$. For m sample size,

$$E\left(F\left(T_{(k)}\right)\right) = \frac{k}{m+1},$$

$$V\left(F\left(T_{(k)}\right)\right) = \frac{k}{(m+1)^2(m+2)},$$

and

$$Cov\left(F\left(T_{(k)}\right),F\left(T_{(j)}\right)\right) = \frac{k(m-j+1)}{(m+1)^2(m+2)};\ \text{ for } k<j.$$

For more specifics, see [18]. The estimators are obtained by minimizing

$$\sum_{k=1}^{m}\left(F\left(T_{(k)}\right) - \frac{k}{m+1}\right)^2,$$

with respect to the unknown parameters. Therefore, for $EGpz$ distribution the LS estimator of θ,α and β, say $\hat{\theta}_{\mathrm{LSE}}$, $\hat{\alpha}_{\mathrm{LSE}}$ and $\hat{\beta}_{\mathrm{LSE}}$ respectively, can be determined via minimizing

$$\sum_{k=1}^{m}\left(\left(1-e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta} - \frac{k}{m+1}\right)^2, \tag{8}$$

with respect to θ, α and β. The $\hat{\theta}_{\mathrm{LSE}}$ can be found by differentiating (8) with respect to θ:

$$\sum_{k=1}^{m}\left(1-e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta}\left[\left(1-e^{-\alpha\left(e_i^{\beta t_{(k)}}-1\right)}\right)^{\theta} - \frac{k}{m+1}\right] \times\log\left(1-e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right) = 0. \tag{9}$$

The $\hat{\alpha}_{\mathrm{LSE}}$ can be found by differentiating (8) with respect to α:

$$\theta\sum_{k=1}^{m}\left(e^{\beta t_{(k)}}-1\right)e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\left(1-e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta-1} \times\left[\left(1-e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta} - \frac{k}{m+1}\right] = 0. \tag{10}$$

The $\hat{\beta}_{\mathrm{LSE}}$ can be found by differentiating (8) with respect to β:

$$\theta\alpha\sum_{k=1}^{m}t_{(k)}e^{\beta t_{(k)}}e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\left(1-e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta-1} \times\left[\left(1-e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta} - \frac{k}{m+1}\right] = 0. \tag{11}$$

Then, the $\hat{\theta}_{\mathrm{LSE}}$, $\hat{\alpha}_{\mathrm{LSE}}$, and $\hat{\beta}_{\mathrm{LSE}}$ can be found numerically by solving (9), (10), and (11) with respect to θ, α and β.

4 Weighted Least Squares Estimators

Weighted least squares (WLS) are a special case of generalized least squares. This estimator can be determined by minimizing

$$\sum_{k=1}^{m} w_k \left(F(T_{(k)}) - \frac{k}{m+1} \right)^2,$$

with respect to the unknown parameters, where

$$w_k = \frac{1}{V\left(F\left(T_{(k)}\right)\right)} = \frac{(m+1)^2 (m+2)}{k(m-k+1)}.$$

Thus, for $EGpz$ distribution the WLS estimators of θ, α, and β, says $\hat{\theta}_{\mathrm{WLSE}}$, $\hat{\alpha}_{\mathrm{WLSE}}$, and $\hat{\beta}_{\mathrm{WLSE}}$ respectively, can be found by minimizing

$$\sum_{k=1}^{m} w_k \left(\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta} - \frac{k}{m+1} \right)^2, \tag{12}$$

with respect to θ, α, and β. The $\hat{\theta}_{\mathrm{WLSE}}$ can be found by differentiating (12) with respect to θ:

$$\sum_{k=1}^{m} w_k \left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta} \left\{ \left(1 - \exp\left[-\alpha\left(e_i^{\beta t_{(k)}} - 1\right)\right]\right)^{\theta} - \frac{k}{m+1} \right\} \log\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right) = 0. \tag{13}$$

The $\hat{\alpha}_{\mathrm{WLSE}}$ can be found by differentiating (12) with respect to α:

$$\theta \sum_{k=1}^{m} w_k \left(e^{\beta t_{(k)}} - 1\right) e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)} \left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta-1} \times \left[\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta} - \frac{k}{m+1} \right] = 0. \tag{14}$$

The $\hat{\beta}_{WLSE}$ can be found by differentiating (12) with respect to β:

$$\theta\alpha \sum_{k=1}^{m} w_k t_{(k)} e^{\beta t_{(k)}} e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)} \left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta-1} \times \left[\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{\theta} - \frac{k}{m+1} \right] = 0. \tag{15}$$

Then, the $\hat{\theta}_{\mathrm{WLSE}}$, $\hat{\alpha}_{\mathrm{WLSE}}$, and $\hat{\beta}_{\mathrm{WLSE}}$ can be found numerically by solving (13), (14), and (15) with respect to θ, α, and β.

5 Percentile Estimation

The percentile estimation method was primarily discovered by [19,20]. This method has been applied very successfully for Weibull distribution, generalized exponential distribution, exponentiated gamma distribution, and exponentiated Pareto distribution [21,22,10,11].

Assume the unknown parameters θ, α, and β of EGpz distribution can be estimated via the percentile method of equating the sample percentile points with the population percentile points. If p_k means an estimate of $G(t_{(k)};\theta,\alpha,\beta)$, then the percentile estimators of θ, α, and β can be determined via minimizing

$$\sum_{k=1}^{m} \left[\log p_k - \theta \log\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right) \right]^2, \tag{16}$$

with respect to θ, α, and β. Here $t_{(k)}$'s are ordered samples and the maximization must be completed iteratively. Some estimators of p_k's can be used. As $p_k = \frac{k}{m+1}$ is the most used estimator, it is an unbiased estimator of $G(t_{(k)};\theta,\alpha,\beta)$. Some other selections of p_k's are $\frac{k-\frac{3}{8}}{m+\frac{1}{4}}$ and $\frac{k-\frac{1}{2}}{m}$. In this paper $p_k = \frac{k}{m+1}$ is used in which the expected value is of $G(t_{(k)})$.

The percentile estimator of θ denoted by $\hat{\theta}_{\mathrm{PCE}}$ can be obtained by

$$\sum_{k=1}^{m} \left\{ \log p_k - \theta \log\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right) \right\} \times \log\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right) = 0. \tag{17}$$

The percentile estimator of α denoted by $\hat{\alpha}_{\mathrm{PCE}}$ can be obtained by

$$\theta \sum_{k=1}^{m} \left(e^{\beta t_{(k)}} - 1\right) e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)} \left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{-1} \times \left[\log p_k - \theta \log\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right) \right] = 0. \tag{18}$$

The percentile estimator of β denoted by $\hat{\beta}_{\mathrm{PCE}}$ can be obtained by

$$\alpha\theta \sum_{k=1}^{m} t_{(k)} e^{\beta t_{(k)}} e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)} \left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right)^{-1} \times \left[\log p_k - \theta \log\left(1 - e^{-\alpha\left(e^{\beta t_{(k)}}-1\right)}\right) \right] = 0. \tag{19}$$

Then, the $\hat{\theta}_{\mathrm{PCE}}$, $\hat{\alpha}_{\mathrm{PCE}}$, and $\hat{\beta}_{\mathrm{PCE}}$ can be determined by solving the three non-linear equations of (17), (18), and (19) with respect to θ, α, and β.

6 Monte Carlo Simulation Study and Conclusions

Monte Carlo simulation study is performed to compare the methods of ML, LS, WLS, and PC estimators. All calculations are executed using Mathematica 9.0.

The next subsections define the steps for obtaining ML, LS, WLS, and PC estimators for three-parameter numerically.

6.1 Maximum Likelihood Estimators

estimators for θ, α, and β are obtained numerically by the following steps.

Step 1For given values of the parameters θ=2.13668, α=1.78059, and β=0.2968, generate a complete sample of size m from the generation random variables

$$T=\frac{1}{\beta}\log\left[1-\frac{1}{\alpha}\log\left(1-U^{\frac{1}{\theta}}\right)\right],$$

where T is $EGpz(\theta,\alpha,\beta)$ and U is a uniform (0,1) distribution.

Step 2The ML estimator of the parameters θ, α, and β are calculated by solving nonlinear equations (5), (6), and (7), respectively.

6.2 Least Squares, Weighted Least Squares, and Percentile Estimators.

The least squares, weighted least squares, and percentile estimators for θ, α, and β are obtained numerically by completing Step 1 outlined above followed by Step 2, which follows.

Step 3Step2. The LS estimator of the parameters θ, α, and β are calculated by solving nonlinear equations (9), (10), and (11), respectively. The WLS estimator of the parameters θ, α, and β are computed by solving nonlinear equations (13), (14), and (15), respectively, and the PC estimator of the parameters θ, α, and β are calculated by solving nonlinear equations (17), (18), and (19), respectively.

All of the above steps for the estimations of parameters θ, α, and β using ML, LS, WLS, and PC estimators were repeated 1000 times to evaluate the mean square error (MSE). The simulations were carried out for complete samples from $EGpz$ distribution for different sample size m. The outcomes are shown in (Table 1).

7 Concluding Remarks

A complete sample from the three-parameter EGpz distribution was considered for obtaining ML LS, WLS, and PC estimators for parameters. For given values of θ=2.13668, α=1.78059, and β=0.2968, a complete sample of size m was generated from EGpz distribution. The ML LS, WLS, and PC estimators were obtained by using Mathematica 9.0. The performances of the estimates were conducted by using the MSE.

Monte Carlo simulation studies were carried out in different sample sizes. From the results in (Table 1), the following were observed:

1.The MSEs of the estimates decreased as the sample size increased.
2.The parameter β was overestimated for all estimates with the exception of LS and WLS estimators, which were underestimated.
3.The parameter α was overestimated for all estimates with the exception of PC estimators, which were underestimated.
4.The parameter θ was underestimated for all estimates with the exception of ML estimator which was overestimated.
5.The ML estimators generally had smaller MSEs compared to the other estimators.

8 Real Data

We provided a real data set to illustrate all of the estimation methods described in the preceding sections. All of the computations were performed using Mathematica code. The data have been taken from [23], and denote the lifetimes of 50 devices. The data are given as follows: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86. This set was investigated by [15] to find the different estimators for the shape parameter of $EGpz$ distribution. The $EGpz(\theta,\alpha,\beta)$ distribution was fitted to this set. We applied the model selection using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (consistent Akaike information criteria), and the HQIC (Hannan-Quinn information criterion) to verify which methods estimator of θ, α, and β made the $EGpz$ distribution be a better fit to this data. For more details, see [24].

$$\left.\begin{array}{c} AIC=-2L\left(\underline{\hat{\gamma}}\right)+2z \\ BIC=-2L\left(\underline{\hat{\gamma}}\right)+z\log m \\ HQIC=-2L\left(\underline{\hat{\gamma}}\right)+2z\log\left(\log m\right) \\ CAIC=-2L\left(\underline{\hat{\gamma}}\right)+\frac{2zm}{m-z-1} \end{array}\right\} \tag{20}$$

where $L\left(\underline{\hat{\gamma}}\right)$ denotes the log-likelihood function, z is the number of parameters, and m is the sample size. Here $\underline{\hat{\gamma}}$ is

Table 1: Estimators and MSEs (between parentheses) of ML, LS, WLS, and PC estimates for three parameters of EGpz distribution

n	Par	ML	LS	WLS	PC
10	$\hat{\beta}$	0.41790	0.28087	0.32231	0.41225
		(0.18196)	(0.04131)	(0.07486)	(0.15583)
	$\hat{\alpha}$	2.03532	2.43619	2.33409	1.89123
		(2.41412)	(2.51097)	(2.69431)	(2.08982)
	$\hat{\theta}$	2.56907	1.96225	1.85752	1.73829
		(1.83416)	(1.61074)	(1.53421)	(1.42442)
30	$\hat{\beta}$	0.32760	0.26235	0.27491	0.44856
		(0.05072)	(0.02985)	(0.03859)	(0.12351)
	$\hat{\alpha}$	2.09118	2.46079	2.47657	1.53301
		(1.75776)	(2.23459)	(2.25372)	(1.64381)
	$\hat{\theta}$	2.37608	1.85609	1.95153	1.74065
		(0.78159)	(1.08687)	(0.78190)	(0.83733)
50	$\hat{\beta}$	0.31693	0.24217	0.25971	0.46893
		(0.01520)	(0.02493)	(0.02164)	(0.10316)
	$\hat{\alpha}$	2.06829	2.53094	2.51784	1.34362
		(1.34259)	(2.19784)	(2.24837)	(1.49417)
	$\hat{\theta}$	2.31435	1.76811	1.93627	1.76056
		(0.39526)	(0.96091)	(0.56430)	(0.39039)

Table 2: ML, PC, LS, and WLS estimations and AIC, BIC, CAIC, and HQIC measures for three parameters of EGpz distribution.

	Estimates			Measures			
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	*AIC*	*BIC*	*CAIC*	*HQIC*
ML	0.021004	0.345465	0.614046	464.716	463.813	460.097	465.238
LS	0.018603	0.541985	1.20088	487.938	487.035	483.319	488.459
PC	0.055813	0.010494	0.328076	452.28	451.377	447.661	452.802
WLS	0.024557	0.121182	0.424109	467.57	466.667	462.951	468.092

assumed to represent the unknown parameters, i.e. $\underline{\gamma}=(\theta,\alpha,\beta)$. The log-likelihood function given by (4) was calculated and relation (20) was applied for *ML*, *PC*, *LS*, and *WLS* estimations. The method by smallest *AIC* or *BIC*, *CAIC*, and *HQIC* value was selected as the best methods estimator to fit the data. The *ML*, *PC*, *LS*, and *WLS* estimations of the three parameters and the *AIC*, *BIC*, *CAIC*, and *HQIC* value for *EGpz* distribution are given in (Table 2). From (Table 2) it is concluded that the WLS estimation has the minimum value of *AIC*, *BIC*, *CAIC*, and *HQIC* in comparison with *ML*, *PC*, and *LS* estimations. The *WLS* estimation is the best estimation of the parameters θ, α, and β, thereby making the *EGpz* distribution fit better to this data.

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