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Fixed Point Theorem For Integral Type C^*-Valued Contraction

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Abstract: Recently, Z. Ma et al., introduced the notion of C^*-algebra valued metric spaces and proved some related fixed point theorems in these spaces. In this paper, we introduce the concept of Branciari integral type contractive condition for C^*-algebra valued metric spaces. Also we provide an example to support our main result.

Keywords: Metric space, C^*-algebra valued metric spaces, Branciari contractive mapping, fixed point result.

1 Introduction

We are familiar with the well known result Banach-Caccippoli theorem [1] first introduced by S. Banach, a French mathematician in 1922. This theorem is also called Banach contractive theorem or principle, which is stated as follows; **Theorem 1.1.** Let (X,d) be a complete metric space, $\delta \in (0,1)$ and $f : X \to X$, then f is said to be a contractive mapping such that for all $y,z \in X$,

$$d(fy, fz) \leq \delta d(y,z).$$

Then f has a unique fixed point.

Banach contraction principle plays an important role for solving nonlinear problems. Kannan [6] used the Banach contractive principle for analyzing new type of contractive condition. In 2002, Branciari [3] introduced the concept of integral type contractive mapping to generalized the concept of Banach contraction principle. In 2010, F. Khojasteh et al. [7] used the Branciari integral type contractive mapping for the cone metric space and proved some fixed point theorems.

Recently in 2014, Z. Ma et al. [9] established the notion of C^*-algebra valued metric spaces, and proved some fixed point theorems for contractive and expansive mappings. For more details and basic definitions of the C^* algebra we refer [2,4,5,8,11].

In this paper we introduce the integral type C^*-valued contractive mapping for the C^*-algebra valued metric spaces and prove some fixed point theorems.

2 Preliminaries

We recollected some basic definitions, notations and results of C^*-algebra that may observe [4,11]. A $*$-algebra $\mathscr{A}$ is a complex algebra with linear involution $*$ such that $y^{**} = y$ and $(yz)^* = z^*y^*$, for any $y,z \in \mathscr{A}$. If $*$-algebra together with complete sub multiplicative norm satisfying $||y^*|| = ||y||$ for all $y \in \mathscr{A}$, then $*$-algebra is said to be a Banach $*$-algebra. A C^*-algebra is a Banach $*$-algebra such that $||y^*y|| = ||y||^2$ for all $y \in \mathscr{A}$. An element of $\mathscr{A}$ is called positive element, if $\mathscr{A}_+ = \{y^* = y | y \in \mathscr{A}\}$ and $\sigma(y) \subset \mathbb{R}_+$, where $\sigma(y)$ is the spectrum of an element $y \in \mathscr{A}$, i.e. $\sigma(y) = \{\lambda \in \mathbb{R} : \lambda I - y \text{ is not invertible}\}$. There is a natural partial ordering on $\mathscr{A}_+$ given by $y \preceq z$ if and only if $y - z \in \mathscr{A}_+$.

Definition 1.*Suppose that X be a nonempty set, and the mapping $d : X \times X \to \mathbb{A}$ is satisfying the following conditions:*

1. *$d(y,z) \geq 0$ for all $y,z \in X$ and $d(y,z) = 0 \Leftrightarrow y = z$;*
2. *$d(y,z) = d(z,y)$ for all $y,z \in X$;*
3. *$d(y,z) \leq d(y,x) + d(x,z)$ for all $x,y,z \in X$.*

Then d is C^*-algebra valued metric on X, and $(X, \mathbb{A}, d)$ is C^*-algebra valued metric space.

It is clear that C^*-algebra valued metric spaces is the generalization of the metric space by substituting $\mathbb{A}$ instead of $\mathbb{R}$.

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Definition 2. *Let $(X,\mathbb{A},d)$ is C^*-algebra valued metric space and let $\{y_n\}$ be a sequence in X. If*

1. *for any $\varepsilon > 0$, there is N such that for all $n > N$, $||d(y_n,y)|| \leq \varepsilon$, then the sequence $\{y_n\}$ is said to be convergent, and we denote it as $\lim_{n\to\infty} y_n = y$.*
2. *for any $\varepsilon > 0$, there is N such that for all $m,n > N$, $||d(y_m,y_n)|| \leq \varepsilon$, then the sequence $\{y_n\}$ is said to be Cauchy sequence.*
3. *C^*-algebra valued metric space is said to be complete if every Cauchy sequence in X with respect to $\mathbb{A}$ is convergent.*

Example 1. Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define

$$d(y,z) = \begin{pmatrix} |y-z| & 0 \\ 0 & \delta|y-z| \end{pmatrix} \text{ for all } y,z \in \mathbb{R} \text{ and } \delta \geq 0.$$

It is essay to verify that d is a C^*-algebra valued metric space and $(X,M_2(\mathbb{R}),d)$ is complete C^*-algebra valued metric spaces.

Definition 3. *Let $(X,\mathbb{A},d)$ be a C^*-valued metric spaces. A mapping f from X into X is said to be a C^*-valued contractive if there exists an $c \in \mathbb{A}$ with $||c|| < 1$ such that*

$$d(fy,fz) \leq c^* d(y,z)c,$$

for all $y,z \in X$.

3 Main results

Branciari in 2002, introduced the general integral type contraction which stated as follows.

Let Ψ be the class of all mappings ψ from $\mathbb{R}_+$ into $\mathbb{R}_+$ which is Lebesgue integrable, summable on each compact subset of $\mathbb{R}_+$, nonnegative and for each $\varepsilon > 0$, $\int_0^\varepsilon \psi(z)dz > 0$.

Theorem 3.1. Let (X,d) be a complete metric space, $\delta \in (0,1)$ and let $h : X \to X$ be a mapping such that for each $y,z \in X$,

$$\int_0^{d(hy,hz)} \psi(z)dz \leq \delta \int_0^{d(y,z)} \psi(z)dz, \tag{1}$$

where ψ from $\mathbb{R}_+$ into $\mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $\mathbb{R}_+$, nonnegative and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \psi(z)dz > 0$. Then h has a unique fixed point $y \in X$ such that for each $y \in X$, $\lim_{n\to\infty} h^n y = y$.

Motivated by the work of Z. Ma et al. [9] and Branciari [3], we introduce the following definition.

Definition 4. *Let $(X,\mathbb{A},d)$ be a C^*-valued metric space. A mapping $h : X \to X$ is said to be a integral C^*-valued contraction mapping on X if there exists an $c \in \mathbb{A}$ with $||c|| < 1$ such that*

$$\int_0^{d(hy,hz)} \psi(z)dz \leq c^* (\int_0^{d(y,z)} \psi(z)dz)c,$$

for all $y,z \in X$ and $\psi \in \Psi$.

Now we define a subclass of integral type C^*-valued contraction which we will use in our main result. We call this class a sub additive integral type C^*-contraction. Let Θ be the set of all mappings $\psi \in \Psi$ satisfying the following;

$$\int_0^{a+b} \psi(z)dz \leq \int_0^a \psi(z)dz + \int_0^b \psi(z)dz,$$

for all $a,b \geq 0$.

Theorem 3.2. Let $(X,\mathbb{A},d)$ be complete C^*-algebra valued metric space, if there exists $c \in \mathbb{A}$ with $||c|| < 1$ and $h : X \to X$ be a integral C^*-valued contractive mapping such that for all $x,y \in X$,

$$\int_0^{d(hx,hy)} \psi(z)\, dz \leq c^* \Big(\int_0^{d(x,y)} \psi(z)\, dz \Big) c, \tag{2}$$

where $\psi \in \Psi$. Then h has a unique fixed point.

Proof. Choose $x_0 \in X$ and setting $x_{n+1} = hx_n = h^{n+1}x_0$. Then we have

$$\begin{aligned}
\int_0^{d(x_{n+1},x_n)} \psi\, dz &= \int_0^{d(hx_n,hx_{n-1})} \psi\, dz \\
&\leq c^* \Big(\int_0^{d(x_n,x_{n-1})} \psi dz \Big) c \\
&\leq c^* c^* \Big(\int_0^{d(x_{n-1},x_{n-2})} \psi dz \Big) cc \\
&\leq (c^*)^2 \Big(\int_0^{d(x_{n-1},x_{n-2})} \psi dz \Big) (c)^2 \\
&\cdot \\
&\cdot \\
&\cdot \\
&\leq (c^*)^n \Big(\int_0^{d(x_0,x_1)} \psi dz \Big) (c)^n.
\end{aligned}$$

For $n > m$ and by triangular inequality and sub additive property in C^*-algebra metric space, we get

$$\begin{aligned}
\int_0^{d(hx_m,hx_n)} \psi dz &\leq \int_0^{d(hx_n,hx_{n+1})+d(hx_{n+1},hx_{n+2})+\ldots+d(hx_{m-1},hx_m)} \psi dz. \\
&\leq \int_0^{d(hx_{n+1},hx_n)} \psi dz \\
&\quad + \cdots + \int_0^{d(hx_{m-1},hx_m)} \psi dz \\
&\leq (c^*)^n \int_0^{d(x_0,x_1)} \psi dz (c)^n \\
&\quad + \cdots + (c^*)^m \int_0^{d(x_0,x_1)} \psi dz (c)^m \\
&\leq \{(c^*)^n (c)^n + \cdots + (c^*)^m (c)^m\} \int_0^{d(x_0,x_1)} \psi dz \\
&\leq \{(c^n)^* (c)^n + \cdots + (c^n)^* (c)^m\} \int_0^{d(x_0,x_1)} \psi dz
\end{aligned}$$

$$\leq \sum_{i=n}^{m} |c^i|^2 \int_0^{d(x_0,x_1)} \psi dz$$
$$\leq \Big|\Big| \sum_{i=n}^{m} |c^i|^2 \int_0^{d(x_0,x_1)} \psi dz \Big|\Big| I$$
$$\leq \Big|\Big| \sum_{i=n}^{m} |c^i|^2 \Big|\Big| \Big|\Big| \int_0^{d(x_0,x_1)} \psi dz \Big|\Big| I$$
$$\leq \sum_{i=n}^{m} ||c||^{2i} \Big|\Big| \int_0^{d(x_0,x_1)} \psi dz \Big|\Big| I$$
$$\leq \frac{||c||^{2m}}{1-||c||} \Big|\Big| \int_0^{d(x_0,x_1)} \psi dz \Big|\Big| I$$
$$\leq \frac{||c||^{2m}}{1-||c||} \Big|\Big| \int_0^{d(x_0,x_1)} \psi dz \Big|\Big| I.$$

Thus,

$$\int_0^{d(hx_m,hx_n)} \psi dz \to 0, \text{ as } m,n \to \infty,$$

which implies that

$$\lim_{n,m\to\infty} ||d(hx_m,hx_n)|| = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Hence $\{x_n\}$ converges to $x \in X$. i.e.,

$$\lim_{n\to\infty} x_n = x.$$

Now for fixed point of h.

$$\int_0^{d(x_{n+1},hx)} \psi\, dz = \int_0^{d(hx_n,hx)} \psi\, dz$$
$$\leq c^* \int_0^{d(x_n,x)} \psi\, dz c.$$

Thus,

$$\lim_{n\to\infty} ||d(x_{n+1},hx)|| = 0.$$

Now, for the unique fixed point of h. Let y be another fixed point of h, then

$$\int_0^{d(x,y)} \psi\, dz = \int_0^{d(hx,hy)} \psi\, dz$$
$$\leq c^* \int_0^{d(x,y)} \psi\, dz c$$
$$< \int_0^{d(x,y)} \psi\, dz.$$

Which is contradiction. Thus h has a unique fixed point $x \in X$.

Remark 3.3. This theorem is the generalization of the C^*-algebra valued contractive mapping, by setting $\psi(z) = 1$,

$$\int_0^{d(hx,hy)} \psi(z)\, dz = d(hx,hy) \leq c^* d(x,y) c = \int_0^{d(x,y)} \psi(z)\, dz.$$

Example 2. Let $X = [0,1]$ be any non empty set and d be metric space defined as

$$d(x,y) = ||x-y|| I,$$

and define $h : X \to X$, $\psi : [0,\infty) \to [0,\infty)$ by

$$h(z) = \begin{cases} \dfrac{z}{1+qz} & if \quad z = \dfrac{1}{m}, \\ 0 & if \quad z \neq \dfrac{1}{m} \end{cases} \tag{3}$$

and

$$\phi(t) = \begin{cases} t^{\frac{1}{t}-2}(1-\log t) & if \quad t > 0, \\ 0 & if \quad t = 0, \end{cases} \tag{4}$$

for all $m \in \mathbb{N}$ and q be any positive integer. As we know that (1) is equivalent to

$$||d(hx,hy)||^{\frac{1}{||d(hx,hy)||}} \leq ||c|| ||d(x,y)||^{\frac{1}{||d(x,y)||}} \quad \text{for all } x,y \in X. \tag{5}$$

Now our next target is to show that (5) is satisfied for $c = ||c|| = \frac{1}{\sqrt{2}} < 1$. For this let us consider $x = \frac{1}{m+1}$ and $y = \frac{1}{m}$ for $m \in \mathbb{N}$, then we have

$$||d(hx,hy)||^{\frac{1}{||d(hx,hy)||}} = \Big|\Big| \frac{1}{m+1+p} - \frac{1}{m+p} \Big|\Big|^{\frac{1}{||\frac{1}{m+1+p} - \frac{1}{m+p}||}}$$
$$= \Big[\frac{1}{(m+1+p)(m+p)}\Big]^{(m+1+p)(m+p)} \tag{6}$$

Now, R.H.S of (5) implies that,

$$||d(x,y)||^{\frac{1}{||d(x,y)||}} = \Big|\Big| \frac{1}{m} - \frac{1}{m+1} \Big|\Big|^{\frac{1}{||\frac{1}{m} - \frac{1}{m+1}||}}$$
$$= \Big[\frac{1}{m(m+1)}\Big]^{m(m+1)}. \tag{7}$$

Putting value of (6) and (7) in (5), then we get

$$\Big[\frac{1}{(m+1+p)(m+p)}\Big]^{(m+1+p)(m+p)} \leq ||c|| \Big[\frac{1}{m(m+1)}\Big]^{m(m+1)} \tag{8}$$

Therefore (8) is true for $||c|| = \frac{1}{\sqrt{2}} < 1$, so h is an integral C^*-valued contraction with contraction constant $||c|| = \frac{1}{\sqrt{2}} < 1$. Thus all the condition of **Theorem 3.2.** is satisfied and h has a unique fixed point 0.

4 Conclusion

The idea of an integral type C^*-valued contraction is not only the extension of C^*-valued contraction, but it develops the inequality (1). Whereas, the notion of sub additive C^*-valued contraction extends the idea of C^*-valued contraction but it slightly generalizes the inequality (1).

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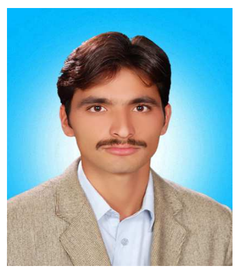

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