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Transmuted Exponentiated Frêchet Distribution: Properties and Applications

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Abstract: Transmuted probability distribution corresponding to a distribution function $G(x)$ is defined as $F(x) = (1+\lambda)G(x) - \lambda G(x)^2$; $|\lambda| \leq 1$. In this paper we study some general properties of the transmuted probability distribution function in relation to the base distribution $G(x)$. In particular the transmuted exponentiated Frêchet (TEF) distribution is studied in detail. The different methods of estimation of parameters such as, weighted least squares and the maximum likelihood estimates of this distribution are studied. Finally, a real time data analysis is performed for this distribution and it is found that this class is more flexible class and it shown that the TEF distribution is much better fit for data's originally fitted and analysed using Frêchet or Exponentiated Frêchet distribution.

Keywords: Transmuted Exponentiated Frêchet Distribution; Hazard function; Moments; Maximum likelihood estimation; Data analysis.
Mathematics Subject Classifications: 62N05; 60E99.

1 Introduction and Literature

A random variable X is said to have transmuted distribution if its cumulative distribution (cdf) is given by

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \ |\lambda| \leq 1 \tag{1}$$

where $G(x)$ is the cumulative distribution function of the base random variable. By differentiation of (1) yields,

$$f(x) = g(x)\left[(1+\lambda) - 2\lambda G(x)\right] \tag{2}$$

where $f(x)$ and $g(x)$ are the corresponding probability density functions with cumulative distribution functions $F(x)$ and $G(x)$ respectively. This is the quadratic rank transmutation map, studied extensively in Shaw et. al. [1]. Observe that at $\lambda = 0$ we have the distribution of the base random variable.

1.1 Transmutation Map

In this subsection we demonstrate transmuted probability distribution. Let F and G be the cumulative distribution functions, of two distributions with a common support '$\mathscr{S}$'. The general rank transmutation as given by Shaw et. al. [1] is defined as

$$G_R(u) = G(F^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(u) = inf_{x\in R}\{F(x) \geq u\} \ \ for \ u \in [0,1].$$

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The function $G_R(u)$ map the unit interval $I=[0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_R(0)=0$ and $G_R(1)=1$. A Quadratic Rank Transmutation Map $(QRTM)$ is defined as

$$G_R(u)=u+\lambda u(1-u),\ |\lambda|\leq 1,\ 0\leq u\leq 1, \tag{3}$$

from which it follows that the cumulative distribution functions satisfy the relationship

$$F(x)=(1+\lambda)G(x)-\lambda G(x)^2 \tag{4}$$

when $G(x)$ is absolutely continuous on differentiation of (4) yields,

$$f(x)=g(x)[(1+\lambda)-2\lambda G(x)] \tag{5}$$

where $f(x)$ and $g(x)$ are the corresponding probability density functions associated with cumulative distribution functions $F(x)$ and $G(x)$ respectively. An extensive information about the quadratic rank transmutation map is given in Shaw et. al. [1]. Observe that at $\lambda=0$ we have the distribution of the base random variable. The following Lemma proved that the function $f(x)$ given in (5) satisfies the property of probability density function.

Lemma 1.1. $f(x)$ given in (5) is a well defined probability density function.
Proof. Rewriting $f(x)$ as $f(x)=g(x)[(1-\lambda(2G(x)-1))]$ we observe that $f(x)$ is non-negative. Also,

$$\begin{aligned}\int_{\mathscr{S}} f(x)dx &= \int_{\mathscr{S}}(1+\lambda)g(x)dx-2\lambda\int_{\mathscr{S}} g(x)dx\\ &=1+\lambda-\lambda\\ &=1\end{aligned}$$

Hence $f(x)$ is a well defined probability density function. We call $f(x)$ the transmuted probability density function with base density $g(x)$. Further properties of the transmuted distribution is studied in Section 2.

Also many authors have worked with the generalization of some well-known distributions. Aryal and Tsokos [2] defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Also Aryal and Tsokos [3] presented a new generalization of Weibull distribution called the transmuted Weibull distribution.

Recently, Aryal [4] proposed and studied the various structural properties of the transmuted Log-Logistic distribution. Khan and King [5] introduced the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal et. al. [3] and they also studied the mathematical properties and maximum likelihood estimation of the unknown parameters. Subsequently, Elbatal and Aryal [6] presented on the Transmuted additive Weibull distribution. Also, Elbatal [7] studied the Transmuted Modified Inverse Weibull distribution. Merovci [8] introduced the Transmuted Rayleigh distribution, Transmuted generalized Rayleigh distribution [9], and Transmuted Lindley distribution [10]. Elbatal and Elgarhy [11] studied the Transmuted Quasi Lindley distribution.

Here we make an attempt to establish a generalization for Exponentiated Frêchet distribution. Frêchet distribution was introduced by a French mathematician named Maurice Frêchet (1878-1973) who had identified before one possible limit distribution for the largest order statistic in [12] . The Frêchet distribution has been shown to be useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rain fall, sea currents and wind speeds. Therefore Frêchet distribution is well suited to characterize random variables of large features. Applications of the Frêchet distribution in various fields given in Harlow [13] showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications. Nadarajah and Kotz [14] discussed the sociological models based on Frêchet random variables. Further, Zaharim et. al. [15] applied Frêchet for analyzing the wind speed data. Mubarak [16] studied the Frêchet progressive type-II censored data with binomial removals. The Frêchet distribution is a special case of the generalized extreme value distribution. This type-II extreme value distribution (Frêchet) case is equivalent to taking the reciprocal of values from a standard Weibull distribution. The cumulative distribution function (CDF) and probability density function (PDF) for Frêchet distribution are given by

$$F(x,\theta,\beta)=e^{-\left(\frac{\theta}{x}\right)^{\beta}},\ x>0,\ \theta>0,\ \beta>0.$$

where the parameter $\beta>0$ determines the shape of the distribution and $\theta>0$ is the scale parameter. And

$$f(x,\theta,\beta)=\frac{\beta}{\theta}\left(\frac{\theta}{x}\right)^{\beta+1}e^{-\left(\frac{\theta}{x}\right)^{\beta}}$$

Recently, a new three parameter distribution, named as Exponentiated Frêchet (EF) distribution has been introduced by Nadarajah and Kotz [17] as a generalization of the standard Frêchet distribution. The exponentiated Frêchet distribution is considered to be one of the newest model of lifetime models. There are over fifty applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds and track race records (Kotz and Nadarajah [18]). We define the new distribution by the cumulative distribution function

$$G(x,\theta,\beta,\alpha) = 1 - \left[1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]^{\alpha}, \; x > 0 \tag{6}$$

where the shape parameter $\alpha > 0$. The corresponding probability density function is given by

$$g(x,\theta,\beta,\alpha) = \alpha\beta\theta^{\beta}x^{-(1+\beta)}e^{-\left(\frac{\theta}{x}\right)^{\beta}}\left[1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]^{\alpha-1}, x > 0. \tag{7}$$

In this article we present a new generalization of Exponentiated Frêchet distribution called the Transmuted Exponentiated Frêchet (TEF) distribution. We will derive the subject distribution using the quadratic rank transmutation map given in (3) Shaw et. al. [1]. The present paper is organized as follows. The Exponentiated Frêchet distribution is introduced in Section 3. Various aging properties are also studied. In Section 4 we discuss the statistical properties include quantile functions, moments, moment generating function. The distribution of the order statistics is expressed in Section 5. Reliability characteristics of the distribution is studied in Section 6. The least squares and weighted least squares estimators of the parameters of the TEF distribution are introduced in Section 7. In Section 8 we employed the maximum likelihood estimation to estimate the unknown parameters. A simulation study has been conducted at Section 9 and a real time applications are shown in Section 10. We conclude our study by observing that the TEF is a better fit for data's originally fitted and analysed using Frêchet or Exponentiated Frêchet distribution.

2 General Properties

In this section we study the properties of the transmuted variable in relation to the base random variable. Many characteristics of the transmuted distribution function is assured by the behaviour of the baseline distribution function. The next theorem shows the relationship between moments for the transmuted distribution once the baseline moments exists.

Theorem 2.1. Let $\Phi(X)$ be a non-degenerate measurable function of the random variable X with transmuted distribution as in (4). If $E_F(\Phi(X))$ denotes $\int\limits_{\mathscr{S}} \Phi(X)f(x)dx$, then

$$E_F(\Phi(X)) = (1+\lambda)E_G(\Phi(X)) - 2\lambda E_G[\Phi(X)G(X)] \tag{8}$$

Proof. From (5)

$$\begin{aligned} E_F(\Phi(X)) &= \int\limits_{\mathscr{S}} \Phi(X)[(1+\lambda)g(x) - 2\lambda g(x)G(x)]dx \\ &= (1+\lambda)E_G(\Phi(X)) - 2\lambda \int\limits_{\mathscr{S}} \Phi(X)g(x)G(x)dx \\ &= (1+\lambda)E_G(\Phi(X)) - 2\lambda E_G[\Phi(X)G(X)] \end{aligned}$$

Corollary 2.1. If $L_G(t)$ denotes the Laplace transform of the base distribution G; then the Laplace transform of the transmuted distribution F is,

$$L_F(t) = (1+\lambda)L_G(t) - 2\lambda E_G[e^{-Xt}G(X)]; \; |t| < 1.$$

Corollary 2.2. If $\mu_r(F) = \int x^r f(x)dx$ then $\mu_r(F) = (1+\lambda)\mu_r(G) - 2\lambda E_G[X^r G(X)]$.

Theorem 2.2. For $\lambda > 0$,
(i). If F is a convex distribution function implies that G is also a convex distribution function.
(ii). Conversely, if G is a convex distribution then F is convex if and only if,

$$f(x) \geq \frac{2\lambda g^3(x)}{g'(x)}, \; for\,all \; x \in \mathscr{S},$$

where

$$g'(x)=\frac{dg(x)}{dx}.$$

Proof. Let F be a convex distribution function. Then by definition, for $\lambda>0$, $f'(x)>0$ which in turn implies

$$\begin{aligned}(1+\lambda)g'(x)-2\lambda g'(x)G(x)-2\lambda g^2(x)>0&\\ \Leftrightarrow \frac{g'(x)}{g(x)}f(x)>2\lambda g^2(x)&\\ \Leftrightarrow g'(x)>2\,\frac{g^3(x)}{f(x)}&\\ \Rightarrow g'(x)>0\ for\, all\ x\in\mathscr{S}.&\end{aligned}$$

Hence proving (i). To prove (ii) observe that G is a convex distribution function implies $g'(x)>0$ for all $x\in\mathscr{S}$. Hence from (i) it follows that F is a convex distribution function if and only if

$$f(x)\geq\frac{2\lambda g^3(x)}{g'(x)}.$$

Hence the result.

The next few result studies the ageing properties of the transmuted distribution $F(x)$ in relation to $G(x)$. One of the characteristic in reliability analysis is the hazard rate function (HF). For an absolutely continuous general transmuted distribution it is defined by

$$h_{TD}=\frac{f}{1-F}=\frac{(1+\lambda)g(x)-2\lambda G(x)g(x)}{1-(1+\lambda)G(x)+\lambda G(x)^2}\tag{9}$$

The following results are now immediate.

Theorem 2.3. For $\lambda<0$ $(\lambda>0)$ the transmuted distribution $F(x)$ has a increasing failure rate distribution (decreasing failure rate distribution) if and only if $G(x)$ is an increasing failure rate distribution (decreasing failure rate distribution).

Proof. From (9)

$$h_{TD}=h_G(x)\left[1+\frac{\lambda\overline{G}(x)}{1-\lambda G(x)}\right]\tag{10}$$

where, $h_G(x)=\frac{g(x)}{\overline{G}(x)}$ and $\overline{G}(x)=1-G(x)$.

The result follows by observing that $\left[1+\frac{\lambda\overline{G}(x)}{1-\lambda G(x)}\right]$ is increasing whenever for $\lambda\leq 0$, $\lambda=1$.

Hence the result.

Hence it is evident that in general the transmuted distribution functions do not behave in a similar manner as the base distribution. Hence it is of interest to study the transmuted distributions on specifying the base distribution. Motivated by this fact, we look into particular transmuted distributions.

3 Transmuted Exponentiated Frêchet Distribution

In this section we introduce the Transmuted Frêchet (TEF) distribution. Now combining (6) and (7) we have the cumulative distribution function of Transmuted Exponentiated Frêchet distribution

$$F_{TEF}(x,\theta,\beta,\alpha,\lambda)=\left[1-\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right]\left[1+\lambda\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right]\tag{11}$$

where λ is the transmuted parameter. The corresponding probability density function is given by

$$\begin{aligned}f_{TEF}(x,\theta,\beta,\alpha,\lambda)=\alpha\beta\theta^{\beta}x^{-(1+\beta)}e^{-\left(\frac{\theta}{x}\right)^{\beta}}\times(q(x))^{\alpha-1}&\\ \times\left[(1-\lambda)+2\lambda\,(q(x))^{\alpha}\right]&\end{aligned}$$

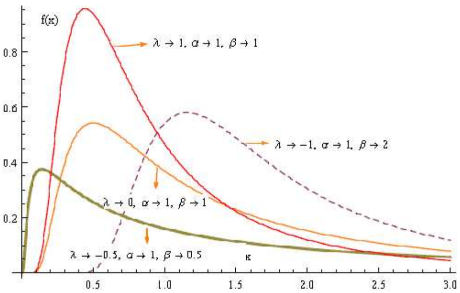

Fig. 1: Probability density function of Transmuted Exponentiated Frêchet distribution for $\theta = 1$ and different values of λ, α and β
.

where, $q(x) = \left(1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)$

$$\begin{aligned} f_{TEF}(x,\theta,\beta,\alpha,\lambda) =&(1-\lambda)\alpha\beta\theta^{\beta}x^{-(1+\beta)}e^{-\left(\frac{\theta}{x}\right)^{\beta}}(q(x))^{\alpha-1} \\ &+2\lambda\alpha\beta\theta^{\beta}x^{-(1+\beta)}e^{-\left(\frac{\theta}{x}\right)^{\beta}}(q(x))^{2\alpha-1} \end{aligned} \tag{12}$$

It is observed that the Transmuted Frêchet distribution is an extended model to analyse data from complex situations and it generalizes some of the widely used distributions. For instance when $\beta = 1$ it reduces to transmuted exponentiated inverted exponential distribution as discussed in Elbatal et. al. [19]. The exponentiated Frêchet distribution is clearly a special case for $\lambda = 0$. When $\beta = \lambda = 1$ and $\alpha = 0.5$ then the resulting distribution is an inverted exponential distribution with parameter θ. Figure 1 illustrates some of the possible shapes of the probability density function of a Transmuted Exponentiated Frêchet distribution for selected values of the parameters α, β, λ and for $\theta = 1$.

4 Statistical Properties

This section is devoted to studying statistical properties of the (TEF) distribution, more specifically quantile function, moments and moment generating function.

4.1 Quantile Function and Random Number Generation

We present a method for simulating from the TEF distribution (11). The quantile function corresponding to (11) is

$$\begin{aligned} Q(u) =&F^{-1}(u) \\ =&\theta\left\{-\ln\left[1-\left(\frac{(\lambda-1)+\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda}\right)^{\frac{1}{\alpha}}\right]\right\}^{\frac{-1}{\beta}} \end{aligned} \tag{13}$$

Simulating the TEF random variable is straight forward process. Let U be a uniform variate on the unit interval (0,1). Thus, by means of the inverse transformation method, we consider the random variable X is given by

$$X = \theta\left\{-\ln\left[1-\left(\frac{(\lambda-1)+\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda}\right)^{\frac{1}{\alpha}}\right]\right\}^{\frac{-1}{\beta}},$$

which follows (12), that is $X \sim TEF(\theta,\beta,\alpha,\lambda)$. This process is explained in Section 9 through a simulation study.

4.2 Skewness and Kurtosis

In this subsection we present the shortcomings of the classical kurtosis measure are well-known. There are many heavy tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. The Bowley skewness (Kenney and Keeping, [20]) is one of the earliest skewness measures defined by

$$S_K = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}},$$

and the Moors Kurtosis (see Moors [21]) based on octiles is defined by

$$K_u = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}.$$

Where $Q(u)$ represents the quantile function.

4.3 Moments

In this subsection we discuss r^{th} moment for TEF distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., central tendency, dispersion, skewness and kurtosis).
Theorem 4.1. If X has $TEF(\theta,\beta,\alpha,\lambda)$, then the r^{th} moment of X (μ_r) is given by the following

$$\mu_r' = \sum_{j=0}^{\infty} (-1)^j \theta^r (1+j)^{-\left(1-\frac{r}{\beta}\right)} \Gamma(1-\frac{r}{\beta}) \left[(1-\lambda)\binom{\alpha-1}{j} + 2\lambda\binom{2\alpha-1}{j}\right] \tag{14}$$

Proof. Let X be a random variable with density function (12). The r^{th} ordinary moment of the (TEF) distribution is given by

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x)dx$$

$$\begin{aligned}\mu_r' =&(1-\lambda)\alpha\beta\theta^\beta \int_0^{\infty} x^{r-\beta-1} e^{-\left(\frac{\theta}{x}\right)^\beta} \left[1-e^{-\left(\frac{\theta}{x}\right)^\beta}\right]^{\alpha-1} dx \\ &+ 2\lambda\alpha\beta\theta^\beta \int_0^{\infty} x^{r-\beta-1} e^{-\left(\frac{\theta}{x}\right)^\beta} \left[1-e^{-\left(\frac{\theta}{x}\right)^\beta}\right]^{2\alpha-1} dx\end{aligned} \tag{15}$$

Setting

$$\left[1-e^{-\left(\frac{\theta}{x}\right)^\beta}\right]^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} e^{-j\left(\frac{\theta}{x}\right)^\beta} \tag{16}$$

substituting from (16) into (15) we get

$$\begin{aligned}\mu_r' =&(1-\lambda)\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\alpha\beta\theta^\beta \int_0^{\infty} x^{r-\beta-1} e^{-(j+1)\left(\frac{\theta}{x}\right)^\beta} dx \\ &+ 2\lambda\sum_{j=0}^{\infty}(-1)^j\binom{2\alpha-1}{j}\alpha\beta\theta^\beta \int_0^{\infty} x^{r-\beta-1} e^{-(j+1)\left(\frac{\theta}{x}\right)^\beta} dx.\end{aligned} \tag{17}$$

let $(j+1)\left(\frac{\theta}{x}\right)^\beta = t$, we get

$$\mu_r' = \sum_{j=0}^{\infty} (-1)^j \theta^r (1+j)^{-\left(1-\frac{r}{\beta}\right)} \Gamma(1-\frac{r}{\beta}) \times K(j)$$

where,

$$K(j)=\left[(1-\lambda)\binom{\alpha-1}{j}+2\lambda\binom{2\alpha-1}{j}\right]$$

Which completes the proof.

Based on the first four moments of the TEF distribution, the coefficient of variation, the measures of skewness and kurtosis of the TEF distribution can be obtained according to the following relation

$$CV_{TEF}=\sqrt{\frac{\mu_2}{\mu_1}-1} \tag{18}$$

$$MS_{TEF}=\frac{\mu_3-3\mu_1\mu_2+2\mu_1^3}{[\mu_2-\mu_1^2]^{\frac{3}{2}}} \tag{19}$$

and

$$MK_{TEF}=\frac{\mu_4-4\mu_1\mu_3+6\mu_1^2\mu_2-3\mu_1^4}{[\mu_2-\mu_1^2(\theta)]^2} \tag{20}$$

4.4 Moment Generating Function

In this subsection we derived the moment generating function of TEF distribution.

Theorem 4.2. If X has TEF distribution, then the moment generating function $M_X(t)$ has the following form.

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^r}{r!}(-1)^j\theta^r(1+j)^{-(1-\frac{r}{\beta})}\Gamma(1-\frac{r}{\beta})\times K(j) \tag{21}$$

where,

$$K(j)=\left[(1-\lambda)\binom{\alpha-1}{j}+2\lambda\binom{2\alpha-1}{j}\right]$$

Proof. We start with the well known definition of the moment generating function given by

$$\begin{aligned}M_X(t)&=\int_0^{\infty}e^{tx}f_{TEF}(x)dx\\&=\sum_{r=0}^{\infty}\tfrac{t^r}{r!}x^r f_{TEF}(x)dx=\sum_{r=0}^{\infty}\tfrac{t^r}{r!}\mu_r'\\&=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\tfrac{t^r}{r!}(-1)^j\theta^r(1+j)^{-(1-\frac{r}{\beta})}\Gamma(1-\tfrac{r}{\beta})\\&\times\left[(1-\lambda)\binom{\alpha-1}{j}+2\lambda\binom{2\alpha-1}{j}\right]\end{aligned}$$

which completes the proof.

5 Distribution of the Order Statistics

In fact, the order statistics have many applications in reliability and life testing. The order statistics arise in the study of reliability of a system. Let $X_1,X_2,...,X_n$ be a simple random sample from $TEF(\theta,\beta,\alpha,\lambda,x)$ with cumulative distribution function and probability density function as in (11) and (12), respectively. Let $X_{(1:n)}\leq X_{(2:n)}\leq...\leq X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i:n)}$ denote the lifetime of an $(n-i+1)-$ out$-$ of$-$ n system which consists of n independent and identical components. Then the pdf of $X_{(i:n)}$ $, 1 \le i \le n$ is given by

$$f_{i:n}(x) = \frac{1}{\beta(i, n-i+1)} [F(x, \Phi)]^{i-1} [1 - F(x, \Phi)]^{n-i} f(x, \Phi) \tag{22}$$

where $\Phi = (\theta, \beta, \alpha, \lambda)$. Also, the joint pdf of $X_{(i:n)}$, $X_{(j:n)}$ and $1 \le i \le j \le n$ is

$$\begin{aligned} f_{i:j:n}(x_i, x_j) = & C\ [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-i-1} \\ & \times [1 - F(x_j)]^{n-j} f(x_i) f(x_j) \end{aligned} \tag{23}$$

where

$$C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

We defined the first order statistics $X_{(1)} = Min(X_1, X_2, ..., X_n)$, the last order statistics as $X_{(n)} = Max(X_1, X_2, ..., X_n)$ and median order X_{m+1} .

5.1 Distribution of Minimum , Maximum and Median

Let $X_1, X_2, ..., X_n$ be independently identically distributed order random variables from the transmuted Exponentiated Frêchet distribution having first, last and median order probability density function are given by the following

$$\begin{aligned} f_{1:n}(x) = & n [1 - F(x, \Phi)]^{n-1} f(x, \Phi) \\ = & n \left\{ 1 - \left[1 - h_{(1)}^{\alpha} \right] \left[1 + \lambda h_{(1)}^{\alpha} \right] \right\}^{n-1} \\ & \times \alpha \beta \theta^{\beta} x_{(1)}^{-(1+\beta)} (1 - h_{(1)}) h_{(1)}^{\alpha-1} \\ & \times \left[(1-\lambda) + 2\lambda h_{(1)}^{\alpha} \right] \end{aligned} \tag{24}$$

$$\begin{aligned} f_{n:n}(x) = & n \left[F(x_{(n)}, \Phi) \right]^{n-1} f(x_{(n)}, \Phi) \\ = & n \left\{ \left[1 - h_{(n)}^{\alpha} \right] \left[1 + \lambda h_{(n)}^{\alpha} \right] \right\}^{n-1} \\ & \times \alpha \beta \theta^{\beta} x_{(n)}^{-(1+\beta)} (1 - h_{(n)}) h_{(n)}^{\alpha-1} \left[(1-\lambda) + 2\lambda h_{(n)}^{\alpha} \right] \end{aligned} \tag{25}$$

and

$$\begin{aligned} f_{m+1:n}(\widetilde{x}) = & \frac{(2m+1)!}{m!m!} (F(\widetilde{x}))^m (1 - F(\widetilde{x}))^m f(\widetilde{x}) \\ = & \frac{(2m+1)!}{m!m!} \left\{ \left[1 - h_{(m+1)}^{\alpha} \right] \left[1 + \lambda h_{(m+1)}^{\alpha} \right] \right\}^m \\ & \times \left\{ 1 - \left[1 - h_{(m+1)}^{\alpha} \right] \left[1 + \lambda h_{(m+1)}^{\alpha} \right] \right\}^m \\ & \times \alpha \beta \theta^{\beta} x_{(m+1)}^{-(1+\beta)} (1 - h_{(m+1)}) h_{(m+1)}^{\alpha-1} \\ & \times \left[(1-\lambda) + 2\lambda h_{(m+1)}^{\alpha} \right] \end{aligned} \tag{26}$$

where

$$h_{(s)} = \left(1 - e^{-\left(\frac{\theta}{x_{(s)}} \right)^{\beta}} \right)$$

5.2 Joint Distribution of the i^{th} and j^{th} Order Statistics

The joint distribution of the i^{th} and j^{th} order statistics from transmuted Exponentiated Frêchet (TEF) distribution is

$$\begin{aligned} f_{i:j:n}(x_i,x_j) =& C\ [F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-i-1} \\ &\times [1-F(x_j)]^{n-j} f(x_i) f(x_j) \\ =& C\left\{\left[1-h_{(i)}^{\alpha}\right]\left[1+\lambda h_{(i)}^{\alpha}\right]\right\}^{i-1} \\ &\times \left\{\left[1-h_{(j)}^{\alpha}\right]\left[1+\lambda h_{(j)}^{\alpha}\right]-\left[1-h_{(i)}^{\alpha}\right]\left[1+\lambda h_{(i)}^{\alpha}\right]\right\}^{j-i-1} \\ &\times \left\{1-\left[1-h_{(j)}^{\alpha}\right]\left[1+\lambda h_{(j)}^{\alpha}\right]\right\}^{n-j} \\ &\times \alpha\beta\theta^{\beta}x_{(i)}^{-(1+\beta)}(1-h_{(i)})h_{(i)}^{\alpha-1}\left[(1-\lambda)+2\lambda h_{(i)}^{\alpha}\right] \\ &\times \alpha\beta\theta^{\beta}x_{(j)}^{-(1+\beta)}(1-h_{(j)})h_{(j)}^{\alpha-1}\left[(1-\lambda)+2\lambda h_{(j)}^{\alpha}\right] \end{aligned} \tag{27}$$

6 Reliability Characteristics

The reliability function (RF) of the transmuted Exponentiated Frêchet distribution is denoted by $\overline{F}_{TEF}(x)$ also known as the survivor function and is defined as

$$\begin{aligned} \overline{F}_{TEF}(x) =& 1-F_{TEF}(x) \\ =& 1-\left[1-\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right]\left[1+\lambda\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right] \end{aligned} \tag{28}$$

It is important to note that $\overline{F}_{TEF}(x)+F_{TEF}(x)=1$. For the Transmuted Exponentiated Frêchet (TEF) distribution the hazard function is given by

$$h_{TEF}(x)=\frac{f_{TEF}(x)}{1-F_{TEF}(x)}$$

$$h_{TEF}(x)=\frac{A(\underline{\omega})\times B(\underline{\omega})}{C(\underline{\omega})} \tag{29}$$

where,

$$\underline{\omega}=(\alpha,\beta,\theta,\lambda),$$

$$A(\underline{\omega})=\alpha\beta\theta^{\beta}x^{-(1+\beta)}e^{-\left(\frac{\theta}{x}\right)^{\beta}}\left[1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]^{\alpha-1},$$

$$B(\underline{\omega})=\left[(1-\lambda)+2\lambda\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right],$$

and

$$C(\underline{\omega})=1-\left[1-\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right]\left[1+\lambda\left(1-e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right]$$

Figure 2 illustrates the behaviour of the hazard rate function of a transmuted Exponentiated Frêchet distribution for the different choices of parameters λ,α,β and fixing $\theta=1$.

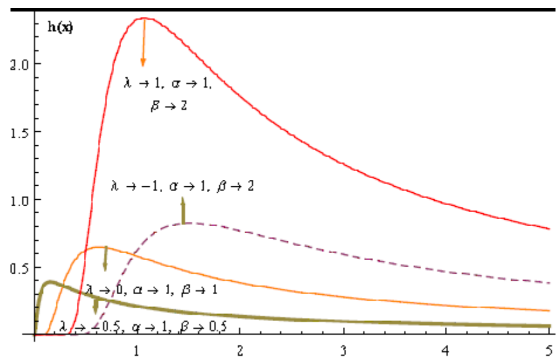

Fig. 2: Hazard rate function of Transmuted Exponentiated Frêchet distribution for $\theta = 1$ and different values of λ, α and β
.

Theorem 6.1. If $\alpha = \theta = \lambda = 1$, then the failure rate is increasing if $\beta < 0$ and is decreasing if $\beta > 0$.

Proof. If $\alpha = \theta = \lambda = 1$ then

$$h(x) = \frac{2\beta\left(\frac{1}{x}\right)^{1+\beta}}{\left(e^{\left(\frac{1}{x}\right)^{\beta}} - 1\right)}$$

which is increasing for $\beta < 0$ and is decreasing for $\beta > 0$.

Theorem 6.2. If $\beta = \alpha = 1$ then the failure rate is monotonically decreasing for both $\lambda < 0$ and $\lambda > 0$.

Proof. If $\beta = \alpha = 1$ then we have (Figure 3)

$$h(x) = \frac{\theta}{x^2}\left[\frac{1}{\left(e^{\left(\frac{\theta}{x}\right)} - 1\right)} + \frac{\lambda}{\left(e^{\left(\frac{\theta}{x}\right)} - \lambda\right)}\right]$$

It can be easily verified that $h(x)$ is decreasing for both $\lambda < 0$ and $\lambda > 0$. Note that

$$h(0) = \infty \;\; and \;\; h(\infty) = 0$$

The cumulative hazard function of the transmuted Exponentiated Frêchet distribution is denoted by $H_{TEF}(x)$ and is defined as

$$H_{TEF}(x) = -\ln\left|\left[1 - \left(1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right]\left[1 + \lambda\left(1 - e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right)^{\alpha}\right]\right| \tag{30}$$

Similar to the hazard rate function, we can also illustrate the behaviour of the cumulative hazard rate function for different choices of parameters.

7 Least Squares and Weighted Least Squares Estimators

In this section we provide the regression based estimators of the unknown parameters of the transmuted Exponentiated Frêchet distribution which was originally suggested by Swain, Venkatraman and Wilson [22] to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from a distribution function $G(.)$ and suppose $Y_{(i)}$; $i = 1, 2, \ldots, n$ denotes the ordered sample. The proposed method uses the distribution of $G(Y_{(i)})$. For a sample of size n, we have $E(G(Y_{(j)})) = \frac{j}{n+1}$, $V((G(Y_{(j)})) = \frac{j(n-j+1)}{(n+1)^2(n+2)}$ and

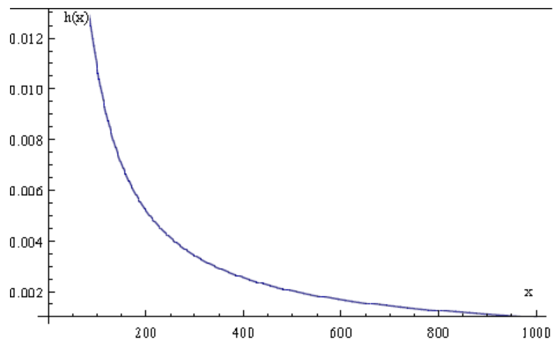

Fig. 3: Hazard rate function of Transmuted Exponentiated Frêchet distribution for $\alpha=\beta=1$

.

$Cov(G(Y_{(j)}),G(Y_{(k)}))=\frac{j(n-k+1)}{(n+1)^2(n+2)}$; for $j<k$, see Johnson, Kotz and Balakrishnan [23]. Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1 (Least Squares Estimators). Obtain the estimators by minimizing

$$\sum_{j=1}^{n}\left(G(Y_{(j)}-\frac{j}{n+1}\right)^2 \tag{31}$$

with respect to the unknown parameters. Therefore in case of TQL distribution the least squares estimators of θ,β,α and λ, say $\widehat{\theta}_{LSE},\widehat{\beta}_{LSE},\widehat{\alpha}_{LSE}$, and $\widehat{\lambda}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left\{\left[1-e^{-\theta x_{(j)}}\left[1+\frac{\theta x_{(j)}}{\alpha+1}\right]\right]\times N(j)-\frac{j}{n+1}\right\}^2$$

where,

$$N(j)=\left[(1-\lambda)+2\lambda e^{-\theta x_{(j)}}(1+\frac{\theta x_{(j)}}{\alpha+1})\right]$$

with respect to θ,β,α and λ.

Method 2 (Weighted Least Squares Esimators). The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left(G(Y_{(j)}-\frac{j}{n+1}\right)^2 \tag{32}$$

with respect to the unknown parameters, where

$$w_j=\frac{1}{V(G(Y_{(j)}))}=\frac{(n+1)^2(n+2)}{j(n-k+1)}$$

Therefore, in case of TEF distribution the weighted least squares estimators of θ,β,α and λ, say, $\widehat{\theta}_{WLSE},\widehat{\beta}_{WLSE},\widehat{\alpha}_{WLSE}$, and $\widehat{\lambda}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left\{\left[1-e^{-\theta x_{(j)}}\left[1+\frac{\theta x_{(j)}}{\alpha+1}\right]\right]\times P(j)-\frac{j}{n+1}\right\}^2$$

where,

$$P(j)=\left[(1-\lambda)+2\lambda e^{-\theta x_{(j)}}(1+\frac{\theta x_{(j)}}{\alpha+1})\right]$$

with respect to the unknown parameters only.

8 Estimation and Inference

In this section, we determine the maximum likelihood estimates $(MLEs)$ of the parameters of the TEF distribution from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size n from $TEF(\theta,\beta,\alpha,\lambda)$ is given by

$$\begin{aligned}
LogL =& n\log\alpha + n\log\beta + n\beta\log\theta \\
&-(1+\beta)\sum_{i=1}^{n}\log x_{(i)} - \theta^{\beta}\sum_{i=1}^{n}x_{(i)}^{-\beta} \\
&+(\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right] \\
&+\sum_{i=1}^{n}\log\left[(1-\lambda)+2\lambda\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha}\right]
\end{aligned} \tag{33}$$

The log-likelihood can be maximized either directly or by solving the non-linear likelihood equations obtained by differentiating (33). The components of the score vector are given by

$$\begin{aligned}
\frac{\partial LogL}{\partial\theta} =& \frac{n\beta}{\theta} - \beta\theta^{\beta-1}\sum_{i=1}^{n}x_i^{-\beta} + \beta(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\left(\frac{\theta}{x_{(i)}}\right)^{\beta-1}}{x_{(i)}\left[1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right]} \\
&+\sum_{i=1}^{n}\frac{2\lambda\beta\alpha\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha-1}e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\left(\frac{\theta}{x_{(i)}}\right)^{\beta-1}}{x_{(i)}\left[(1-\lambda)+2\lambda\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha}\right]},
\end{aligned} \tag{34}$$

$$\begin{aligned}
\frac{\partial LogL}{\partial\alpha} =& \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right] \\
&+\sum_{i=1}^{n}\frac{2\lambda\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha}\log\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)}{\left[(1-\lambda)+2\lambda\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha}\right]},
\end{aligned} \tag{35}$$

$$
\begin{aligned}
\frac{\partial LogL}{\partial \beta} = & \frac{n}{\beta} + n\log\theta - \sum_{i=1}^{n}\log x_{(i)} - \sum_{i=1}^{n}\log\left(\frac{\theta}{x_{(i)}}\right)\left(\frac{\theta}{x_{(i)}}\right)^{\beta} \\
& + (\alpha-1)\sum_{i=1}^{n}\frac{\log\left(\frac{\theta}{x_{(i)}}\right)e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}{\left[1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right]} \\
& + 2\lambda\alpha\sum_{i=1}^{n}\frac{Q(x_{(i)})}{\left[(1-\lambda)+2\lambda\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha}\right]},
\end{aligned} \tag{36}
$$

where,

$$
Q(x_{(i)}) = \left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha-1}\log\left(\frac{\theta}{x_{(i)}}\right)e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\left(\frac{\theta}{x_{(i)}}\right)^{\beta}
$$

and

$$
\frac{\partial LogL}{\partial \lambda} = \sum_{i=1}^{n}\frac{2\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha}-1}{\left[(1-\lambda)+2\lambda\left(1-e^{-\left(\frac{\theta}{x_{(i)}}\right)^{\beta}}\right)^{\alpha}\right]} = 0. \tag{37}
$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting the above non-linear equations (34) and (37) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. This has been explained through simulation study and a real time data analysis in the following sections. Also, all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$
\begin{pmatrix}\hat{\theta}\\ \hat{\beta}\\ \hat{\alpha}\\ \hat{\lambda}\end{pmatrix} \sim N\left[\begin{pmatrix}\theta\\ \beta\\ \alpha\\ \lambda\end{pmatrix}, \begin{pmatrix}\widehat{V_{\theta\theta}} & \widehat{V_{\theta\beta}} & \widehat{V_{\theta\alpha}} & \widehat{V_{\theta\lambda}}\\ \widehat{V_{\beta\theta}} & \widehat{V_{\beta\beta}} & \widehat{V_{\beta\alpha}} & \widehat{V_{\beta\lambda}}\\ \widehat{V_{\alpha\theta}} & \widehat{V_{\alpha\beta}} & \widehat{V_{\alpha\alpha}} & \widehat{V_{\alpha\lambda}}\\ \widehat{V_{\lambda\theta}} & \widehat{V_{\lambda\beta}} & \widehat{V_{\lambda\alpha}} & \widehat{V_{\lambda\lambda}}\end{pmatrix}\right] \tag{38}
$$

$$
V^{-1} = -E\begin{bmatrix} V_{\theta\theta} & V_{\theta\beta} & V_{\theta\alpha} & V_{\theta\lambda}\\ & V_{\beta\beta} & V_{\beta\alpha} & V_{\beta\lambda}\\ & & V_{\alpha\alpha} & V_{\alpha\lambda}\\ & & & V_{\lambda\lambda}\end{bmatrix}
$$

where

$$
\begin{aligned}
V_{\theta\theta} &= \frac{\partial^2 LogL}{\partial\theta^2}, V_{\beta\beta} = \frac{\partial^2 LogL}{\partial\beta^2}, V_{\alpha\alpha} = \frac{\partial^2 LogL}{\partial\alpha^2}, V_{\lambda\lambda} = \frac{\partial^2 LogL}{\partial\lambda^2}\\
V_{\alpha\theta} &= \frac{\partial^2 LogL}{\partial\alpha\partial\theta}, V_{\alpha\beta} = \frac{\partial^2 LogL}{\partial\alpha\partial\beta}, V_{\theta\beta} = \frac{\partial^2 LogL}{\partial\theta\partial\beta}, V_{\theta\lambda} = \frac{\partial^2 LogL}{\partial\theta\partial\lambda}
\end{aligned}
$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariances of these ML estimators for $\hat{\lambda}, \hat{\theta}, \hat{\alpha}$and $\hat{\beta}$. Using (38), we approximate $100(1-\alpha)\%$ confidence intervals for λ, β, θ and α are determined respectively as

$$
\hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\theta\theta}}},\ \hat{\beta} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\beta\beta}}},\ \hat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\alpha\alpha}}}\ and\ \hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\lambda\lambda}}}
$$

where z_{γ} is the upper 100_{γ} the percentile of the standard normal distribution.

9 Simulation Study

Here a simulation study to determine the biases, standard deviations, and Root Mean Squared Errors(RSMEs) of the estimators discussed in previous section is presented. We consider a simulate sample of sizes $n = 50, 100, 150, 200$ for the initial values of $\theta = 1.25$, $\alpha = 0.75$, $\beta = 1.5$ and $\lambda = 0.4$. For each combination of β, θ,α, λ,and n we performed 100 replications of the simulation. The results are presented in Table 1.

Examining from Table 1, we observe that as the sample size increases, biases, standard deviations and RSMEs of $\hat{\theta}$, $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\lambda}$ decrease steadily.

Table 1: Mean, Bias,Standard Deviation and RSME of θ, α, β and λ Based on 100 Replications for Transmuted Exponentiated Frêchet distribution obtained using *MaximumLikelihood* method

Sample Size (n)	Parameter Estimates	Mean	Bias	Standard Deviation	RSME
50	$\widehat{\theta}$	1.590291	0.340291	1.560374	2.22248620
	$\widehat{\alpha}$	0.913857	0.163857	1.105894	1.43035176
	$\widehat{\beta}$	1.767180	0.267180	0.914291	1.98758547
	$\widehat{\lambda}$	0.619600	0.219600	0.416374	0.74534361
100	$\widehat{\theta}$	1.445665	0.195665	2.120037	2.55725595
	$\widehat{\alpha}$	0.689613	0.060387	0.412042	0.80227600
	$\widehat{\beta}$	1.666431	0.166431	0.626099	1.77906600
	$\widehat{\lambda}$	0.555016	0.155016	0.455785	0.71637200
150	$\widehat{\theta}$	1.752899	0.502899	4.948343	5.2265015
	$\widehat{\alpha}$	0.824261	0.074261	1.902299	2.06453875
	$\widehat{\beta}$	1.836286	0.336286	0.724880	1.97286495
	$\widehat{\lambda}$	0.571191	0.171191	0.289819	0.63986206
200	$\widehat{\theta}$	1.79253	0.54253	4.27201	4.61330057
	$\widehat{\alpha}$	0.647814	0.102186	1.154068	1.318464614
	$\widehat{\beta}$	1.753049	0.253049	0.837058	1.940852968
	$\widehat{\lambda}$	0.477469	0.077469	0.414893	0.631196067

10 Data Analysis

In this section we will analyse a real data set to explain the appropriateness of the transmuted Exponentiated Frêchet distribution for modelling wind speed data. We will provide comparison of the results with the Frêchet and exponentiated Frêchet distribution. Note that Frêchet distribution is a sub-model of both the transmuted Exponentiated Fêchet and Exponentiated Frêchet distribution.

The data used for the present study were obtained from a yearly published book at Permerhatian Cuaca Harian Pusat Pengajian Sosial, Pembangunan & Persekitaran (PPSPP), Fakulti Sains Sosial, & Kemanusiaan (FSSK), Universiti Kebangsaan Malaysia (UKM) during the year 2004 to 2006, Zaharim et. al.[15]. The data observation was done by positioning a rotating cup type anemometer on the station in open spaces free of obstacle at 3 meters height up on the Cameron Highland, Malaysia. Wind speeds were observed every 10 seconds and averaged over 5 minutes period. The 5-minutes averaged data were further averaged over one hour. At the end of each hour, the hourly mean wind speed was calculated and stored sequentially in a permanent memory. Knowledge of the statistical properties of wind speed is essential for predicting the energy output of a wind energy conversion system.

We fit the data into Frêchet distribution, Exponentiated Frêchet distribution, Transmuted Exponentiated Frêchet distribution and made comparison with the estimated parameter values. The results are presented in Table 2.

Table 2: Estimated Parameters of Frêchet, Exponentiated Frêchet and Transmuted Exponentiated Frêchet Distributions

Year	Distribution	Parameter Estimates				Log-likelihood
		$\widehat{\beta}$	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\lambda}$	
2004	Frêchet	3.251	1.569			-10.811
	Exponentiated Frêchet	23.4546	5.79241	1.00154		-7.19111×10^{18}
	Transmuted Exponentiated Frêchet	1.12663	4.64288	10.6478	0.720222	-9.32402
2005	Frêchet	3.000	1.262			-13.333
	Exponentiated Frêchet	93.4345	9.24088	1.01293		-2.81892×10^{72}
	Transmuted Exponentiated Frêchet	28.15	1.59569	0.162663	0.380465	-1.12647
2006	Frêchet	1.9226	1.024			-19.4303
	Exponentiated Frêchet	90.5402	9.7208	1.003		-2.4794×10^{71}
	Transmuted Exponentiated Frêchet	1.9135	3.4811	9.88677	0.386237	-6.65192

Transmuted Exponentiated Frêchet distribution fits the subject data better than the 2-parameter Frêchet and Exponentiated Frêchet distribution. The Kolmogorov-Smirnov test confirms that with test statistics for the year 2004 is 0.113489, for 2005 is 0.186626 and for 2006 is 0.170071. There fore for all the 3 years the wind speed data fits well for transmuted Exponentiated Frêchet distribution.

11 Conclusions

In the present paper, we have proposed a new generalization of the Exponentiated Frêchet distribution called the transmuted Exponentiated Frêchet distribution. The distribution of interest is generated by using the general rank transmutation map and taking Exponentiated Frêchet distribution as the base distribution. Maximum Likelihood estimation method is used to estimate the parameters involved. The reliability behaviour of the subject distribution is studied. We have studied the Malaysian wind speed data set published in the literature to show the usefulness of the transmuted Exponentiated Frêchet distribution and make comparison with Frêchet and exponentiated Frêchet distribution. The Kolmogorov-Smirnov test revealed that the transmuted Exponentiated Frêchet distribution fits well for the Malaysian wind speed data. We are hoping that the present study will help as a reference and serve to enhance future research in this direction.

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