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Two Dimensional Harmonic Quasi Convex Functions and Integral Inequalities

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Abstract: The objective of this paper is to introduce the notion of two-dimensional harmonic quasi-convex functions. We obtain some new Hermite-Hadamard type integral inequalities for two-dimensional harmonic quasi convex functions. Some special cases are also discussed.

Keywords: Convex functions, harmonic quasi convex, two dimensional, Hermite-Hadamard inequality.
2010 AMS Subject Classification: 26D15, 26A51

1 Introduction and Preliminaries

In recent years, much attention has been given to convexity theory, which plays an important and crucial role in the development of various fields of pure and applied sciences. Several new generalizations and extensions of classical convexity theory have been proposed using novel and innovative ideas, see [1,2,4,5,6,7,8]. Iscan [9] introduced the harmonic convex functions and derived some integral inequalities of Hermite-Hadamard type.

Definition 1([9]). *A function $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x)+tf(y), \quad \forall x,y \in I, t \in [0,1].$$

It is worth to mention here that harmonicity plays a vital role in different fields of pure and applied sciences. For example, in [3] authors have discussed the significance of harmonic mean in Asian options of stock. See also [10]. It is worth mentioning that harmonic means have applications in electrical circuits theory. To be more precise, the total resistance of a set of parallel resistors is just half of harmonic means of the total resistors. For example, if R_1 and R_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula: $R_T = \frac{1}{2}\mathscr{H}(R_1,R_2)$, which is half the harmonic mean. Harmonic means also play crucial role in the development of parallel algorithms for solving non-linear problems. Noor [11] used the harmonic means and harmonic convex functions to suggest some iterative methods for solving linear and nonlinear system of equations. Another significant and interesting application of harmonic means is in the field of numerical analysis, as one can obtain variants of iterative methods for solving nonlinear equations using harmonic means. For more details, see [13,14].
Recently Noor et al. [12] extended the concept of harmonic convexity to two-dimensions.

Definition 2([12]). *Consider the rectangle $\Delta = [a,b]\times[c,d] \subset \mathbb{R}^2_+$. A function $f : \Delta \to \mathbb{R}$ is said to be harmonic convex function on Δ, if*

$$f\left(\frac{xy}{(1-\lambda)x+\lambda y}, \frac{uw}{(1-\lambda)u+\lambda w}\right) \leq \lambda f(x,u)+(1-\lambda)f(y,w),$$

whenever $x,y \in [a,b]$, $x,w \in [c,d]$ and $\lambda, r \in [0,1]$.

Zhang et al. [15] introduced the class of harmonic quasi convex functions.

Definition 3([15]). *A function $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic quasi convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \max\{f(x), f(y)\},$$

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$$\forall x,y \in I, t \in [0,1].$$

For more details and interesting results pertaining to harmonic convexity of the functions, see [1,2,10,13].

Hermite-Hadamard's integral inequality has attracted several researchers due to its simplicity. It is one of the most intensively-studied result of classical convexity. It provides a necessary and sufficient condition for a function to be convex. This result reads as:

Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Iscan [9] extended this result for harmonic convex function.

Let $f : I = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an harmonic convex function, then

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

For more details on Hermite-Hadamard like inequalities, see [6,7,8].

Motivated by the research going on, we introduce two-dimensional harmonic quasi convex functions. We also derive some new integral inequalities of Hermite-Hadamard type via this new class of harmonic convexity. First of all, we define the class of two-dimensional harmonic quasi convex functions.

Definition 4. *Consider the rectangle $\Delta = [a,b]\times[c,d] \subset \mathbb{R}^2_+$. A function $f : \Delta \to \mathbb{R}$ is said to be two-dimensional harmonic quasi convex function on Δ, if*

$$f\left(\frac{xz}{(1-r)x+rz}, \frac{yw}{(1-r)y+rw}\right) \leq \max\{f(x,y), f(z,w)\},$$

whenever $(x,y),(z,w) \in \Delta$ and $r \in [0,1]$.

The following auxiliary result plays an important part in obtaining our main results.

Lemma 1([12]). *Let $f : \Delta \to \mathbb{R}$ be partial differentiable function on $\Delta = [a,b]\times[c,d]$ in $\mathbb{R}^2_+$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial\lambda\partial r} \in L_1(\Delta)$, then*

$$\begin{aligned}
&\Xi(a,b,c,d,x,y;\Delta)\\
&= \frac{ab(b-a)cd(d-c)}{4}\\
&\times \int_0^1\int_0^1 \left(\frac{1-2\lambda}{(\lambda b+(1-\lambda)a)^2}\right)\left(\frac{1-2r}{(rd+(1-r)c)^2}\right)\\
&\times \frac{\partial^2 f}{\partial r\partial\lambda}\left(\frac{ab}{\lambda b+(1-\lambda)a}, \frac{cd}{rd+(1-r)c}\right)\mathrm{d}r\mathrm{d}\lambda,
\end{aligned}$$

where

$$\begin{aligned}
&\Xi(a,b,c,d,x,y;\Delta)\\
&= \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\\
&-\frac{1}{2}\left[\frac{ab}{b-a}\left\{\int_a^b \frac{f(x,c)}{x^2}\mathrm{d}x + \int_a^b \frac{f(x,d)}{x^2}\mathrm{d}x\right\}\right.\\
&\left.+\frac{cd}{d-c}\left\{\int_c^d \frac{f(a,y)}{y^2}\mathrm{d}y + \int_c^d \frac{f(b,y)}{y^2}\mathrm{d}y\right\}\right]\\
&+\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{f(x,y)}{x^2y^2}\mathrm{d}x.
\end{aligned}$$

2 Integral Inequalities

In this section, we derive some new Hermite-Hadamard type integral inequalities essentially using Lemma 1 for two-dimensional harmonic quasi convex functions.

Theorem 1. *Let $f : \Delta \to \mathbb{R}$ be partial differentiable function on $\Delta = [a,b]\times[c,d]$ in $\mathbb{R}^2_+$ with $a < b$ and $c < d$ and $\frac{\partial^2 f}{\partial\lambda\partial r} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial\lambda}\right|$ is two-dimensional harmonic quasi convex function on Δ, then*

$$\begin{aligned}
&\left|\Xi(a,b,c,d,x,y;\Delta)\right|\\
&\leq \frac{ab(b-a)cd(d-c)}{4}\\
&\times\left\{\left(\frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right)\right)\right.\\
&\left.\times\left(\frac{1}{cd} - \frac{2}{(d-c)^2}\ln\left(\frac{(c+d)^2}{4cd}\right)\right)\right\}\\
&\times\max\left\{\left|\frac{\partial^2 f}{\partial r\partial\lambda}(a,b)\right|, \left|\frac{\partial^2 f}{\partial r\partial\lambda}(c,d)\right|\right\}.
\end{aligned}$$

Proof. Using Lemma 1 and the fact that $\left|\frac{\partial^2 f}{\partial r\partial\lambda}\right|$ is two-dimensional harmonic quasi convex function, we have

$$\begin{aligned}
&\left|\Xi(a,b,c,d,x,y;\Delta)\right|\\
&= \left|\frac{ab(b-a)cd(d-c)}{4}\right.\\
&\times\int_0^1\int_0^1 \left(\frac{1-2\lambda}{(\lambda b+(1-\lambda)a)^2}\right)\left(\frac{1-2r}{(rd+(1-r)c)^2}\right)\\
&\left.\times\frac{\partial^2 f}{\partial r\partial\lambda}\left(\frac{ab}{\lambda b+(1-\lambda)a}, \frac{cd}{rd+(1-r)c}\right)\mathrm{d}r\mathrm{d}\lambda\right|
\end{aligned}$$

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$$
\begin{aligned}
&\leq \frac{ab(b-a)cd(d-c)}{4}\\
&\times \int_0^1\int_0^1 \left(\frac{|1-2\lambda|}{(\lambda b+(1-\lambda)a)^2}\right)\left(\frac{|1-2r|}{(rd+(1-r)c)^2}\right)\\
&\times \left|\frac{\partial^2 f}{\partial r\partial\lambda}\left(\frac{ab}{\lambda b+(1-\lambda)a},\frac{cd}{rd+(1-r)c}\right)\right| \mathrm{d}r\mathrm{d}\lambda\\
&\leq \frac{ab(b-a)cd(d-c)}{4}\\
&\times \int_0^1\int_0^1 \left(\frac{|1-2\lambda|}{(\lambda b+(1-\lambda)a)^2}\right)\left(\frac{|1-2r|}{(rd+(1-r)c)^2}\right)\\
&\qquad\times \max\left\{\left|\frac{\partial^2 f}{\partial r\partial\lambda}(a,b)\right|,\left|\frac{\partial^2 f}{\partial r\partial\lambda}(c,d)\right|\right\}\mathrm{d}r\mathrm{d}\lambda\\
&= \frac{ab(b-a)cd(d-c)}{4}\\
&\times \left\{\left(\frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right)\right)\right.\\
&\times \left.\left(\frac{1}{cd}-\frac{2}{(d-c)^2}\ln\left(\frac{(c+d)^2}{4cd}\right)\right)\right\}\\
&\qquad\times \max\left\{\left|\frac{\partial^2 f}{\partial r\partial\lambda}(a,b)\right|,\left|\frac{\partial^2 f}{\partial r\partial\lambda}(c,d)\right|\right\}.
\end{aligned}
$$

This completes the proof. □

Theorem 2. *Let $f:\Delta\to\mathbb{R}$ be partial differentiable function on $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2_+$ with $a<b$ and $c<d$ and $\frac{\partial^2 f}{\partial\lambda\partial r}\in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial\lambda}\right|^q$ is two-dimensional harmonic quasi convex function, where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, then*

$$
\begin{aligned}
&\left|\Xi(a,b,c,d,x,y;\Delta)\right|\\
&\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\\
&\quad\times\left(\left(\frac{b^{1-2q}-a^{1-2q}}{(b-a)(1-2q)}\right)\times\left(\frac{d^{1-2q}-c^{1-2q}}{(d-c)(1-2q)}\right)\right.\\
&\quad\times\left.\left[\max\left\{\left|\frac{\partial^2 f}{\partial r\partial\lambda}(a,b)\right|^q,\left|\frac{\partial^2 f}{\partial r\partial\lambda}(c,d)\right|^q\right\}\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Using Lemma 1, Hölder's inequality and the fact that $\left|\frac{\partial^2 f}{\partial r\partial\lambda}\right|^q$ is two-dimensional harmonic quasi convex function, we have

$$
\begin{aligned}
&\left|\Xi(a,b,c,d,x,y;\Delta)\right|\\
&=\left|\frac{ab(b-a)cd(d-c)}{4}\right.\\
&\quad times\int_0^1\int_0^1\left(\frac{1-2\lambda}{(\lambda b+(1-\lambda)a)^2}\right)\left(\frac{1-2r}{(rd+(1-r)c)^2}\right)\\
&\qquad\times\left.\frac{\partial^2 f}{\partial r\partial\lambda}\left(\frac{ab}{\lambda b+(1-\lambda)a},\frac{cd}{rd+(1-r)c}\right)\mathrm{d}r\mathrm{d}\lambda\right|\\
&\leq \frac{ab(b-a)cd(d-c)}{4}\left(\int_0^1\int_0^1|(1-2\lambda)(1-2r)|^p\mathrm{d}r\mathrm{d}\lambda\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\int_0^1\frac{1}{(\lambda b+(1-\lambda)a)^{2q}}\frac{1}{(rd+(1-r)c)^{2q}}\right.\\
&\times\left.\left|\frac{\partial^2 f}{\partial r\partial\lambda}\left(\frac{ab}{\lambda b+(1-\lambda)a},\frac{cd}{rd+(1-r)c}\right)\right|^q\mathrm{d}r\mathrm{d}\lambda\right)^{\frac{1}{q}}\\
&\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\\
&\times\left(\int_0^1\int_0^1\left(\frac{1}{(\lambda b+(1-\lambda)a)^{2q}}\right)\left(\frac{1}{(rd+(1-r)c)^{2q}}\right)\right.\\
&\quad\times\left.\left[\max\left\{\left|\frac{\partial^2 f}{\partial r\partial\lambda}(a,b)\right|^q,\left|\frac{\partial^2 f}{\partial r\partial\lambda}(c,d)\right|^q\right\}\right]\mathrm{d}r\mathrm{d}\lambda\right)^{\frac{1}{q}}\\
&\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\\
&\times\left(\left(\frac{b^{1-2q}-a^{1-2q}}{(b-a)(1-2q)}\right)\times\left(\frac{d^{1-2q}-c^{1-2q}}{(d-c)(1-2q)}\right)\right.\\
&\quad\times\left.\left[\max\left\{\left|\frac{\partial^2 f}{\partial r\partial\lambda}(a,b)\right|^q,\left|\frac{\partial^2 f}{\partial r\partial\lambda}(c,d)\right|^q\right\}\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof. □

3 Conclusion

In this paper, we have introduced and studied a new class of two dimensional harmonic quasi convex functions. We have derived some new integral inequalities of Hermite-Hadamard type involving two-dimensional harmonically quasi convex functions.

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