Uni-soft Substructures of Rings and Modules

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Abstract: In this paper, we introduce union soft subrings and union soft ideals of a ring and union soft submodules of a left module and investigate their related properties with respect to soft set operations, anti image and lower α-inclusion of soft sets. We also obtain significant relation between soft subrings and union soft subrings, soft ideals and union soft ideals, soft submodules and union soft submodules.

Keywords: Soft sets, union soft subrings (ideals), union soft submodules, anti image, α-inclusion.

1 Introduction

The notion of soft set was introduced in 1999 by Molodtsov [28] as a new mathematical tool for dealing with uncertainties. Since its inception, it has received much attention in the mean of algebraic structures such as groups [2], semirings [11], rings [1], BCK/BCI-algebras [16,17,18], d-algebras [19], ordered semigroups [20], BL-algebras [33], BCH-algebras [22] and near-rings [31]. Moreover, Xiao et al. [32] proposed the notion of exclusive disjunctive soft sets and studied some of its operations and Gong et al. [15] studied the bijective soft set with its operations. Atagün and Sezgin defined the concepts of soft subrings and ideals of a ring, soft subfields of a field, soft submodules of a module. They studied their properties especially with respect to soft set operations in more detail. In this paper, first we extend Atagün and Sezgin’s study [4] by focusing on soft subrings and ideals of a ring and soft submodules of a module with respect to image, preimage and upper α-inclusion of soft sets. We then introduce union soft subrings and ideals of a ring and union soft submodules of a left module and investigate their related properties with respect to soft set operations, anti image and lower α-inclusion of soft sets. Moreover, we obtain relations between soft subrings and union soft subrings, soft ideals and union soft ideals and soft submodules and union soft submodules. The union soft set theory (in a few algebraic structures) is also studied in the following papers [21,23,24].

2 Preliminaries

Throughout this paper, R will always denote a ring with zero 0R, M a left R-module with identity 0M and N a left submodule of M. Let U be an initial universe set, E be a set of parameters, P(U) be the power set of U and A ⊆ E.
Definition 1. [28] If $F$ is a mapping given by $F : A \rightarrow P(U)$, then the set $F_A = \{(x, F(x)) : x \in A\}$ is called a soft set over $U$.

Definition 2. [3] The relative complement of a soft set $F_A$ over $U$ is denoted by $F_A^c$, where $F_A^c : A \rightarrow P(U)$ is a mapping given as $F_A^c(\alpha) = U \setminus F_A(\alpha)$ for all $\alpha \in A$.

Definition 3. [8, 9] Let $F_A$ and $G_B$ be soft sets over $U$ and $\Psi$ be a function from $A$ to $B$. Image of $F_A$ under $\Psi$ and anti image of $F_A$ under $\Psi$ are the soft sets $\Psi(F_A)$ and $\Psi^*(F_A)$, where $\Psi(F_A) : B \rightarrow P(U)$ and $\Psi^*(F_A) : B \rightarrow P(U)$ are set-valued functions defined as if $\Psi^{-1}(b) \neq \emptyset$, then $\Psi(F_A)(b) = \bigcup\{F(a) : a \in A \text{ and } \Psi(a) = b\}$, otherwise $\Psi(F_A)(b) = \emptyset$, and if $\Psi^{-1}(b) = \emptyset$, then $\Psi^*(F_A)(b) = \bigcap\{F(a) : a \in A \text{ and } \Psi(a) = b\}$, otherwise $\Psi^*(F_A)(b) = \emptyset$ for all $b \in B$, respectively. Preimage (or inverse image) of $G_B$ under $\Psi$ is the soft set $\Psi^{-1}(G_B)$, where $\Psi^{-1}(G_B) : A \rightarrow P(U)$ is a set-valued function defined by $\Psi^{-1}(G_B)(\alpha) = G(\Psi(\alpha))$ for all $\alpha \in A$.

Definition 4. [3] Let $F_A$ and $G_B$ be two soft sets over $U$ such that $A \cap B \neq \emptyset$. The restricted union of $F_A$ and $G_B$ is denoted by $F_A \cup_B G_B$, and is defined as $F_A \cup_B G_B = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cup G(c)$.

Theorem 1. [9] Let $F_I$ and $T_K$ be soft sets over $U$, $F_{I_T}$ be their relative soft sets, respectively and $\Psi$ be a function from $I$ to $K$. Then, $i)$ $\Psi^{-1}(T_K) = (\Psi^{-1}(T_K))^\prime$, $ii)$ $\Psi(F_{I_T}) = (\Psi(F_I))^\prime$ and $\Psi^*(F_{I_T}) = (\Psi(F_K))^\prime$.

Definition 5. [10] Let $F_A$ be a soft set over $U$ and $\alpha$ be a subset of $U$. Then, upper $\alpha$-inclusion of $F_A$, denoted by $F^\leq \alpha$ is defined as $F^\leq \alpha = \{x \in A \mid F(x) \supseteq \alpha\}$. Similarly, $F^\geq \alpha = \{x \in A \mid F(x) \subseteq \alpha\}$, respectively.

Definition 6. [4] Let $S$ be a subring of $R$ and let $F_S$ be a soft set over $R$. Then, $F_S$ is called a soft subring of $R$, denoted by $F_S \subseteq R$, if for all $x, y \in S$, $F(x - y) \supseteq F(x) \cap F(y)$ and $F(xy) \supseteq F(x) \cap F(y)$.

Definition 7. [4] Let $I$ be an ideal of $R$ and let $F_I$ be a soft set over $R$. Then, $F_I$ is called a soft ideal of $R$, denoted by $F_I \subseteq R$, if for all $x, y \in I$ and $r \in R$, $F(x - y) \supseteq F(x) \cap F(y)$, $F(rx) \supseteq F(x)$ and $F(xr) \supseteq F(x)$.

Definition 8. [4] Let $N$ be a submodule of $M$ and $F_N$ be a soft set over $M$. Then, $F_N$ is called a soft submodule of $M$, denoted by simply $F_N \subseteq M$, if for all $x, y \in N$ and $r \in R$, $F(x - y) \supseteq F(x) \cap F(y)$ and $F(rx) \supseteq F(x)$.

3 Some characterizations for soft subrings and soft ideals

In this section, we obtain some significant characterizations for soft subrings and soft ideals of a ring with respect to image, preimage and upper $\alpha$-inclusion of soft sets.
Then, 
\[(\Psi(F_1))(r_j) = \bigcup \{F(i) : i \in I, \Psi(i) = r_j\} = \bigcup \{F(i) : i \in I, i = \Psi^{-1}(r_j)\} = \bigcup \{F(i) : i \in I, i = \Psi^{-1}(\Psi(r_j))\} = \bigcup \{F(i) : i \in I, \Psi(i) = r_j\} = \bigcup \{F(i) : i \in I, i = (\Psi(F_1))(r_j)\} \]

Similarly, one can show that \((\Psi(F_1))(r_j) \subseteq (\Psi(F_2))(r_j)\) for all \(r \in R\) and \(j \in J\). Hence, \(\Psi(F_1)\) is a soft ideal of \(R\).

**Theorem 7.** Let \(F_1\) and \(G_1\) be soft sets over \(R\), where \(I\) and \(J\) are ideals of \(R\) and \(\Psi\) be a ring epimorphism from \(I\) to \(J\). If \(G_2\) is a soft ideal of \(R\), then so is \(\Psi^{-1}(G_2)\).

**Proof.** Let \(i_1, i_2 \in I\), then \(\Psi^{-1}(G_2))((i_1 - i_2) \subseteq (\Psi^{-1}(G_2)))((i_1) \cap (\Psi^{-1}(G_2)))((i_2)\) is satisfied as shown in Theorem 4. Now, let \(r \in R\) and \(i \in I\). Since \(\Psi\) is surjective, there exists \(j \in J\) such that \(\Psi(i) = j\). Then, \(\Psi^{-1}(G_2))((r)) = \Psi(G((r)) = G(\Psi(\Psi(r)) = G(\Psi(\Psi(r)) = (\Psi^{-1}(G_2)))((i))\) and \((\Psi^{-1}(G_2)))((i) \cap (\Psi^{-1}(G_2)))((i)\) is satisfied. Hence, \(\Psi^{-1}(G_2)\) is a soft ideal of \(R\).

**4 Union soft subrings and union soft ideals**

In this section, we introduce union soft subrings and union soft ideals of a ring, investigate their basic properties and establish the relation between soft subrings and union soft subrings as well as soft ideals and union soft ideals.

**Definition 9.** Let \(S\) be a subring of \(R\) and \(F_S\) be a soft set over \(R\). \(F_S\) is called a union soft subring of \(R\), denoted \(F_S \subseteq_u R\), if \(F(x \cdot y) \subseteq F(x) \cup F(y)\) and \(F(xy) \subseteq F(x) \cup F(y)\) for all \(x, y \in S\).

**Example 1.** Given the ring \(R = (\mathbb{Z}_8, +,.)\), \(S = \{0, 2, 4, 6\} \subseteq R\) and the soft set \(S_1\) over \(R\), where \(F : S_1 \rightarrow P(R)\) is a set-valued function defined by \(F(x) = \{y \in \mathbb{Z}_8 : y < x\}\), then \(F(0) = \{0\}, F(2) = F(4) = \{0, 2, 4, 6\}\) and \(F(6) = \{0, 4\}\). Then, one can easily show that \(F_S \subseteq_u R\).

Now, the subring of \(R\) be given as \(S_2 = \{0, 4\}\) and the soft set \(G_{S_2}\) over \(R\), where \(G : S_2 \rightarrow P(R)\) is a set-valued function defined by \(G(0) = \{0, 1, 3, 4, 5\}\) and \(G(1) = \{0, 3, 5\}\). Then, \(G(0) \subseteq G(4)\) and \(G(0) \not\subseteq G(4)\). It follows that \(G_{S_2}\) is not a union soft subring of \(R\).

**Example 2.** Given the ring \(R = M_2(\mathbb{Z}_8)\), i.e., \(2 \times 2\) matrices with \(\mathbb{Z}_8\) terms, with the operations addition and multiplication of matrices. Let \(S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\). It is obvious that \(S\) is a subring of \(R\). Let the soft set \(T_S\) over \(R\), where \(T : S \rightarrow P(R)\) is a set-valued function defined by \(T(0, 0) = \{0, 1, 2, 3, 4, 5\}\) and \(T(3, 0) = \{12, 3, 40, 0, 12\}\).

Then, one can easily show that \(T_S \subseteq_u R\). However, if we define a soft set \(H_S\) over \(R\) such that

\[H(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = \{0, 1, 10, 11\}, H(\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}) = \{0, 1, 11, 12, 04\},\]

then,

\[H(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}) \neq H(\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}) \cup H(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}).\]

Thus, \(H_S\) is not a union soft subring of \(R\).

In [4], Atagün and Sezgin showed that the restricted intersection, the sum and the product of two soft subrings of \(R\) is a soft subring of \(R\). Here, we show that the restricted union of two union soft subrings of \(R\) is a union soft subring of \(R\).

**Theorem 8.** If \(F_{S_1} \subseteq_u R\) and \(F_{S_2} \subseteq_u R\), then \(F_{S_1} \cup F_{S_2} \subseteq_u R\).

**Proof.** Since \(F_{S_1}\) and \(F_{S_2}\) are subrings of \(R\), then \(F_{S_1} \cap F_{S_2}\) is a subring of \(R\). By Definition 4, let \(F_{S_1} \cup F_{S_2} = (F_{S_1} \cap F_{S_2}) = (H(x) \cup H(y))\) for all \(x, y \in S_1 \cap S_2\). Hence, \(F_{S_1} \cup F_{S_2}\) is a union soft subring of \(R\).

**Theorem 9.** If \(F_S \subseteq_u R\), then \(F(0_R) \subseteq F(x)\) for all \(x \in S\).

**Proof.** Since \(F_S\) is a union soft subring of \(R\), then \(F(0_R) \subseteq F(x) \cup F(y) = F(x)\) for all \(x \in S\). Therefore, \(F_S\) is a subring of \(S\).

**Theorem 11.** Let \(F_S\) be a soft set over \(R\) and \(\alpha \subseteq R\) be a subset of \(R\) such that \(F(0_R) \subseteq \alpha\). If \(F_S\) is a union soft subring of \(R\), then \(F_\alpha \subseteq R\) is a subring of \(R\).

**Proof.** Since \(F(0_R) \subseteq \alpha\), then \(0_R \in F_\alpha \subseteq R\) and \(0 \neq F_\alpha \subseteq R\). We need to show that \(x - y \in F_\alpha \subseteq R\) for all \(x, y \in S\). Since \(x, y \in S\), then \(F(x) = F(0_R)\) and \(F(y) = F(0_R)\). By Theorem 9, \(F(0_R) \subseteq F(x) \cup F(y) = F(x)\). Since \(F_S\) is a union soft subring of \(R\), then \(F(x) \subseteq F(x) \cup F(y) = F(0_R)\) and \(F(0_R) \subseteq F(x) \cup F(y) = F(0_R)\), the following theorem gives the relation between soft subrings and union soft subrings of a ring.

**Theorem 12.** Let \(F_S\) be a soft set over \(R\). Then, \(F_S\) is a union soft subring of \(R\) iff \(F_S\) is a soft subring of \(R\).
Proof. Let $F_S$ be a union soft subring of $R$. Then for all $x, y \in R$, $F^*(x - y) = R \setminus (F(x) \cup F(y)) = (R \setminus F(x)) \cap (R \setminus F(y)) = F^*(x) \cap F^*(y)$ and $F^*(xy) = R \setminus (F(xy) \cup F(x) \cup F(y)) = (R \setminus F(x)) \cap (R \setminus F(y)) = F^*(x) \cap F^*(y)$. Thus, $F_S$ is a union soft subring of $R$. The converse can be proved similarly.

**Theorem 13.** Let $F_S$ and $G_T$ be soft sets over $R$, where $S$ and $T$ are subrings of $R$ and $\Psi$ be a ring homomorphism from $S$ to $T$. If $G_T$ is a union soft subring of $R$, then so is $\Psi^{-1}(G_T)$.

Proof. Let $G_T$ be a union soft subring of $R$. Then, $G_T$ is a soft subring of $R$ by Theorem 12 and $\Psi^{-1}(G_T)$ is a soft subring of $R$ by Theorem 4. Thus, $\Psi^{-1}(G_T) = (\Psi^{-1}(G_T))^\prime$ is a soft subring of $R$ by Theorem 1 (i). Therefore, $\Psi^{-1}(G_T)$ is a union soft subring of $R$ by Theorem 12.

**Theorem 14.** Let $F_S$ and $G_T$ be soft sets over $R$, where $S$ and $T$ are subrings of $R$ and $\Psi$ be a ring isomorphism from $S$ to $T$. If $F_S$ is a union soft subring of $R$, then so is $\Psi(F_S)$.

Proof. Let $F_S$ be a union soft subring of $R$. Then, $F_S$ is a soft subring of $R$ by Theorem 12 and $\Psi(F_S)$ is a soft subring of $R$ by Theorem 3. Thus, $\Psi(F_S) = (\Psi(F_S))^\prime$ is a soft subring of $R$ by Theorem 1 (ii). So, $\Psi(F_S)$ is a union soft subring of $R$ by Theorem 12.

**Theorem 15.** Let $R_1$ and $R_2$ be two rings and $F_{S_1} \subseteq_{u} R_1$, $H_{S_2} \subseteq_{u} R_2$. If $f : S_1 \rightarrow S_2$ is a ring homomorphism, then $i) H_{f(S_1)} \subseteq_{u} R_2$ and $F_{Ker f} \subseteq_{u} R_1$, $ii) f$ is an epimorphism, $F_{f^{-1}(S_2)} \subseteq_{u} R_1$.

Proof. $i)$ Since $S_1 < R_1$, $S_2 < R_2$ and $f : S_1 \rightarrow S_2$ is a ring homomorphism, then $f(S_1) < R_2$ and as $f(S_1) \subseteq S_2$, the result is obvious by Definition 9. Moreover, since $Ker f < R_1$ and $Ker f \subseteq S_1$, the rest of the proof is clear by Definition 9. $ii)$ Since $S_1 < R_1$, $S_2 < R_2$ and $f : S_1 \rightarrow S_2$ is a ring epimorphism, then it is clear that $f^{-1}(S_2) < R_1$. Since $S_1 < R_1$ and $f^{-1}(S_2) \subseteq S_1$, $F_1(x - y) = F_1(x) \cup F_1(y)$ and $F_1(xy) = F_1(x) \cup F_1(y)$ for all $x, y \in f^{-1}(S_1)$. This completes the proof.

**Corollary 1.** Let $F_{S_1} \subseteq_{u} R_1$, $H_{S_2} \subseteq_{u} R_2$ and $f : S_1 \rightarrow S_2$ is a ring homomorphism, then $H_{f(S_1)} \subseteq_{u} R_2$.

**Definition 10.** Let $I$ be an ideal of $R$ and let $F_I$ be a soft set over $R$. Then, $F_I$ is called a union soft ideal of $R$, denoted by $F_{I} \subseteq_{u} R$. If $F(x - y) \subseteq F(x) \cup F(y)$, $F(rx) \subseteq F(x)$ and $F(xr) \subseteq F(x)$ for all $x, y \in I$ and $r \in R$.

**Example 4.** Consider the ring $R = \mathbb{Z}_{16}$, the ideal of $R$ as $I_1 = \{0, 8\}$ and the soft set $F_{I_1}$ over $R$, where $F : I_1 \rightarrow P(R)$ is a set-valued function defined by $F(0) = \{0, 3, 15\}$ and $F(8) = \{3, 6, 9, 12, 15\}$. It can be easily shown that $F_{I_1} \subseteq_{u} R$. Now, let the ideal of $R$ be $I_2 = \{0, 4, 8, 12\}$ and the soft set $G_{I_2}$ over $R$, where $G : I_2 \rightarrow P(R)$ is a set-valued function defined by $G(0) = \{0, 4, 9, 12\}$, $G(4) = G(12) = \{0, 4, 6, 9, 15\}$ and $G(8) = \{0, 4, 6, 12\}$. Then, $G(2 \cdot 8) = G(0) = \{0, 4, 9, 12\}$ is a submodule of $G(8) = \{0, 4, 6, 12\}$. It follows that $G_{I_2}$ is not a union soft ideal of $R$.

**Theorem 16.** If $F_I \subseteq_{u} R$ and $G_{I_2} \subseteq_{u} R$, then $F_I \cup_{\Psi} G_{I_2} \subseteq_{u} R$.

Proof. Since $I_1 \subseteq_{u} R$, then $I_1 \subseteq_{u} R$. By Definition 4, $F_{I_1} \subseteq_{u} G_{I_2} = H_{I_1} \subseteq_{u} R$, where $H(x) = F(x) \cup G(x)$ for all $x \in I_1 \subseteq_{u} R$. Then for all $x, y \in I_1 \subseteq_{u} R$ and $r \in R$, $H(x - y) = F(x - y) \cup G(x - y) \subseteq (F(x) \cup F(y)) \cup (G(x) \cup G(y)) = (F(x) \cup G(x)) \cup (F(y) \cup G(y)) = H(x) \cup H(y)$, $H(rx) = F(rx) \cup G(rx) \subseteq (F(x) \cup G(x)) \cup (F(y) \cup G(y)) = H(x) \cup H(y)$, $H(xr) = F(xr) \cup G(xr) \subseteq (F(x) \cup G(x)) \cup (F(y) \cup G(y)) = H(x) \cup H(y)$. This completes the proof.

**Theorem 17.** If $F_I \subseteq_{u} R$, then $I_F = \{x \in I \mid F(x) = F(0_R)\}$ is an ideal of $R$.

Proof. The proof follows from Theorem 10 and Definition 10.

**Theorem 18.** Let $F_I$ be a soft set over $R$ and $\alpha$ be a subset of $R$ such that $F(0_R) \subseteq \alpha$. If $F_I$ is a union soft ideal of $R$, then $F_I \subseteq_{\alpha} R$ is an ideal of $R$.

Proof. Let $F_I$ be a union soft ideal of $R$, $x, y \in I$ and $r \in R$. Then, for all $x, y \in I$ and $r \in R$, $F(x - y) = R \setminus F(x - y) \subseteq R \setminus (F(x) \cup F(y)) = (R \setminus F(x)) \cap (R \setminus F(y)) = F^*(x) \cap F^*(y)$. Moreover, $F(rx) = R \setminus F(rx) \subseteq R \setminus F(x) = F(x)$ and $F(xr) = R \setminus F(xr) \subseteq R \setminus F(x) = F(x)$. Thus, $F_I$ is a soft ideal of $R$. The converse can be proved similarly.

**Theorem 19.** Let $F_I$ and $G_J$ be soft sets over $R$, where $I$ and $J$ are ideals of $R$ and $\Psi$ be a ring epimorphism from $I$ to $J$. If $G_J$ is a union soft ideal of $R$, then so is $\Psi^{-1}(G_J)$.

Proof. Follows from Theorem 1 (i), 7 and 19.

**Theorem 20.** Let $F_I$ and $G_J$ be soft sets over $R$, where $I$ and $J$ are ideals of $R$ and $\Psi$ be a ring epimorphism from $I$ to $J$. If $F_I$ is a union soft ideal of $R$, then so is $\Psi^{-1}(G_J)$.

Proof. Follows from Theorem 1 (ii), 6 and 19.

5 Some characterizations for soft submodules

In this section, we obtain some characterizations for soft submodules of a module with respect to image, preimage and upper $\alpha$-inclusion of soft sets.

**Theorem 22.** Let $F_M$ be a soft set over $M$ and $\alpha$ be a subset of $M$ such that $F(0_M) \subseteq \alpha$. If $F_M$ is a soft submodule of $M$, then $F_M \subseteq_{\alpha} M$ is a submodule of $M$. 
Theorem 23. Let $F_N$ and $G_K$ be soft sets over $M$, where $N$ and $K$ are submodules of $M$ and $\Psi$ be a module isomorphism from $N$ to $K$. If $F_N$ is a soft submodule of $M$, then so is $\Psi(F_N)$.

Proof. Let $k_1, k_2 \in K$. Since $\Psi$ is surjective, there exists $n_1, n_2 \in N$ such that $\Psi(n_1) = k_1$ and $\Psi(n_2) = k_2$. Thus, as in the case of Theorem 3, $(\Psi(F_N))(k_1 - k_2) \supseteq (\Psi(F_N))(k_1) \cap (\Psi(F_N))(k_2)$ is satisfied. Now, let $r \in R$ and $k \in K$. Since $\Psi$ is surjective, there exists $\tilde{n} \in N$ such that $\Psi(\tilde{n}) = k$. Then, $(\Psi(F_N))(rk) = \bigcup \{F(n) : n \in N, \Psi(n) = rk\} = \bigcup \{F(n) : n \in N, i = \Psi^{-1}(r\Psi(\tilde{n})))\} = \bigcup \{F(n) : n \in N, i = \Psi^{-1}(r\Psi(\tilde{n}))) = \Psi^{-1}(r\Psi(\tilde{n})))\} = \bigcup \{F(\tilde{n}) : \tilde{n} \in N, \Psi(\tilde{n}) = k\} \supseteq \bigcup \{F(\tilde{n}) : \tilde{n} \in N, \Psi(\tilde{n}) = k\} = (\Psi(F_N))(k)$. Hence, $\Psi(F_N)$ is a soft submodule of $M$.

Theorem 24. Let $F_N$ and $G_K$ be soft sets over $M$, where $N$ and $K$ are submodules of $M$ and $\Psi$ be a module homomorphism from $N$ to $K$. If $G_K$ is a soft submodule of $M$, then so is $\Psi^{-1}(G_K)$.

Proof. Let $n_1, n_2 \in N$. As in the case of Theorem 4, $(\Psi^{-1}(G_K))(n_1 - n_2) \supseteq (\Psi^{-1}(G_K))(n_1) \cap (\Psi^{-1}(G_K))(n_2)$ is satisfied. Now let $r \in R$ and $n \in N$. Then, $(\Psi^{-1}(G_K))(rn) = G(\Psi(rn)) = G(r\Psi(n)) \supseteq G(\Psi(n)) = (\Psi^{-1}(G_K))(n)$. Hence, $\Psi^{-1}(G_K)$ is a soft submodule of $M$.

6 Union soft submodules

In this section, we introduce union soft submodules of a module, investigate its basic properties and establish the relation between soft submodules and union soft submodules.

Definition 11. Let $N$ be a submodule of $M$ and $F_N$ be a soft set over $M$. Then, $F_N$ is called a union soft submodule of $M$, denoted by $(F, N) \subseteq M$ or simply $F_N \subseteq M$, if $F(x - y) \subseteq F(x) \cup F(y)$ and $F(rx) \subseteq F(x)$ for all $x, y \in N$ and $r \in R$.

Example 4. Consider the ring $R = (\mathbb{Z}_{12}, +, \cdot)$, the left $R$-module $M = (\mathbb{Z}_{12}, +, \cdot)$ with natural operation and the submodule $N_1 = \{0, 6\}$ of $M$. Let the soft set $F_{N_1}$ over $M$, where $F : N_1 \to P(M)$ is a set valued function defined by $F(0) = \{0, 4, 9\}$ and $F(6) = \{0, 3, 4, 9, 11\}$. Then, it can be easily seen that $(F, N_1) \subseteq M$. Now, let the submodule of $M$ be $N_2 = \{0, 4, 8\}$ and the soft set $G_{N_2}$ over $M$, where $G : N_2 \to P(M)$ is a set valued function defined by $G(0) = \{0, 3, 9\}$ and $G(4) = \{0, 3, 5, 8, 11\}$ and $G(8) = \{0, 3, 5, 8, 9, 11\}$. Then, $G(2 \cdot 4) = G(8) = \{0, 3, 5, 8, 9, 11\} \supseteq G(4) = \{0, 3, 5, 8, 11\}$. Therefore, $G_{N_2}$ is not a union soft submodule of $M$.

The following theorems are given without their proofs, since one can easily show them in view of Section 5.

Theorem 25. If $F_N$ is a union soft submodule of $M$ and $G_{N_2}$ is a union soft submodule of $M$, then so is $F_N \cup F_{N_2}$.

Theorem 26. If $F_N \subseteq_M$, then $F(0) \subseteq M$ for all $x \in N$.

Theorem 27. If $F_N \subseteq_M$, then $NF = \{x \in N | F(x) = F(0)M\}$ is a submodule of $N$.

Theorem 28. Let $F_N$ be a soft set over $M$ and $\alpha$ be a subset of $M$ such that $(F(\alpha) \subseteq M \forall x \in N)$. If $F_N$ is a union soft submodule of $M$, then $F_N^{\alpha}$ is a submodule of $M$.

Theorem 29. Let $F_N$ be a soft set over $M$. Then, $F_N$ is a union soft submodule of $M$ if and only if $F_n$ is a soft submodule of $M$.

Theorem 30. Let $F_N$ and $G_K$ be soft sets over $M$, where $N$ and $K$ are submodules of $M$ and $\Psi$ be a module isomorphism from $N$ to $K$. If $G_K$ is a union soft submodule of $M$, then so is $\Psi^{-1}(G_K)$.

Theorem 31. Let $F_N$ and $G_K$ be soft sets over $R$, where $N$ and $K$ are submodules of $M$ and $\Psi$ be a module homomorphism from $N$ to $K$. If $F_N$ is a union soft submodule of $M$, then so is $\Psi^{-1}(F_N)$.

Theorem 32. Let $M_1$ and $M_2$ be two $R$-modules, $F_{N_1} \subseteq M_1$, $H_{N_2} \subseteq M_2$. If $f : N_1 \to N_2$ is a module homomorphism, then i) $H_{f(N_1)} \subseteq M_2$ and $F_{\text{Ker}f} \subseteq M_1$, ii) If $f$ is an epimorphism, $F_{f^{-1}(N_2)} \subseteq M_1$.

Corollary 2. Let $F_N \subseteq M_1$, $H_{N_2} \subseteq M_2$ and $f : N_1 \to N_2$ is a module homomorphism, then $H_{f(0_{N_2})} \subseteq M_2$.

7 Conclusion

Atagün and Sezgin in [4] defined soft subrings and soft ideals of a ring, soft subfields of a field and soft submodule of a left module. In this paper, we have introduced union soft subrings and union soft ideals of a ring and union soft submodules of a left module and investigate their related properties with respect to soft set operations, anti image and lower $\alpha$-inclusion of soft sets. We also obtain significant relations between soft subrings and union soft subrings, soft ideals and union soft ideals of a ring and soft submodules and union soft submodules of a left module. To extend this work, one could study the union soft substructures of different algebras.

References


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