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Supra β-Bicontinuous Maps via Topological Ordered Spaces

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Abstract: The author utilizes the notions of supra β-open sets and increasing (decreasing, balancing) sets to introduce and study several types of supra continuous (supra open, supra closed, supra homeomorphism) maps in supra topological ordered spaces. He gives the equivalent conditions for each one of these concepts and illustrate the relationships among them separately. Apart from that, He investigates under what conditions these maps preserve some separation axioms between supra topological ordered spaces.

Keywords: I(D, B)-supra β-continuous map, I(D, B)-supra β-open map, I(D, B)-supra β-homeomorphism map, Ordered supra β-separation axioms.

1 Introduction

A concept of topological ordered spaces $(X,\tau,\preceq)$ is a set endowed with both a topology τ and a partial order relation $\preceq$. This concept was initiated by Nachbin [15] in 1965. McCartan [14] in 1968, introduced and studied ordered separation axioms by utilizing monotone neighborhoods. He also defined strong ordered separation axioms and pointed out that T_i-ordered space is strictly stronger than T_i-space, for $i = 0,1,2,3,4$. Mashhour et al. [13] introduced a notion of supra topological spaces and generalized some properties of topological spaces to supra topological spaces such as continuity and some separation axioms. In 1991, Arya and Gupta [5] utilized semi open sets [12] to introduce semi separation axioms in topological ordered spaces. In 2002, Kumar [11] introduced and studied the concepts of continuity, openness, closeness and homeomorphism between topological ordered spaces. Das [6] introduced ordered separation axioms in supra topological ordered spaces and generalized results obtained by McCartan [14]. Jafari and Tahiliani [10] introduced and studied supra β-open sets and supra β-continuous maps. In 2012, Rao and Chudamani [16] used supra β-open sets to generalize the concepts and results obtained in [11]. In 2016, Abo-elhamayel and Al-shami [1] introduced the concepts of x-supra continuous (x-supra open, x-supra closed, x-supra homeomorphism) maps in supra topological ordered spaces, for x= {I, D, B} and studied their properties. El-Shafei et al. [7] gave a notion of supra R-open sets and presented new types of supra separation axioms. They also initiated some ordered maps via supra topological ordered spaces and investigated their properties in [8] and presented strong separation axioms in supra topological ordered spaces in [9]. They modified McCartan's assertion [14] which state that every strong T_i-order space is a T_i-order space, for $i = 0,1,2,3,4$, in case of $i = 0,1$.

The aim of the present paper is to establish some types of x-supra β-continuous (x-supra β-open, x-supra β-closed, x-supra β-homeomorphism) maps in supra topological spaces, for x= {increasing, decreasing, balancing} (briefly, for x= {I, D, B}). We give the necessary and sufficient conditions for these maps to preserve some separation axioms. As well as, we investigate the properties of these concepts and provide several illustrative examples. Many of the findings that raised at are generalizations of those findings in supra topological ordered spaces which introduced in [1].

2 Preliminaries

From now on, $(X,\tau,\preceq_1)$ and $(Y,\tau,\preceq_2)$ stand for topological ordered spaces and $(X,\mu,\preceq_1)$ and $(Y,\mu,\preceq_2)$

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stand for supra topological ordered spaces. A diagonal relation is denoted by $\triangle$.

In the following, we recall some concepts and notions that will be needed in the sequels.

Definition 2.1.[15] Let B be a subset of a partially ordered set $(X,\preceq)$ and $x\in X$. Then:
(i) $i(x)=\{a\in X : x\preceq a\}$ and $d(x)=\{a\in X : a\preceq x\}$.
(ii) $i(B)=\bigcup\{i(b) : b\in B\}$ and $d(B)=\bigcup\{d(b) : b\in B\}$.
(iii) A set B is called increasing (resp. decreasing), If $B=i(B)(resp.B=d(B))$.

Definition 2.2. A map $f:(X,\preceq_1)\to(Y,\preceq_2)$ is called:
(i) Order preserving (or increasing) if $a\preceq_1 b$, then $f(a)\preceq_2 f(b)$, for each $a,b\in X$.
(ii) Order embedding if $a\preceq_1 b$ if and only if $f(a)\preceq_2 f(b)$, for each $a,b\in X$.

Theorem 2.3. (i) If $f:(X,\preceq_1)\to(Y,\preceq_2)$ is an increasing map, then the inverse image of each increasing (decreasing) is increasing (decreasing).
(ii) If $f:(X,\preceq_1)\to(Y,\preceq_2)$ is a decreasing map, then the inverse image of each increasing (decreasing) is decreasing (increasing).

Definition 2.4. [1] A map $g:(X,\tau)\to(Y,\theta)$ is said to be supra open (resp. supra closed) if the image of any open (resp. closed) subset of X is a supra open (resp. supra closed) subset of Y.

Definition 2.5. [13] (i) A map $g:(X,\tau)\to(Y,\theta)$ is said to be supra continuous if the inverse image of each open subset of Y is a supra open subset of X.
(ii) Let (X,τ) be a topological space and μ be a supra topology on X. We say that μ is associated supra topology with τ if $\tau\subseteq\mu$.

Definition 2.6. [10] A subset E of a supra topological space (X,μ) is called supra β-open if $E\subseteq cl(int(cl(E)))$ and its complement is called supra β-closed.

Definition 2.7. [10] A map $g:(X,\tau)\to(Y,\theta)$ is said to be:
(i) Supra β-continuous if the inverse image of each open subset of Y is a supra β-open subset of X.
(ii) Supra β-open (resp. supra β-closed) if the image of each open (resp. closed) subset of X is a supra β-open (resp. supra β-closed) subset of Y.

In what follows, we give the definition supra β-homeomorphism maps.

Definition 2.8. A map $g:(X,\tau)\to(Y,\theta)$ is said to be supra β-homeomorphism if it is bijective, supra β-continuous and supra β-open.

Definition 2.9.([10], [13]) Let E be a subset of a supra topological space (X,μ). Then:
(i) Supra interior of E, denoted by $sint(E)$, is the union of all supra open sets contained in E.
(ii) Supra closure of E, denoted by $scl(E)$, is the intersection of all supra closed sets containing E.
(iii) Supra β-interior of E, denoted by $s\beta int(E)$, is the union of all supra β-open sets contained in E.
(iv) Supra β-closure of E, denoted by $s\beta cl(E)$, is the intersection of all supra β-closed sets containing E.

Definition 2.10.[15] A topological ordered space $(X,\tau,\preceq)$ is called:
(i) Lower (Upper) strong T_1-ordered if for each $a,b\in X$ such that $a\not\preceq b$, there exists an increasing (a decreasing) open set G containing $a(b)$ such that $b(a)$ belongs to G^c.
(ii) Strong T_1-ordered if it is strong lower T_1-ordered and strong upper T_1-ordered.
(iii) Strong T_0-ordered if it is strong lower T_1-ordered or strong upper T_1-ordered.
(iv) Strong T_2-ordered if for every $a,b\in X$ such that $a\not\preceq b$, there exist disjoint supra open sets W_1 and W_2 containing a and b, respectively, such that W_1 is increasing and W_2 is decreasing.

Definition 2.11.[9] A supra topological ordered space $(X,\mu,\preceq)$ is called:
(i) Lower (Upper) SST_1-ordered if for each $a,b\in X$ such that $a\not\preceq b$, there exists an increasing (a decreasing) supra open set G containing $a(b)$ such that $b(a)$ belongs to G^c.
(ii) SST_1-ordered space if it is both lower SST_1-ordered and upper SST_1-ordered.
(iii) SST_0-ordered space if it is lower SST_1-ordered or upper SST_1-ordered.
(iv) SST_2-ordered if for every $a,b\in X$ such that $a\not\preceq b$, there exist disjoint supra open sets W_1 and W_2 containing a and b, respectively, such that W_1 is increasing and W_2 is decreasing.

3 Supra β-continuous maps in supra topological ordered spaces

The concepts of I-supra β-continuous, D-supra β-continuous and B-supra β-continuous maps in supra topological ordered spaces are presented and their main properties are investigated. The relationships among them are illustrated with the help of examples. Also, we study under what conditions these three types of supra β-continuous maps preserve some of ordered supra β-separation axioms.

Definition 3.1. A subset E of $(X,\mu,\preceq_1)$ is said to be:
(i) I-supra (resp. D-supra, B-supra) β-open if it is supra β-open and increasing (resp. decreasing, balancing).
(ii) I-supra (resp. D-supra, B-supra) β-closed if it is supra

β-closed and increasing (resp. decreasing, balancing).

Definition 3.2. A map $f:(X,\mu,\preceq_1)\to(Y,\tau,\preceq_2)$ is called I-supra (resp. D-supra, B-supra) β-continuous at $p\in X$ if for each open set H containing $f(p)$, there exists an I-supra (resp. a D-supra, a B-supra) β-open set G containing p such that $f(G)\subseteq H$.
Also, the map is called I-supra (resp. D-supra, B-supra) β-continuous if it is continuous at each point $p\in X$.

Theorem 3.3. A map $f:(X,\mu,\preceq_1)\to(Y,\tau,\preceq_2)$ is I-supra (resp. D-supra, B-supra) β-continuous if and only if the inverse image of each open subset of Y is an I-supra (resp. a D-supra, a B-supra) β-open subset of X.
Proof. We only prove the theorem in case of f is an I-supra β-continuous map and the other follow similar lines.
To prove the necessary part, let G be an open subset of Y, Then we have the following two cases:
(i) $f^{-1}(G)=\emptyset$ which is an I-supra β-open subset of X.
(ii) $f^{-1}(G)\neq\emptyset$. By choosing $p\in X$ such that $p\in f^{-1}(G)$, we obtain that $f(p)\in G$. So there exists an I-supra open set H_p containing p such that $f(H_p)\subseteq G$. Since p is chosen arbitrary, then $f^{-1}(G)=\bigcup_{p\in f^{-1}(G))}H_p$. Thus $f^{-1}(G)$ is an I-supra β-open subset of X.
To prove the sufficient part, let G be an open subset of Y containing $f(p)$. Then $p\in f^{-1}(G)$. By hypothesis, $f^{-1}(G)$ is an I-supra β-open set. Since $f(f^{-1}(G))\subseteq G$, then f is an I-supra β-continuous at $p\in X$ and since p is chosen arbitrary, then f is an I-supra β-continuous.■

Remark 3.4. (i) Every I-supra (D-supra, B-supra) β-continuous map is supra β-continuous.
(ii) Every B-supra β-continuous map is I-supra β-continuous and D-supra β-continuous.

The following two examples illustrate that a supra β-continuous map (resp. I-supra β-continuous) need not be I-supra β-continuous or D-supra β-continuous or B-supra β-continuous (resp. B-supra β-continuous).

Example 3.5. Let $\tau=\{\mathscr{R},G\subseteq\mathscr{R}$ such that $1\in G\}$ be a topology on the set of real numbers $\mathscr{R}$ and the discrete topology μ be associated supra topology with τ. Let the partial ordered relation $\preceq_1=\triangle\bigcup\{(1,5),(3,8)\}$ on $\mathscr{R}$ and let the map $f:(\mathscr{R},\mu)\to(\mathscr{R},\tau)$ be the identify map. Obviously, f is supra β-continuous. Now, $\{1,8\}$ is an open subset of Y, whereas $f^{-1}(\{1,8\})=\{1,8\}$ is neither a decreasing nor an increasing supra β-open subset of X. Then f is not I-supra (D-supra, B-supra) β-continuous.

Example 3.6. We replace only the partial order relation in Example 3.5 by $\preceq=\triangle\bigcup\{(3,1)\}$. Then the map f is I-supra β-continuous, but not B-supra β-continuous.

The relationships among types of supra continuous maps are illustrated in the following figure.

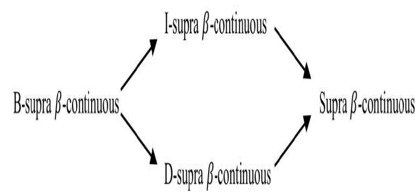

Fig. 1: The relationships among types of supra continuous maps

Definition 3.7. Let E be a subset of $(X,\mu,\preceq_1)$. Then:
(i) $E^{is\beta o}=\bigcup\{G:G$ is an I-supra β-open set contained in $E\}$.
(ii) $E^{ds\beta o}=\bigcup\{G:G$ is a D-supra β-open set contained in $E\}$.
(iii) $E^{bs\beta o}=\bigcup\{G:G$ is a B-supra β-open set contained in $E\}$.
(iv) $E^{is\beta cl}=\bigcap\{H:H$ is an I-supra β-closed set containing $E\}$.
(v) $E^{ds\beta cl}=\bigcap\{H:H$ is a D-supra β-closed set containing $E\}$.
(vi) $E^{bs\beta cl}=\bigcap\{H:H$ is a B-supra β-closed set containing $E\}$.

lemma 3.8. Let E be a subset of $(X,\mu,\preceq)$. Then:
(i) $(E^{ds\beta cl})^c=(E^c)^{is\beta o}$.
(ii) $(E^{is\beta cl})^c=(E^c)^{ds\beta o}$.
(iii) $(E^{bs\beta cl})^c=(E^c)^{bs\beta o}$.
Proof. (i) $(E^{ds\beta cl})^c=\{\bigcup F:F$ is a D-supra β-closed set containing $E\}^c$
$=\bigcap\{F^c:F^c$ is an I-supra β-open set contained in $E^c\}$
$=(E^c)^{is\beta o}$.
The proof of (ii) and (iii) is similar to that of (i).■

Theorem 3.9. Let $g:(X,\mu,\preceq_1)\to(Y,\tau,\preceq_2)$ be a map. Then the following five statements are equivalent:
(i) g is I-supra β-continuous;
(ii) The inverse image of each closed subset of Y is a D-supra β-closed subset of X;
(iii) $(g^{-1}(H))^{ds\beta cl}\subseteq g^{-1}(cl(H))$, for every $H\subseteq Y$;
(iv) $g(A^{ds\beta cl})\subseteq cl(g(A))$, for every $A\subseteq X$;
(v) $g^{-1}(int(H))\subseteq(g^{-1}(H))^{is\beta o}$, for every $H\subseteq Y$.
Proof. $(i)\Rightarrow(ii)$ Consider H is a closed subset of Y. Then H^c is open. Therefore $g^{-1}(H^c)=(g^{-1}(H))^c$ is an I-supra β-open subset of X. So $g^{-1}(H)$ is D-supra β-closed.
$(ii)\Rightarrow(iii)$ For any subset H of Y, we have that $cl(H)$ is closed. Since $g^{-1}(cl(H))$ is a D-supra β-closed subset of X, then $(g^{-1}(H))^{ds\beta cl}\subseteq(g^{-1}(cl(H))^{ds\beta cl}=g^{-1}(cl(H))$.

$(iii) \Rightarrow (iv)$: Consider A is a subset of X. Then $A^{ds\beta cl} \subseteq (g^{-1}(g(A))^{ds\beta cl} \subseteq g^{-1}(cl(g(A))$. Therefore $g(A^{ds\beta cl}) \subseteq g(g^{-1}(cl(g(A))) \subseteq cl(g(A))$.
(iv) $\Rightarrow$ **(v)**: Let H be a subset of Y. By Lemma (3), we obtain that $g(X-(g^{-1}(H))^{is\beta o}) = g(((g^{-1}(H))^c)^{ds\beta cl})$. By (iv) $g(((g^{-1}(H))^c)^{ds\beta cl}) \subseteq cl(g(g^{-1}(H))^c) = cl(g(g^{-1}(H^c))) \subseteq cl(Y-H) = Y - int(H)$. Therefore $(X - (g^{-1}(H))^{is\beta o}) \subseteq g^{-1}(Y - int(H)) = X - g^{-1}(int(H))$. Thus $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{is\beta o}$.
$(v) \Rightarrow (i)$: Consider H is an open subset of Y. Then $g^{-1}(H) = g^{-1}(int(H)) \subseteq (g^{-1}(H))^{is\beta o}$. Since $g^{-1}(H)$ is I-supra β-open, then $(g^{-1}(H))^{is\beta o} \subseteq g^{-1}(H)$. Therefore $g^{-1}(H)$ is an I-supra β-open subset of X. Thus g is I-supra β-continuous.■

Theorem 3.10. Let $g : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:
(i) g is D-supra β-continuous;
(ii) The inverse image of each closed subset of Y is an I-supra β-closed subset of X;
(iii) $(g^{-1}(H))^{is\beta cl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;
(iv) $g(A^{is\beta cl}) \subseteq cl(g(A))$, for every $A \subseteq X$;
(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{ds\beta o}$, for every $H \subseteq Y$.
Proof. The proof is similar to that of Theorem 3.9.■

Theorem 3.11. Let $g : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:
(i) g is B-supra β-continuous;
(ii) The inverse image of each closed subset of Y is a B-supra β-closed subset of X;
(iii) $(g^{-1}(H))^{bs\beta cl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;
(iv)$g(A^{bs\beta cl}) \subseteq cl(g(A))$, for every $A \subseteq X$;
(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{bs\beta o}$, for every $H \subseteq Y$.
Proof. The proof is similar to that of Theorem 3.9.■

Definition 3.12. A supra topological ordered space $(X, \mu, \preceq)$ is called:
(i) Lower (Upper) strong supra β-T_1-ordered (briefly, Lower (Upper) $SS\beta$-T_1-ordered) if for each $a, b \in X$ such that $a \npreceq b$, there exists an increasing (a decreasing) supra β-open set G containing $a(b)$ such that $b(a)$ belongs to G^c.
(ii) $SS\beta$-T_0-ordered space if it is lower $SS\beta$-T_1-ordered or upper $SS\beta$-T_1-ordered.
(iii) $SS\beta$-T_1-ordered space if it is both lower $SS\beta$-T_1-ordered and upper $SS\beta$-T_1-ordered.
(iv) $SS\beta$-T_2−ordered if for every $a, b \in X$ such that $a \npreceq b$, there exist disjoint supra β-open sets W_1 and W_2 containing a and b, respectively, such that W_1 is increasing and W_2 is decreasing.
(v) Lower (upper) supra β-regularly ordered if for every decreasing (increasing) supra β-closed set F and for each $a \notin F$, there exist disjoint supra β-open sets W_1 and W_2 containing F and a, respectively, such that W_1 is decreasing (increasing) and W_2 is increasing (decreasing).
(vi) Supra β-normally ordered if for every disjoint supra β-closed sets F_1 and F_2 such that F_1 is decreasing and F_2 is increasing, there exist disjoint supra β-open sets W_1 and W_2 containing F_1 and F_2, respectively, such that W_1 is decreasing and W_2 is increasing.

Theorem 3.13. Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be I-supra β-continuous and f^{-1} be order preserving. If $(Y, \tau, \preceq_2)$ is lower T_1-ordered, then $(X, \mu, \preceq_1)$ is lower $SS\beta$-T_1-ordered.
Proof. Let $a, b \in X$ such that $a \npreceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a), y = f(b)$. Since f^{-1} is order preserving, then $x \npreceq_2 y$. Since $(Y, \tau, \preceq_2)$ is lower T_1-ordered, then there exists an increasing neighborhood W of x in Y such that $x \in W$ and $y \notin W$. Therefore there exists an open set G such that $x \in G \subseteq W$. Since f is bijective I-supra β-continuous, then $a \in f^{-1}(G)$ which is I-supra β-open and $b \notin f^{-1}(G)$. Thus $(X, \mu, \preceq_1)$ is lower $SS\beta$-T_1-ordered.■

Theorem 3.14. Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be D-supra β-continuous and f^{-1} be order preserving. If $(Y, \tau, \preceq_2)$ is upper T_1-ordered, then $(X, \mu, \preceq_1)$ is upper $SS\beta$-T_1-ordered.
Proof. The proof is similar to that of Theorem 3.13.■

Theorem 3.15. Let a bijective map $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ be B-supra β-continuous and f^{-1} be order preserving. If $(Y, \tau, \preceq_2)$ is T_i-ordered, then $(X, \mu, \preceq_1)$ is $SS\beta$-T_i-ordered, for $i = 0, 1, 2$.
Proof. We prove the theorem in case of $i = 2$. Let $a, b \in X$ such that $a \npreceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f^{-1} is order preserving, then $x \npreceq_2 y$. Since $(Y, \tau, \preceq_2)$ is T_2-ordered, then there exist disjoint balancing neighborhoods W_1 and W_2 of x and y, respectively. Therefore there are disjoint open sets G and H containing x and y, respectively. Since f is bijective B-supra β-continuous, then $a \in f^{-1}(G)$ which is an increasing supra β-open subset of X, $b \in f^{-1}(H)$ which is a decreasing supra β-open subset of X and $f^{-1}(G) \bigcap f^{-1}(H) = \emptyset$. Thus $(X, \mu, \preceq_1)$ is $SS\beta$-T_2-ordered.
In a similar way, we can prove theorem in case of $i = 0, 1$.■

Theorem 3.16. Consider $f : (X, \mu, \preceq_1) \to (Y, \tau, \preceq_2)$ is a bijective supra β-continuous map such that f is ordered embedding. If $(Y, \tau, \preceq_2)$ is strong T_i-ordered, then $(X, \mu, \preceq_1)$ is $SS\beta$-T_i-ordered, for $i = 0, 1, 2$.
Proof. We prove the theorem in case of $i = 2$. Let $a, b \in X$ such that $a \npreceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f is ordered embedding, then $x \npreceq_2 y$. Since $(Y, \tau, \preceq_2)$ is strong T_2-ordered, then there exist disjoint supra open sets G and H containing x and y respectively, such that W_1 is increasing and W_2 is decreasing. Since f is bijective supra β-continuous and

order preserving, then $f^{-1}(G)$ which is an increasing supra β-open set containing a, $f^{-1}(H)$ which is a decreasing supra β-open set containing b and $f^{-1}(G)\bigcap f^{-1}(H) = \emptyset$. Thus $(X,\mu,\preceq_1)$ is $SS\beta$-T_2-ordered.
Similarly, one can prove theorem in case of $i = 0,1$.■

Theorem 3.17. Consider $f : (X,\mu,\preceq_1) \to (Y,\tau,\preceq_2)$ is an injective B-supra β-continuous map. If $(Y,\tau,\preceq_2)$ is a T_i-space, then $(X,\mu,\preceq_1)$ is an $SS\beta$-T_i-ordered space for $i = 1,2$.
Proof. We prove the theorem in case of $i = 2$ and the other case is proven similarly.
Let $a,b \in X$ such that $a \npreceq_1 b$. Then there exist $x,y \in Y$ such that $f(a) = x$, $f(b) = y$ and $x \neq y$. Since $(Y,\tau,\preceq_2)$ is a T_2-space, then there exist disjoint open sets G and H such that $x \in G$ and $y \in H$. Therefore $a \in f^{-1}(G)$ which is an increasing supra β-open subset of X, $b \in f^{-1}(H)$ which is a decreasing supra β-open subset of X and $f^{-1}(G)\bigcap f^{-1}(H) = \emptyset$. Thus $(X,\mu,\preceq_1)$ is an $SS\beta$-T_2-ordered space.■

4 Supra β-open (Supra β-closed) maps in supra topological ordered spaces

In this section, we introduce the concepts of I-supra β-open (I-supra β-closed), D-supra β-open (D-supra β-closed) and B-supra β-open (B-supra β-closed) maps in supra topological ordered spaces. We demonstrate their main properties and illustrate relationships among them with the help of examples. Finally, some results concerning the image and per image of some separation axioms under these maps are presented.

Definition 4.1. A map $g : (X,\tau,\preceq_1) \to (Y,\mu,\preceq_2)$ is said to be:
(i) I-supra (resp. D-supra, B-supra) β-open if the image of any open subset of X is an I-supra (resp. a D-supra, a B-supra) β-open subset of Y.
(ii) I-supra (resp. D-supra, B-supra) β-closed if the image of any closed subset of X is an I-supra (resp. a D-supra, a B-supra) β-closed subset of Y.

Remark 4.2. (i) Every I-supra (D-supra, B-supra) β-open map is supra β-open.
(ii) Every I-supra (D-supra, B-supra) β-closed map is supra β-closed.
(iii) Every B-supra β-open (resp. B-supra β-closed) map is I-supra β-open and D-supra β-open (resp. I-supra β-closed and D-supra β-closed).

The following two examples illustrate that a supra β-open (resp. D-supra β-open) map need not be I-supra β-open or D-supra β-open or B-supra β-open (resp. B-supra β-open).

Example 4.3. Let the topology $\tau = \{\emptyset, X, \{1,2\}\}$ and the partial order relation $\preceq_2 = \triangle\bigcup\{(1,3),(3,2)(1,2)\}$ on $X = \{1,2,3\}$. Let the supra topology associated with τ be $\{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}\}$ on X. The identity map $f : X \to X$ is a supra β-open map. Now, $\{1,2\}$ is an open subset of X. Since $f(\{1,2\}) = \{1,2\}$ is neither an increasing nor a decreasing supra β-open subset of Y, then f is not x-supra β-open map for x={I, D, B}.

Example 4.4. We replace only the partial order relation in Example 4.3 by $\preceq = \triangle\bigcup\{(1,3),\ (1,2)\}$. Then the map f is D-supra β-open, but is not B-supra β-open.

The following two examples illustrate that supra β-closed (resp. an I-supra β-closed) maps need not be I-supra β-closed or D-supra β-closed or B-supra β-closed (resp. a B-supra β-closed).

Example 4.5. Let the topology $\tau = \{\emptyset, X, \{a,b\}\}$ on $X = \{a,b,c\}$, the supra topology associated with τ be $\{\emptyset, X, \{c\}, \{a,b\}\}$ and the partial order relation $\preceq_2 = \triangle\bigcup\{(a,c),(c,b),(a,b)\}$ on X. The map $f : X \to X$ is defined as follows $f(a) = f(c) = c$ and $f(b) = b$. Obviously, f is supra β-closed. Now, $\{c\}$ is a closed subset of X, but $f(\{c\}) = \{c\}$ is neither a decreasing nor an increasing supra β-closed subset of Y. Then f is not I(D, B)-supra β-closed.

Example 4.6. We replace only the partial order relation in Example 4.5 by $\preceq = \triangle\bigcup\{(b,c)\}$. Then the map f is I-supra β-closed, but is not B-supra β-closed.

The relationships among the presented types of supra open (supra closed) maps are illustrated in the following figure.

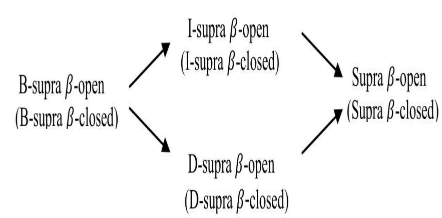

Fig. 2: The relationships among types of supra open (supra closed) maps

Theorem 4.7. The following statements are equivalent, for a map $f : (X,\tau,\preceq_1) \to (Y,\mu,\preceq_2)$:
(i) f is I-supra β-open;
(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{is\beta o})$, for every $H \subseteq Y$;

(iii) $f(int(G)) \subseteq (f(G))^{is\beta o}$, for every $G \subseteq X$.
Proof. $(i) \Rightarrow (ii)$: Since $int(f^{-1}(H))$ is an open subset of X, then $f(int(f^{-1}(H)))$ is an I-supra β-open subset of Y. Since $f(int(f^{-1}(H))) \subseteq f(f^{-1}(H)) \subseteq H$, then $int(f^{-1}(H)) \subseteq f^{-1}(H^{is\beta o})$.
$(ii) \Rightarrow (iii)$: By replacing H by $f(G)$ in **(ii)**, we obtain that $int(f^{-1}(f(G))) \subseteq f^{-1}((f(G))^{is\beta o})$. Since $int(G) \subseteq f^{-1}(f(int(f^{-1}(f(G))))) \subseteq f^{-1}((f(G))^{is\beta o})$, then $f(int(G)) \subseteq (f(G))^{is\beta o}$.
$(iii) \Rightarrow (i)$: Let G be an open subset of X. Then $f(int(G)) = f(G) \subseteq (f(G))^{isp o}$. So $f(G)$ is an I-supra β-open set. Thus f is an I-supra β-open map.■

In a similar way one can prove the following two theorems.

Theorem 4.8. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$:
(i) f is D-supra β-open;
(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{ds\beta o})$, for every $H \subseteq Y$;
(iii) $f(int(G)) \subseteq (f(G))^{ds\beta o}$, for every $G \subseteq X$.

Theorem 4.9. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$:
(i) f is B-supra β-open;
(ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{bs\beta o})$, for every $H \subseteq Y$;
(iii) $f(int(G)) \subseteq (f(G))^{bs\beta o}$, for every $G \subseteq X$.

Theorem 4.10. Let $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be a map. Then we have the following results.
(i) f is I-supra β-closed if and only if $(f(G))^{is\beta cl} \subseteq f(cl(G))$, for any $G \subseteq X$.
(ii) f is D-supra β-closed if and only if $(f(G))^{ds\beta cl} \subseteq f(cl(G))$, for any $G \subseteq X$.
(iii) f is B-supra β-closed if and only if $(f(G))^{bs\beta cl} \subseteq f(cl(G))$, for any $G \subseteq X$.
Proof. (i) Necessity: Consider f is an I-supra β-closed map. Then $f(cl(G))$ is an I-supra β-closed subset of Y. Since $f(G) \subseteq f(cl(G))$, then $(f(G))^{is\beta cl} \subseteq f(cl(G))$.
Sufficiency: Consider B is a closed subset of X. Then $f(B) \subseteq (f(B))^{is\beta cl} \subseteq f(cl(B)) = f(B)$. Therefore $f(B) = (f(B))^{is\beta cl}$ is an I-supra β-closed set. Thus f is an I-supra β-closed map.
The proof of (ii) and (iii) is similar to that of (i).■

Theorem 4.11. Let $f : (X, \tau, \preceq_1) \to (Y, \mu, \preceq_2)$ be a bijective map. Then we have the following results.
(i) f is I-supra β-open if and only if f is D-supra β-closed.
(ii) f is D-supra β-open if and only if f is I-supra β-closed.
(iii) f is B-supra β-open if and only if f is B-supra β-closed.
Proof. (i) Necessity: Let f be an I-supra β-open map and let G be a closed subset of X. Then G^c is open. Since f is bijective, then $f(G^c) = (f(G))^c$ is I-supra β-open. Therefore $f(G)$ is a decreasing supra β-closed subset of Y. Thus f is D-supra β-closed.
Sufficiency: let f be a D-supra β-closed map and let B be an open subset of X. Then B^c is closed. Since f is bijective, then $f(B^c) = (f(B))^c$ is D-supra β-closed. Therefore $f(B)$ is I-supra β-open. Thus f is I-supra β-closed.
The proof of (ii) and (iii) is similar to that of (i).■

Theorem 4.12. The following two statements hold.
(i) If the maps $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is open and $g : (Y, \theta, \preceq_2) \to (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) β-open, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) β-open.
(ii) If the maps $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is closed and $g : (Y, \theta, \preceq_2) \to (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) β-closed, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) β-closed.
Proof. It is clear.■

Theorem 4.13. If the maps $g \circ f$ is I-supra (resp. D-supra, B-supra) β-open and $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is surjective continuous, then a map $g : (Y, \theta, \preceq_2) \to (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) β-open.
Proof. Consider $g \circ f$ is I-supra β-open and let G be an open subset of Y. Then $f^{-1}(G)$ is an open subset of X. Since $g \circ f$ is I-supra β-open and f is surjective, then $(g \circ f)(f^{-1}(G)) = g(G)$ is an I-supra β-open subset of Z. Therefore g is I-supra β-open.

A similar proof can be given for the cases between parentheses.■

Theorem 4.14. If the maps $g \circ f : (X, \tau, \preceq_1) \to (Z, \mu, \preceq_3)$ is closed and $g : (Y, \theta, \preceq_2) \to (Z, \mu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) β-continuous injective, then a map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is D-supra (resp. I-supra, B-supra) β-closed.
Proof. Consider g is I-supra β-continuous and let G be a closed subset of X. Then $(g \circ f)(G)$ is a closed subset of Z. Since g is injective and I-supra β-continuous, then $g^{-1}(g \circ f)(G) = f(G)$ is a D-supra β-closed subset of Y. Therefore f is D-supra β-closed.
A similar proof can be given for the cases between parentheses.■

Theorem 4.15. We have the following results for a bijective map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$.
(i) f is I-supra (resp. D-supra, B-supra) β-open if and only if f^{-1} is I-supra (resp. D-supra, B-supra) β-continuous. (ii) f is D-supra (resp. I-supra, B-supra) β-closed if and only if f^{-1} is I-supra (resp. D-supra, B-supra) β-continuous.
Proof. (i) We prove (i) when f is B-supra β-open, and the other cases follows similar lines.

$' \Rightarrow'$ Let f be a B-supra β-open map and let G be an open subset of X. Then $(f^{-1})^{-1}(G) = f(G)$ is a balancing supra β-open subset of Y. Therefore f^{-1} is a B-supra β-continuous.
$' \Leftarrow'$ let G be an open subset of X and f^{-1} be a B-supra β-continuous. Then $f(G) = (f^{-1})^{-1}(G)$ is a balancing supra β-open subset of Y. Therefore f is B-supra β-open.
(ii) Similarly, one can prove (ii).■

Theorem 4.16. Let a bijective map $f : (X,\tau,\preceq_1) \to (Y,\mu,\preceq_2)$ be I-supra β-open (D-supra β-closed) and order preserving. If $(X,\tau,\preceq_1)$ is lower T_1-ordered, then $(Y,\mu,\preceq_2)$ is lower $SS\beta$-T_1-ordered.
Proof. We prove the theorem when a map f is I-supra β-open and the other can be made similarly.
Let $x,y \in Y$ such that $x \not\preceq_2 y$. Since f is bijective, then there exist $a,b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$ and since f is an order preserving map, then $a \not\preceq_1 b$. By hypotheses $(X,\tau,\preceq_1)$ is lower T_1-ordered, then there exists an increasing neighborhood W in X such that $a \in W$ and $b \notin W$. Therefore there exists an open set G such that $a \in G \subseteq W$. Thus $x \in f(G)$ which is an increasing supra β-open and $y \notin f(G)$. Hence $(Y,\mu,\preceq_2)$ is lower $SS\beta$-T_1-ordered.■

Theorem 4.17. Let a bijective map $f : (X,\tau,\preceq_1) \to (Y,\mu,\preceq_2)$ be D-supra β-open (I-supra β-closed) and order preserving. If $(X,\tau,\preceq_1)$ is upper T_1-ordered, then $(Y,\mu,\preceq_2)$ is upper $SS\beta$-T_1-ordered.
Proof. The proof is similar to that of Theorem 4.16.■

Theorem 4.18. Let a bijective map $f : (X,\tau,\preceq_1) \to (Y,\mu,\preceq_2)$ be B-supra β-open (B-supra β-closed) and order preserving. If $(X,\tau,\preceq_1)$ is T_i-ordered, then $(Y,\mu,\preceq_2)$ is $SS\beta$-T_i-ordered, for $i = 0,1,2$.
Proof. We prove the theorem in case of f is a B-supra β-open map and $i = 2$.
For all $x,y \in Y$ such that $x \not\preceq_2 y$, there are $a,b \in X$ such that $a = f^{-1}(x), b = f^{-1}(y)$. Since f is an order preserving, then $a \not\preceq_1 b$. Since $(X,\tau,\preceq_1)$ is T_2-ordered, then there exist disjoint neighborhoods W_1 and W_2 of a and b, respectively, such that W_1 is increasing and W_2 is decreasing. Therefore there are disjoint open sets G and H such that $a \in G \subseteq W_1$ and $b \in H \subseteq W_2$. Thus $x \in f(G)$ which is a balancing supra β-open, $y \in f(H)$ which is a balancing supra β-open and $f(G) \bigcap f(H) = \emptyset$. Thus $(Y,\mu,\preceq_2)$ is $SS\beta$-T_2-ordered.
In a similar way, we can prove theorem in case of $i = 0,1$.
The proof for a B-supra β-closed map can b made similarly.■

Theorem 4.19. Consider a bijective map $f : (X,\tau,\preceq_1) \to (Y,\mu,\preceq_2)$ is supra β-open such that f and f^{-1} are order preserving. If $(X,\tau,\preceq_1)$ is strong T_i-ordered, then $(Y,\mu,\preceq_2)$ is $SS\beta$-T_i-ordered, for $i = 0,1,2$.
Proof. We prove the theorem in case of $i = 2$. Let $x,y \in Y$ such that $x \not\preceq_2 y$. Then there exist $a,b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since f is an order preserving, then $a \not\preceq_1 b$. Since $(X,\tau,\preceq_1)$ is strong T_2-ordered, then there exist disjoint an increasing open sets G containing a and a decreasing open set H containing b. Since f is bijective supra β-open and f^{-1} is an order preserving, then $f(G)$ is an increasing supra β-open set containing x, $f(H)$ is a decreasing supra β-open set containing y and $f(G) \bigcap f(H) = \emptyset$. Therefore $(Y,\mu,\preceq_2)$ is $SS\beta$-T_2-ordered.
Similarly, one can prove theorem in case of $i = 0,1$.
A similar proof can be given for the case between parentheses.■

Theorem 4.20. Let $f : (X,\tau,\preceq_1) \to (Y,\mu,\preceq_2)$ be a bijective supra open map such that f and f^{-1} are order preserving. If $(X,\tau,\preceq_1)$ is strong T_i-ordered, then $(Y,\mu,\preceq_2)$ is $SS\beta$-T_i-ordered, for $i = 0,1,2$.
Proof. The proof is similar to that of Theorem 4.19.■

5 Supra β-homeomorphism maps in supra topological ordered spaces

The concepts of I-supra β-homeomorphism, D-supra β-homeomorphism and B-supra β-homeomorphism maps are introduced and many of their properties are established. Some illustrative examples are provided.

Definition 5.1. Let $\tau^\star$ and $\theta^\star$ be associated supra topologies with τ and θ, respectively. A map $g : (X,\tau,\preceq_1) \to (Y,\theta,\preceq_2)$ is called I-supra (resp. D-supra, B-supra) β-bicontinuous if it is I-supra β-continuous and I-supra β-open (resp. D-supra β-continuous and D-supra β-open, B-supra β-continuous and B-supra β-open).

Definition 5.2. Let $\tau^\star$ and $\theta^\star$ be associated supra topologies with τ and θ, respectively. A bijective map $g : (X,\tau,\preceq_1) \to (Y,\theta,\preceq_2)$ is called I-supra (resp. D-supra, B-supra) β-homeomorphism if it is I-supra β-bicontinuous (resp. D-supra β-bicontinuous, B-supra β-bicontinuous).

Remark 5.3. (i) Every I-supra (D-supra, B-supra) β-homeomorphism map is supra β-homeomorphism.
(ii) Every B-supra β-homeomorphism map is I-supra β-homeomorphism and D-supra β-homeomorphism.

The following two examples illustrate that supra β-homeomorphism (resp. D-supra β-homeomorphism) maps need not be I-supra β-homeomorphism or D-supra

β-homeomorphism or B-supra β-homeomorphism (resp. B-supra β-homeomorphism).

Example 5.4. Let the topology $\tau = \{\emptyset, X, \{a,c\}\}$ on $X = \{a,b,c\}$, the supra topology associated with τ be $\{\emptyset, X, \{a\}, \{a,c\}\}$ and the partial order relation $\preceq_1 = \triangle \bigcup \{(c,a),(c,b)\}$. Let the topology $\theta = \{\emptyset, Y, \{y,z\}\}$ on $Y = \{x,y,z\}$, the supra topology associated with θ be $\{\emptyset, Y, \{y\}, \{y,z\}\}$ and the partial order relation $\preceq_2 = \triangle \bigcup \{(y,z)\}$ on Y. The map $f : (X,\tau,\preceq_1) \to (Y,\theta,\preceq_2)$ is defined as $f(a) = y$, $f(b) = z$ and $f(c) = x$. Now, f is supra β-homeomorphism, but is not x-supra β-homeomorphism, for $x = \{$I,D,B$\}$.

Example 5.5. We replace only the partial order relation $\preceq_1$ in Example 5.4 by $\preceq = \triangle \bigcup \{(a,c)\}$. Then the map f is D-supra β-homeomorphism, but it is not B-supra β-homeomorphism.

The relationships among the presented types of supra homeomorphism maps are illustrated in the following figure.

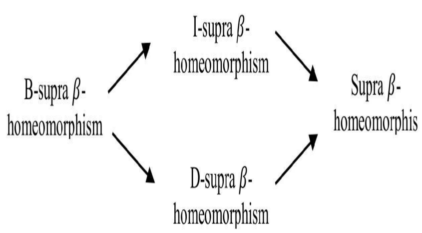

Fig. 3: The relationships among types of supra homeomorphism maps

Theorem 5.6. Let a map $f : X \to Y$ be bijective and I-supra β-continuous. Then the following statements are equivalent:
(i) f is I-supra β-homeomorphism;
(ii) f^{-1} is I-supra β-continuous;
(iii) f is D-supra β-closed.
Proof. $(i) \Rightarrow (ii)$ Let G be an open subset of X. Then $(f^{-1})^{-1}(G) = f(G)$ is an I-supra β-open set in Y. Therefore f^{-1} is I-supra β-continuous.
$(ii) \Rightarrow (iii)$ Let G be a closed subset of X. Then G^c is an open subset of X and $(f^{-1})^{-1}(G^c) = f(G^c) = (f(G))^c$ is an I-supra β-open set in Y. Therefore $f(G)$ is a D-supra β-closed subset of Y. Thus f is D-supra β-closed.
$(iii) \Rightarrow (i)$ Let G be an open subset of X. Then G^c is a closed set and $f(G^c) = (f(G))^c$ is D-supra β-closed. Therefore $f(G)$ is an I-supra β-open subset of Y. Thus f is I-supra β-open. Hence f is an I-supra β-homeomorphism map.■

In a similar way one can prove the following two theorems.

Theorem 5.7. Let a map $f : X \to Y$ be bijective and D-supra β-continuous. Then the following statements are equivalent:
(i) f is D-supra β-homeomorphism;
(ii) f^{-1} is D-supra β-continuous;
(iii) f is I-supra β-closed.

Theorem 5.8. Let a map $f : X \to Y$ be bijective and B-supra β-continuous. Then the following statements are equivalent:
(i) f is B-supra β-homeomorphism;
(ii) f^{-1} is B-supra β-continuous;
(iii) f is B-supra β-closed.

Theorem 5.9. Consider $(X,\tau,\preceq_1)$,$(Y,\theta,\preceq_2)$ are two topological ordered spaces, $\tau^\star$ and $\theta^\star$) are associated supra topologies with τ and θ, respectively. Let $f : X \to Y$ be a supra β-homeomorphism map such that f and f^{-1} are order preserving. If $X(Y)$ is T_i-ordered, then $Y(X)$ is $SS\beta$-T_i-ordered, for $i = 0,1,2$.
Proof. Let $(X,\tau,\preceq_1)$ be strong T_i-ordered, then by Theorem 4.19, $(Y,\theta,\preceq_2)$ is $SS\beta$-T_i-ordered, for $i = 0,1,2$. Let $(Y,\theta,\preceq_2)$ be strong T_i-ordered, then by Theorem 3.16, $(X,\tau,\preceq_1)$ is $SS\beta$-T_i-ordered, for $i = 0,1,2$.

6 Conclusion

In the present paper, the concepts of increasing supra β-continuous (supra β-open, supra β-closed, supra β-homeomorphism) maps, decreasing supra β-continuous (supra β-open, supra β-closed, supra β-homeomorphism) maps and balancing supra β-continuous (supra β-open, supra β-closed, supra β-homeomorphism) maps are given and studied. The sufficient conditions for maps to preserve some separation axioms (which introduced in [6], [9] and [14]) are determined. In particular, we investigate the equivalent conditions for each concept and present their characterizations. Apart from that, we point out the relationships among them with the help of illustrative examples. We plan to use a notion of somewhere dense [4] to define various kinds of maps in topological ordered spaces. In the end, the initiated concepts in this paper are fundamental background for studying several topics in supra topological ordered spaces.

Conflict of interest

The author declares that he has no conflict of interest.

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