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A Two-Warehouse Inventory Model with Quantity Discounts and Maintenance Actions under Imperfect Production Processes

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Abstract: Many earlier studies dealing with the two-warehouse inventory model assumed that the direct cost of the product was irrelevant and that production processes are perfect and stationary. However, in the real world, the purchase cost is some function of the quantity purchased and the production processes may deteriorate and thus defective items will occur. This paper, therefore, aims at developing a new inventory model under an imperfect production process. Three considerations are included in this new model: (1) The supplier may offer quantity discounts to stimulate the retailer into ordering greater lot sizes; (2) The maintenance actions are employed to restore the production process back to the in-control state when the process is out-of-control; (3) A two-warehouse policy is adopted to hold a large amount of stock when a single warehouse would not be sufficient. The unique optimal lot size property and the candidate optimal solution boundaries are derived. An efficient algorithm is developed to help the manager in accurately and quickly determining the order policy. Some numerical examples are given to illustrate the proposed model and algorithm. Some interesting behaviors are also observed.

Keywords: Lot sizing; EOQ and EPQ models; Quantity discounts; Imperfect production system; Maintenance; Two-Warehouse inventory model; Markov chain.

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1 Introduction

In recent years, the issue of economic order quantity (EOQ) or economic production quantity (EPQ) with imperfect quality has received considerable attention from academicians and practitioners because the classical EOQ/EPQ models assume that the production processes are perfect and stationary. Numerous studies have demonstrated that production processes may deteriorate and thus defective items may occur. For example, Rosenblatt and Lee [40] and Porteus [39] are among the first pioneers who explicitly contributed to a significant relationship between quality imperfection and lot size, and showed that the optimal order lot size is smaller than that in the classical EOQ models. Porteus [39] considered an EOQ model and assumed that all items produced are defective when the system is out-of-control. However, Rosenblatt and Lee [40] assumed that a proportion of the items produced are defective once the production process is out-of-control. An interesting variant has been recently proposed by Salameh and Jaber [41] who investigated an EOQ model with imperfect quality in which identifying defective items requires a screening process initiated upon the receipt of an order. Ever since the above model was developed, which is more reasonable than the traditional EOQ model, many extensions were developed. Cárdenas-Barrón [5] corrected an error appearing in the work of Salameh and Jaber [41]. Goyal and Cárdenas-Barrón [19] reconsidered the task performed in Salameh and Jaber [41] and presented a simplified method to determine the optimal lot size. Chan *et al.* [7] proposed a non-shortage model similar to that in Salameh and Jaber [41], wherein products can be classified as good quality, good quality after reworking, imperfect quality and scrap. With respect to the inventory model proposed in Salameh and Jaber [41], Huang [22] investigated a

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single-vendor single-buyer integrated production-inventory problem, where the buyer's inventory policy follows Salameh and Jabers model. Papachristos and Konstantaras [38] questioned the validity of the assumption appearing in Salameh and Jaler's work [41], but failed to provide a solution for this defect. Eroglu and Ozdemir [18], Wee *et al.* [47], and Chang and Ho [9] further extended Salameh and Jaber's work to the case in which shortage backordering is permitted. Building upon the work of Salameh and Jaber, Maddah and Jaber [32] employed renewal theory to correct the flaw in their work and extended the analysis by allowing several batches of defectives to be consolidated and shipped in a single lot. Jaber *et al.* [23] assumed the percentage defective in a shipment reduces in conformance with a learning curve and thus developed two models subject to learning effects. Maddah *et al.* [33] developed two models for news vendor and EOQ-type inventory systems under random yield and items of different quality. A review of the modified EOQ model extensions for imperfect quality items can be found in the work of Khan *et al.* [28]. Following this review, several papers in the literature showed extension or modifications of the work of Salameh and Jaber [41]; for example, Yassine *et al.* [48], Lin [31], Jaber *et al.* [24], and Dey and Giri [17].

More recently, Cárdenas-Barrón [6] proposed a production system with backorders in which the defective items could be reworked. Sana [42] established an economic production lot size model assuming that a certain percent of the total product is defective in the out-of-control state for the imperfect production system. Yoo *et al.* [49] considered an imperfect production and inspection system with customer return and defective disposal. Ouyang and Chang [34] involved the ideas of trade credit and complete backlogging into the EOQ model with imperfect production processes. Pal *et al.* [36] further developed an EOQ buffer for random demand during preventive maintenance or repair of a manufacturing facility with a deteriorating production system. Pal *et al.* [37] explored an EOQ model in an imperfect production system in which the production system may undergo in *out-of-control* state from the *in-control* state after a certain time following a probability density function. This reveals that the deteriorating process follows a two-state discrete-time Markov chain (see, for details, [45]) during production of a lot with a transition occurring with each unit produced. Note that all of the above researchers assumed that it is free to adjust the deteriorating process into the in-control state. However, this assumption may not be true because the maintenance actions that restore the process back to the in-control state require additional costs including labor, material and overhead costs. The work of Hou *et al.* [21] provides a good example. Therefore, it is reasonable that the maintenance costs should be included in the constructed model under the imperfect production system.

It is recognized that quantity discounts can provide economic advantages including lower per-unit purchase cost, lower ordering costs and the decreased likelihood of shortages for both the buyer and vendor (Burwell *et al.* [4], Ji and Shao [26], Lin [31], Chung *et al.* [12]). The supplier usually offers quantity discounts (*i.e*., the unit purchase cost is the function of the order quantities) to stimulate the retailer into ordering larger lot sizes. The retailer may then order larger lot sizes than usual when he/she receives an attractive price discount for purchasing. In this case a single warehouse would not always be sufficient and thus the retailer may employ extra storage space to hold a large stock. Therefore, two-levels of storage, owned warehouse (OW) and rented warehouse (RW) are explored by researchers. Hartley [20] was one of the pioneers in discussing the two warehouse inventory model. Sarma [43] later considered a deterministic inventory model with two levels of storage and infinite production rate. Sarma [44] further extended Hartleys model to explore the deterioration effects in both owned and rented warehouses. Pakkala and Achary [35] proposed a two-warehouse model for deteriorating items with shortages and finite replenishment, in which the rates of item deterioration in the two warehouses are different. A two-warehouse inventory model for items with different deterioration rates, linearly increasing demand and shortages during the finite period was proposed by Bhunia and Maiti [2]. Kar *et al.* [27] further consider the replenishment cost is dependent on the lot size of the current replenishment and thus established a two-warehouse model for non-perishable items with a linear trend in demand and shortages over a fixed and finite time horizon. Zhou [50] extended the existing two-warehouse models to the case with multiple warehouses, in which a type of partial lost sale was taken into an inventory system by assuming it to be a function of shortages already backlogged. Chung *et al.* [10] further took the idea of imperfect quality into the existing two warehouse model to generalize Salameh and Jaber's model [41] in which the storage of rented warehouse was assumed unlimited. Lin [30] studied the economic order quantity mode with imperfect quality and all-unit quantity discounts under two-warehouse consideration and developed two algorithms to determine the optimal lot size and purchasing cost. Dem and Singh [16] developed a two-warehouse manufacturing model for deteriorating items following a time dependent demand pattern in which the systems shift from the in-control state to the out-of-control state, leading to the production of imperfect quality items. Agrawal *et al.* [1] explored an inventory model for deteriorating items following ramp-type demand with flexibility to operate as a two or single warehouse system depending on the model parameters. Some researchers developed a two-warehouse inventory model considering trade credit financing, for example, Liao *et al.* [29], Bhunia *et al.* [3], and Jaggi *et* *al.* [25]. Several other related works on the subject of this paper include (for example) the recent papers [8], [11], [13], [14] and [15].

Based on the above discussions, this study develops a cost minimization two-warehouse inventory model with quantity discounts and maintenance actions under imperfect production processes to explicitly obtain the optimal lot sizing. The main purpose of this paper is four fold:

(1) This paper develops a two-warehouse inventory model with quantity discounts and imperfect production process in which maintenance actions were employed to restore the process back to the in-control state when the process is out-of-control.
(2) This study proves that the expected total cost function has convexity. The closed forms based upon the upper and lower bounds for the candidate optimal solutions are further derived.
(3) An efficient algorithm is provided to help the manager in determining order policy accurately and quickly. Some numerical examples are given to illustrate the proposed model and algorithm.
(4) Managerial insights are drawn.

2 Problem, Definitions and Notations

Consider a deteriorating production system for a product manufactured on a single machine in which the production system operating condition at any time can be classified into one of two states, that is, in-control and out-of-control. While initially producing a lot in the in-control state, a process can go out-of-control with a certain probability q or stay in the in-control state with alternative probability of $1-q$. Once the system is out-of-control it remains in this state until the entire lot is produced. This assumption has been employed by many researchers (see, *e.g.*, Hou *et al.* [21], Maddah *et al.* [33], Porteus [39], and Rosenblatt and Lee [40]). That is, the deteriorating process follows a two-state discrete-time Markov chain during lot production. At the end of a lot produced, the production process is then checked to confirm the state of the process. If the process is out-of-control, it is then restored back to the in-control state with maintenance cost for the production run. Unlike the works of Porteus [39] and Maddah *et al.* [33], this paper assumes that the production system may produce imperfect-quality items with probability q if the production process is out of control. The imperfect-quality items will eventually be reworked at a cost of R such that the production system capacity is completely identical. The ordering cost can be found similar to that of the classical EOQ model. A two-warehouse inventory policy is employed in which the owned warehouse storage capacity is limited and the rented warehouse storage capacity is unlimited. Furthermore, this paper assumes that quantity discounts are offered by the supplier to stimulate the buyer into ordering larger lot sizes. In all-unit quantity discounts, the discount price applies to all units in the order quantity. Let c_j be the unit price of the jth level and Q_j be the jth lowest quantity ($Q_{j-1} < Q_j$). If

$$Q_{j-1} \leqq Q < Q_j,$$

then the unit price is c_j . The price discount schedule is shown in Table (1). That is, the purchasing cost is a function of the ordering quantity, which also influences the holding cost stored at different warehouses (that is, the owned warehouse and the rented warehouse).

Table 1: Price discount structure

j	$Q_{j-1} \leqq Q < Q_j$	c_j
1	$0 < Q < Q_1$	c_1
2	$Q_1 \leqq Q < Q_2$	c_2
.	.	.
.	.	.
.	.	.
n	$Q_{n-1} \leqq Q < Q_n$	c_n
.	.	.
.	.	.
.	.	.
z	$Q_{z-1} \leqq Q < Q_z$	c_z

To establish a two-warehouse inventory model with quantity discounts and an imperfect production process under maintenance actions, the following notations similar to those in Porteus [39] are used.

Notations:

- m demand rate
- K setup cost for each production run
- w the storage capacity of the owned warehouse
- I_r the holding cost per unit time for the rented warehouse, expressed as a fraction of dollar value.
- I_w the holding cost per unit time for the owned warehouse, expressed as a fraction of dollar value, $I_w < I_r$
- n the nth price break for OW in the price schedule in which
$$n = \{ j \mid Q_{j-1} \leqq w < Q_j \}$$
- c_r rework cost for a defective item
- c_j unit cost of jth level corresponding to the cost discount structure
- R maintenance cost for restoring the process from out-of-control back to in-control
- q the probability that the system from in-control state shifts to out-of-control state
- $\overline{q}$ the probability that the system stays in-control state during the production of an item and $\overline{q} = 1-q$

θ the percentage of defective items produced when the process is in the out-of-control state
Q lot size for each production run
N number of defective items produced in a lot of size Q
$f(Q)$ the expected total cost function per unit time for a lot of size Q
$EAC(Q)$ expected annual total cost for a lot of size Q

3 Formulation of the Proposed Model

In this section, a two-warehouse inventory model with quantity discounts and imperfect production process is developed where maintenance actions were employed to restore the production system back to the in-control state. The following results are explored before the model is developed.

Lemma. *The expected number of imperfect-quality items in a lot of size Q is given by*

$$E(N) = \theta \left\{ Q - \frac{\overline{q}(1-\overline{q}^Q)}{q} \right\}. \tag{1}$$

Proof. The proof of the above Lemma is given in Appendix A.

Given that the lot size is Q, the expected annual total cost per production cycle is composed of setup cost, inventory holding cost stored at owned and rented warehouses, purchasing cost, maintenance cost and rework cost, which are derived as follows.

(1) Setup cost:

The setup cost in a production cycle is given by

$$\text{Setup cost} = K. \tag{2}$$

(2) Inventory holding cost stored at owned warehouse (OW) and rented warehouse (RW):

It is recognized that the inventory holding cost per unit time at owned and rented warehouses, denoted by I_w and I_r, respectively, is different and $I_w < I_r$. Based upon this assumption, it is economical to store in OW first and after it is filled, RW is used. RW storage is used first followed by OW. This implies that the following two cases may occur:

(i) The order quantity is equal to or less than the capacity of the owned warehouse (that is, $Q \leqq w$) and thus no additional rented warehouse is needed. Therefore, the inventory holding cost in a production cycle is given by

$$H_{1j} = \frac{I_w c_j Q^2}{2m} \qquad (j = 1, 2, \cdots, n). \tag{3}$$

(ii) The order quantity is equal to or greater than the capacity of the owned warehouse (that is, $Q \geqq w$) and thus additional warehouse capacity is rented. Therefore, the inventory holding cost in a production cycle is given by

$$\begin{aligned} H_{2j} = {} & \frac{I_r c_j \{(Q-w)^2\}}{2m} \\ & + \frac{I_w c_j \{2w(Q-w)^2 + w^2\}}{2m} \qquad (3') \\ & (j = n, n+1, n+2, \cdots, z). \end{aligned}$$

(3) Purchasing cost:

This paper assumes that the supplier offers the all-units-discount method to stimulate the buyer into ordering more quantities. This implies that the purchase price is some function of the quantity purchased. The purchasing cost corresponding to the unit invoice cost c_j in a production run is given as follows:

$$\text{Purchasing cost} = c_j Q. \tag{4}$$

(4) Maintenance cost:

The maintenance cost occurs only when the production process is out-of-control at the end of a production cycle for a lot of size Q. Therefore, the expected maintenance cost per production run is given by

$$\text{Maintenance cost} = R\left(1 - \overline{q}^Q\right). \tag{5}$$

(5) Rework cost:

Because the expected number of imperfect-quality items in a lot size Q is $E(N)$ given in Equation (1), the expected total rework cost per production run is given as follows:

$$c_r E(N) = c_r \theta \left\{ Q - \frac{\overline{q}(1-\overline{q}^Q)}{q} \right\}. \tag{6}$$

Combining the above costs, we know that the expected total cost of a production run becomes the sum of the setup cost, inventory holding cost stored at owned and rented warehouses, purchasing cost, maintenance cost and rework cost. The following two cases may now occur:

Case 1. $Q \leqq w$
The order lot size is equal to or less than the capacity of the owned warehouse. Thus there is no need to rent additional warehouse capacity. Therefore, the total cost of a production cycle in this case is given by

$$\begin{aligned} TC_{1j}(Q) = {} & K + \frac{I_w c_j Q^2}{2m} + c_j Q + R(1 - \overline{q}^Q) \\ & + c_r \theta \left\{ Q - \frac{\overline{q}(1-\overline{q}^Q)}{q} \right\} \qquad (7) \\ & (j = 1, 2, \cdots, n). \end{aligned}$$

Furthermore, the time duration of a production run T is equal to Q/m. When the lot size is Q, the expected total cost corresponding to the purchasing cost c_j is given by

$$\begin{aligned} ATC_{1j}(Q) &= \frac{TC_{1j}(Q)}{T} \\ &= \frac{Km}{Q} + \frac{I_w c_j Q}{2} + c_j m + \frac{Rm(1-\overline{q}^Q)}{Q} \\ &\quad + c_r\theta m - \frac{c_r\theta m\overline{q}(1-\overline{q}^Q)}{qQ} \qquad (8) \\ &\quad (j = 1,2,\cdots,n). \end{aligned}$$

We note that Equation (8) reduces to the model shown in Porteus [39] when the following holds true: (1) We do not take the purchasing cost into account (that is, $c_j Q = 0$); (2) The supplier does not offer a quantity discount and thus $h = I_w c_j$; (3) The maintenance cost is not considered (that is, $R = 0$); (4) All items produced are defective when the process is in the out-of-control state (that is, $\theta = 1$). Furthermore, if the production system does not deteriorate and thus items produced are all perfect quality (that is, $q = 0$), then Equation (8) reduces to the traditional EOQ model.

Case 2. $Q \geqq w$

This case indicates that the order lot size is equal to or greater than the owned warehouse capacity. The rented warehouse is employed to store the additional inventory. Therefore, the total cost of a production run in this case is given by

$$\begin{aligned} TC_{2j}(Q) &= K + \frac{I_r c_j\left\{(Q-w)^2\right\}}{2m} \\ &\quad + \frac{I_w c_j\left\{2w(Q-w)+w^2\right\}}{2m} \\ &\quad + c_j Q + R(1-\overline{q}^Q) \\ &\quad + c_r\theta\left\{Q - \frac{\overline{q}(1-\overline{q}^Q)}{q}\right\} \qquad (9) \\ &\quad (j = n, n+1, n+2, \cdots, z). \end{aligned}$$

Similar to Case 1, we have

$$\begin{aligned} ATC_{2j}(Q) &= \frac{TC_{2j}(Q)}{T} \\ &= \frac{Km}{Q} + \frac{I_r c_j\left\{(Q-w)^2\right\}}{2Q} \\ &\quad + \frac{I_w c_j\left\{2w(Q-w)+w^2\right\}}{2Q} + c_j m + \frac{Rm(1-\overline{q}^Q)}{Q} \\ &\quad + c_r m\theta - \frac{c_r m\theta\overline{q}(1-\overline{q}^Q)}{qQ}, \qquad (10) \\ &\quad (j = n, n+1, n+2, \cdots, z). \end{aligned}$$

We note that, when

$h_r = c_j I_r$, $h_w = c_j I_w$, $q = 0$, $R = 0$, $c_j m = 0$ and $c_r = 0$,

then Equation (10) reduces to the Hartley's model in [20].

Combining Equations (8) and (10), we have

$$ATC_j(Q) = \begin{cases} ATC_{1j}(Q) & (11a) \\ (Q \leqq w;\ j = 1,2,\cdots,n) & \\ ATC_{2j}(Q) & (11b) \\ (Q \geqq w;\ j = n, n+1,\cdots,z) & \end{cases}$$

and

$$ATC_{1n}(w) = ATC_{2n}(w). \qquad (12)$$

Therefore, $ATC_j(Q)$ is continuous on $Q_j > 0$ for $j = 1,2,3,\cdots,z$.

4 The Optimal Solution and Algorithm

In order to minimize the expected total cost per unit time, let us take the first-order derivatives of $ATC_{1j}(Q)$ and $ATC_{2j}(Q)$, respectively. We note that the order quantity corresponding to different purchasing costs should be different. Therefore, in order to clarify this illustration, we take Q_j to replace Q in the following statement.

$$ATC_{1j}{}'(Q_{1j}) = \begin{cases} -\frac{m(K+R)}{Q_j{}^2} + \frac{I_w c_j}{2} & (13a) \\ (q = 1;\ j = 1,2,\cdots,n) & \\ -\frac{mK}{Q_j{}^2} + \frac{I_w c_j}{2} - \frac{m\alpha}{Q_j{}^2}(1-\overline{q}^{Q_j} + Q_j\overline{q}^{Q_j}\ln\overline{q}) & (13b) \\ \left(\alpha = R - \frac{c_r\theta\overline{q}}{q};\ 0<q<1;\ j = 1,2,\cdots,n\right) & \\ -\frac{mK}{Q_j{}^2} + \frac{I_w c_j}{2} \qquad (q = 0;\ j = 1,2,\cdots,n) & (13c) \end{cases}$$

and

$$ATC_{2j}{}'(Q_{2j}) = \begin{cases} -\frac{m(K+R)+c_j w^2\frac{(I_r-Iw)}{2}}{Q_j{}^2} + \frac{I_r c_j}{2} & (14a) \\ (q = 1;\ j = j = n, n+1, n+2,\cdots,z) & \\ -\frac{mK+c_j w^2\frac{(I_r-Iw)}{2}}{Q_j{}^2} + \frac{I_r c_j}{2} & \\ \quad -\frac{m\alpha}{Q_j{}^2}(1-\overline{q}^{Q_j} + Q_j\overline{q}^{Q_j}\ln\overline{q}) & (14b) \\ \left(\alpha = R - \frac{c_r\theta\overline{q}}{q};\ 0<q<1;\ j = n, n+1, n+2,\cdots,z\right) & \\ -\frac{mK+c_j w^2\frac{(I_r-Iw)}{2}}{Q_j{}^2} + \frac{I_r c_j}{2} & (14c) \\ (q = 0;\ j = n, n+1, n+2,\cdots,z). & \end{cases}$$

Unfortunately, it is not easy to determine whether the second derivatives of $ATC_{1j}(Q_1j)$ and $ATC_{2j}(Q_2j)$ are positive or negative. Therefore, we employ such results as Theorem 1 below in order to show that a unique $Q_{1j}{}^*$ exists such that

$$ATC_{1j}{}'(Q_{1j}^*) = 0 \qquad (j = 1,2,\cdots,n).$$

This implies that $ATC_{1j}(Q_1 j)$ is convex on $Q_{1j} > 0$. Alternatively, Theorem 2 illustrates that a unique ${Q_{2j}}^*$ exists satisfying the following condition:

$$ATC_{2j}(Q_{2j}^*) = 0 \qquad (j = n, n+1, n+2, \cdots, z).$$

This indicates the fact that $ATC_{2j}(Q_{2j})$ is convex on $Q_{2j} > 0$. Furthermore, Equations (11a) and (11b) imply that $ATC_j(Q_j)$ is piecewise convex on
$Q_j > 0$ for $j = 1, 2, 3, \cdots, z$.

Theorem 1. *The optimal lot size Q_{1j}^* of $ATC_{1j}(Q_{1j})$ exists and is unique for $j = 1, 2, \cdots, n$.*

Proof. The proof of Theorem 1 is given in Appendix B.

Theorem 2. *The optimal lot size Q_{2j}^* of $ATC_{2j}(Q_{2j})$ exists and is unique for $j = n, n+1, n+2, \cdots, z$.*

Proof. For the proof of Theorem 2, we choose to refer the reader to Appendix C.

In this paper, we treat both of the domains $ATC_{1j}(Q_{1j})$ and $ATC_{2j}(Q_{2j})$ as $(0, \infty)$. Theorem 1 indicates that $ATC_{1j}(Q_{1j})$ is a convex function and thus ${ATC_{1j}}'(Q_{1j})$ increases on $(0, w]$. Alternatively, Theorem 2 indicates that $ATC_{2j}(Q_{2j})$ is a convex function and thus ${ATC_{2j}}'(Q_{2j})$ increases on $[w, \infty)$. Although Theorem 1 and Theorem 2 show that the optimal lot sizes exist for $ATC_{1j}(Q)$ and $ATC_{2j}(Q)$ and are unique, we cannot easily find the closed-form expressions for Q_{1j}^* and Q_{2j}^* for these two cases. Fortunately, however, we can derive the closed forms for the upper and the lower bounds on the candidate optimal lot size. In order to find the bounds for the candidate optimal solutions Q_{1j}^* and Q_{2j}^* when $0 < q < 1$, we introduce the following notations:
(a) The bounds for Q_{1j} when $0 < q < 1$ are denoted by

$$Q_{1j0} = \sqrt{\frac{2mK}{c_j I_w}} \qquad (j = 1, 2, \cdots, n) \tag{15a}$$

and

$$Q_{1j1} = \sqrt{\frac{2m(K+R)}{c_j I_w}} \qquad (j = 1, 2, \cdots, n). \tag{15b}$$

(b) The bounds for Q_{2j} when $0 < q < 1$ are denoted by

$$Q_{2j0} = \sqrt{\frac{2mK + c_j w^2(I_r - I_w)}{c_j I_r}} \tag{16a}$$

$$(j = n, n+1, n+2, \cdots, z)$$

and

$$Q_{2j1} = \sqrt{\frac{2m(K+R) + c_j w^2(I_r - I_w)}{c_j I_r}} \tag{16b}$$

$$(j = n, n+1, n+2, \cdots, z)$$

In view of Equations (B2) and (C2) given in Appendix B and Appendix C, respectively, we have the same term

$$(1 - \overline{q}^{Q_j} + Q_j \overline{q}^{Q_j} \ln \overline{q})$$

presented as a $B(Q_j)$ in the bracket. Before determining the bounds for the candidate optimal solutions for $ATC_{1j}(Q_{1j})$ and $ATC_{2j}(Q_{2j})$, we need Theorem 3 in order to obtain the $B(Q_j)$ property.

Theorem 3. *The following assertion holds true*:

$$0 < B(Q_j) = (1 - \overline{q}^{Q_j} + Q_j \overline{q}^{Q_j} \ln \overline{q}) < 1$$
$$(Q_j > 0; \; j = 1, 2, \cdots, z).$$

Proof. The proof of Theorem 3 is given in Appendix D.

From Theorem 3, we readily obtain

$$0 < B(Q_j) = (1 - \overline{q}^{Q_j} + Q_j \overline{q}^{Q_j} \ln \overline{q}) < 1.$$

Furthermore, by employing Equations (15a), (15b), (16a) and (16b), we have Theorem 4 and Theorem 5 for determining the optimal solutions for $ATC_{1j}(Q_{1j})$ and $ATC_{2j}(Q_{2j})$, respectively.

Theorem 4.

(A) *If $\alpha \leqq 0$, then*

$$0 < {Q_{1j}}^* \leqq Q_{1j0} \leqq Q_{1j1} \qquad (j = 1, 2, \cdots, n).$$

(B) *If $\alpha > 0$, then*

$$0 < Q_{1j0} < {Q_{1j}}^* < Q_{1j1} \qquad (j = 1, 2, \cdots, n).$$

Proof. For the proof of Theorem 4, we refer the reader to Appendix E.

Theorem 5.

(A) *If $\alpha \leqq 0$, then*

$$0 < {Q_{2j}}^* \leqq Q_{2j0} \leqq Q_{2j1} \qquad (j = n, n+1, n+2, \cdots, z).$$

(B) *If $\alpha > 0$, then*

$$0 < Q_{2j0} < {Q_{2j}}^* < Q_{2j1} \qquad (j = n, n+1, n+2, \cdots, z).$$

Proof. The proof of Theorem 5 is given in Appendix F.

Theorems 4 and 5 indicate that the bisection method based upon the *Intermediate Value Theorem* (see, *e.g.*, Varberg *et al.* [46]) is appropriately employed in order to find Q_{1j}^* and Q_{2j}^*, respectively.

As we mentioned above, both $ATC_{1j}(Q_{1j})$ and $ATC_{2j}(Q_{2j})$ are concave on $Q_j > 0$. However, we cannot directly determine the overall optimal solution from the above discussion because each unit purchasing cost corresponds to a different total cost curve. This implies that the optimal order quantity may occur at the break point for the total cost curve (see Hartley [20]). Therefore, in order to find the overall optimal solution, an algorithm is developed as follows to help the manager in making his decision quickly and correctly.

Step 1 Compute α from Equation (13b) or Equation (14b).

Step 2 Compute Q_{1j0} and Q_{1j1} from Equations (15a) and (15b) if $j \leqq n$ and compute Q_{2j0} and Q_{2j1} from Equations (16a) and (16b) if $j \geqq n$ for the unit cost associated with each discount category.

Step 3 If $\alpha \leqq 0$, set

$$Q_{1jL} = 0 \quad \text{and} \quad Q_{1jU} = Q_{1j0} \quad (j \leqq n)$$

and

$$Q_{2jL} = 0 \quad \text{and} \quad Q_{2jU} = Q_{2j0} \quad (j \geqq n).$$

Alternatively, if $\alpha > 0$, set

$$Q_{1jL} = Q_{1j0} \quad \text{and} \quad Q_{1jU} = Q_{1j1} \quad (j \leqq n)$$

and set

$$Q_{2jL} = Q_{2j0} \quad \text{and} \quad Q_{2jU} = Q_{2j1} \quad (j \geqq n).$$

Step 4 Find $Q^*_{1j} \in [Q_{1jL}, Q_{1jU}]$ such that

$$f_{1j}(Q^*_{1j}) = 0$$

by using the bisection method suggested by Varberg *et al.* [46]. Similarly, find $Q^*_{2j} \in [Q_{2jL}, Q_{2jU}]$ such that

$$f_{2j}(Q^*_{2j}) = 0$$

by using the aforecited bisection method again.

Step 5 If Q^*_{1j} is less than the minimum for discount or Q^*_{2j} is less than the minimum for discount, adjust the quantity to $Q =$ the minimum for discount. Alternatively, if Q^*_{1j} is greater than the minimum for discount or Q^*_{2j} is greater than the minimum for discount, the optimal solution would not occur in this range and no additional computational procedures corresponding to the discount category are needed.

Step 6 Compute the expected total cost by means of Equation (9) or Equation (10) for each Q^*_{1j} and Q^*_{2j} or *adjust* Q'. These are associated with the order quantity belonging to the discount category.

Step 7 The lowest expected total cost in Step 6 gives the optimal solution.

We note that, as already mentioned in Section 3, Porteus's work in [39], Hartley's model in [20], and the traditional EOQ model are all special cases of the proposed model when certain specified conditions are satisfied. This helps for validating our model.

5 Numerical Examples

Each of the following examples are presented in order to illustrate the proposed algorithm and model.

Example 1. The parameters needed for analyzing the models developed in this paper are given below:

Demand rate $m = 10000$ units/year;

Ordering cost $K = \$600$/cycle;

Rework cost $c_r = \$5$/unit;

Maintenance cost $R = \$300$/cycle.

Furthermore, we have $\theta = 0.85$ and $q = 0.04$. In addition, we assume that the holding cost is 20% of the purchase price per year in RW and 10% of the purchase price per year in OW. The owned warehouse storage capacity is 2600 units. The rented warehouse storage capacity is unlimited. The supplier offers the price discount schedule as in Table 2.

Table 2: Price discount structure in Example 1

j	$Q_{j-1} \leqq Q < Q_j$	c_j
1	$0 < Q < 600$	$c_1 = 20.20$
2	$600 \leqq Q < 1500$	$c_2 = 20.15$
3	$1500 \leqq Q < 2700$	$c_2 = 20.10$
4	$2700 \leqq Q < 4200$	$c_2 = 20.05$
5	$Q \geqq 4200$	$c_2 = 20.00$

According to the proposed algorithm developed in Section 4, we obtain the optimal order quantity and the minimum cost as follows:

Step 1 Computing α from Equation (13b), we have $\alpha = 198$.

Step 2 Computing Q_{1j0} and Q_{1j1} from Equations (15a) and (15b) if $j \leqq n$, we have

$$Q_{110} = 2437.3 \quad \text{and} \quad Q_{111} = 2985.1;$$

$$Q_{120} = 2440.4 \quad \text{and} \quad Q_{121} = 2988.8;$$

$$Q_{130} = 2443.4 \quad \text{and} \quad Q_{131} = 2992.5.$$

Alternatively, from Equations (16a) and (16b), we have Q_{2j0} and Q_{2j1} when $j \geqq n$ as follows:

$$Q_{230} = 2522.9 \quad \text{and} \quad Q_{231} = 2803.1;$$

$$Q_{240} = 2524.4 \quad \text{and} \quad Q_{241} = 2805.1;$$

$$Q_{250} = 2525.9 \quad \text{and} \quad Q_{251} = 2807.1.$$

Step 3 Since $\alpha = 198 > 0$, we have

$$Q_{11L} = 2437.3 \quad \text{and} \quad Q_{11U} = 2985.1;$$

$$Q_{12L} = 2440.4 \quad \text{and} \quad Q_{12U} = 2988.8;$$

$$Q_{13L} = 2443.4 \quad \text{and} \quad Q_{13U} = 2992.5.$$

and

$$Q_{23L} = 2522.9 \quad \text{and} \quad Q_{23U} = 2803.1;$$

$$Q_{24L} = 2524.4 \quad \text{and} \quad Q_{24U} = 2805.1;$$

$$Q_{25L} = 2525.9 \quad \text{and} \quad Q_{25U} = 2807.1.$$

Step 4 Employing the bisection method, we obtain $Q_{1j}^* \in [Q_{1jL}, Q_{1jU}]$ such that

$$f_{1j}(Q_{1j}^*) = 0$$

as follows:

$$Q_{11}^* = 2799; \quad Q_{12}^* = 2814; \quad Q_{13}^* = 2818.$$

Similarly, we have

$$Q_{23}^* = 2711; \quad Q_{24}^* = 2713; \quad Q_{25}^* = 2715.$$

Step 5 Since Q_{11}^*, Q_{12}^* and Q_{13}^* obtained in Step 4 are greater than the allowable range for discount schedule corresponding to $j = 1$, 2 and 3 in Table 2, the optimal solution would not occur at this range and additional computational procedures are not needed. Focusing on Q_{23}^*, Q_{24}^* and Q_{25}^* obtained in Step 4, we have the following results:

The Q_{23}^* value is above the allowable discount range corresponding to Table 2 for $j = 3$, so there is no need to adjust and no optimal solution occurs; The Q_{24}^* value is between 2700 and 4200 and does not have to be adjusted; The Q_{25}^* value is below the value of 4200 and must be adjusted to 4200 units.
After this step, we test the adjusted quantities:

$$Q_{24}^* = 2713 \quad \text{and} \quad Q_{25}^* = 4200$$

for the total expected cost equation.

Step 6 The expected total cost is computed by using Equation (10) in this example, and we thus have

$$ATC_{24}(2713) = 248666 \quad \text{and} \quad ATC_{25}(4200) = 249210.$$

Step 7 Since

$$ATC_{24}(2713) < ATC_{25}(4200),$$

an order quantity of 2713 units associated with a unit purchasing cost of \$20.05, minimizing the total expected cost \$248666, which is obtained from Equation (10).

Example 2. If the holding cost is 15% of the purchase price per year in OW and the supplier offers another price discount schedule as shown in Table 3 below:

Table 3: Price discount structure in Example 2

j	$Q_{j-1} \leqq Q < Q_j$	c_j
1	$0 < Q < 1000$	$c_1 = 20.20$
2	$1000 \leqq Q < 2200$	$c_2 = 20.15$
3	$2200 \leqq Q < 3400$	$c_2 = 20.10$
4	$3400 \leqq Q < 4600$	$c_2 = 20.05$
5	$Q \geqq 4600$	$c_2 = 20.00$

The same procedure can be performed by using the proposed algorithm. Thus, clearly, an order quantity of 2301 units associated with a unit purchasing cost of \$20.1minimizes the total expected cost \$250437, which is obtained from Equation (9). Thus, in this case, there is no need to rent another warehouse.

Example 3. If the parameters are the same as in Example 2 except that the demand rate increases to 20000 units, then we employ the proposed algorithm again in order to find optimal solution. Therefore, an order quantity of 4600 units (adjusted quantity) minimizes the total expected cost \$495804, which is obtained from Equation (10). This reveals that the unit purchasing cost is \$20.

6 Concluding Remarks and Observations

This paper incorporates the two warehouses OW and RW, quantity discounts and maintenance actions incurred for imperfect production system concepts in order to generalize a Markovian EOQ model which was proposed by Porteus [39]. We have shown that the expected total cost function per unit time is piecewise convex and a unique optimal order lot size exists such that the expected total cost is minimized. The candidate optimal lot size boundaries have been explored to help develop solution procedures. An efficient algorithm was developed to help the manager in accurately and quickly determining the order policy. Some numerical examples were given in order to illustrate the proposed model and algorithm. Our numerical results have demonstrated the following assertions:

(1) The lowest unit purchasing cost corresponding to the cost discount schedule may not guarantee that the buyer could obtain the minimum expected total cost.
(2) The lower the holding cost rate for the owned warehouse, the larger the optimal order quantity and the lower the expected total cost.
(3) As the annual demand increases, the order lot size and the expected total cost also increase.
(4) The higher the storage capacity of the owned warehouse, the larger the order lot size and the lower the expected total cost.

We have also observed that Porteuss work in [39], Hartleys model in [20] and the traditional EOQ model are special cases of the proposed model under certain established conditions.

Appendix A

Proof of the Lemma. Let $\overline{q} = 1-q$ and, in a lot of size Q, the probability distribution of the number of items produced in the in-control state X is

$$P\{X=j\} = \begin{cases} q\overline{q}^{j} & (j=1,2,\cdots,Q-1) \\ \overline{q}^{Q} & (j=Q). \end{cases}$$

Then the first moment of X is given by

$$E(X) = q\sum_{j=1}^{Q-1} j\overline{q}^{j} + Q\overline{q}^{Q} = \frac{\overline{q}(1-\overline{q}^{Q})}{q}.$$

Furthermore, the number of imperfect-quality items in a lot of size Q becomes $N=\theta(Q-X)$. Therefore, the expected value of N is given by

$$E(N) = \theta\left\{Q - \frac{\overline{q}(1-\overline{q}^{Q})}{q}\right\}.$$

This completes the proof of the Lemma.

Appendix B

Proof of Theorem 1. From Equations (13a) to (13c), we know that Theorem 1 holds true when $q=0$ or $q=1$. For the case when $0<q<1$, let

$$\begin{aligned} f_{1j}(Q_{1j}) &= {Q_{1j}}^{2} {ATC_{1j}}'(Q_{1j}^{*}) \\ &= -mk + \frac{I_w c_j {Q_{1j}}^{2}}{2} \\ &\quad - m\alpha(1-\overline{q}^{Q_{1j}} + Q_{1j}\overline{q}^{Q_{1j}}\ln\overline{q}) \qquad \text{(B1)} \\ &\quad (j=1,2,\cdots,n), \end{aligned}$$

since $f_{1j}(Q_1 j)$ is a continuous function with

$$\lim_{Q_{1j}\to 0^{+}} f_{1j}(Q_{1j}) = -mk_0 < 0$$

and

$$\lim_{Q_{1j}\to\infty} f_{1j}(Q_{1j}) = \infty > 0 \qquad (j=1,2,...,n).$$

Furthermore, the first derivative of $f_{1j}(Q_{1j})$ is given by

$$ {f_{1j}}'(Q_{1j}) = Q_{1j}[I_w c_j - m\alpha(\ln\overline{q})^2\overline{q}^{Q_{1j}}] \qquad \text{(B2)}$$
$$(j=1,2,\cdots,n).$$

In order to illustrative the uniqueness property of the optimal lot size for $ATC_{1j}(Q_{1j})$, two cases are discussed as follows:

(i) If $\alpha \leqq 0$, then we have

$${f_{1j}}'(Q_{1j}) > 0$$

for $Q_{1j}>0$ and $j=1,2,\cdots,n$. In this case, $f_{1j}(Q_{1j})$ is strictly increasing on $Q_{1j}>0$. This implies that $ATC_{1j}(Q_{1j})$ is a convex function. Therefore, if

$$\alpha \leqq 0 \qquad \text{and} \qquad f_{1j}(Q_{1j}) = 0,$$

we have a unique optimal solution Q_{1j}^{*} such that

$$ATC'_{1j}(Q_{1j}) = 0.$$

(ii) If $\alpha>0$, we have

$${f_{1j}}'(Q_{1j}) \leqq 0 \quad \text{for} \quad Q_{1j} \leqq \overline{Q_{1j}} = \frac{1}{\ln\overline{q}}\ln\left(\frac{I_w c_j}{m\alpha(\ln\overline{q})^2}\right)$$
$$(j=1,2,\cdots,n).$$

This indicates that $f_{1j}(Q_{1j})$ is decreasing first and then is increasing to infinity as Q_{1j} increases for all $j=1,2,\cdots,n$. In this case, $f_{1j}(Q_{1j})$ is strictly increasing on $(\overline{Q_{1j}},\infty)$. This implies that $ATC_{1j}(Q_{1j})$ is a convex function given

$$Q_{1j} > \overline{Q_{1j}} \qquad \text{for} \qquad j=1,2,\cdots,n.$$

Therefore, if

$$\alpha > 0 \qquad \text{and} \qquad f_{1j}(Q_{1j}) = 0,$$

we have a unique optimal solution Q_{1j}^{*} of $ATC_{1j}(Q_{1j})$ with respect to $Q_{1j}^{*} > \overline{Q_{1j}}$, and thus

$${ATC_{1j}}'(Q_{1j}) = 0.$$

Based upon the above discussions, we now know that Theorem 1 holds true. We have thus completed the proof of Theorem 1.

Appendix C

Proof of Theorem 2. From Equations (14a) to (14c), just as in Appendix B, Theorem 2 holds true when $q=0$ or $q=1$. For the case when $0<q<1$, let

$$\begin{aligned} f_{2j}(Q_{2j}) &= Q_{2j}{}^2 ATC_{2j}{}'(Q_{2j}^*) \\ &= -mk + \frac{I_r c_j Q_{2j}{}^2 - c_j w^2 (I_r - I_w)}{2} \\ &\quad - m\alpha(1-\overline{q}^{Q_2 j} + Q_{2j}\overline{q}^{Q_{2j}} \ln \overline{q}) \qquad \text{(C1)} \\ &\quad (j=n,n+1,n+2,\cdots,z). \end{aligned}$$

Since $f_{2j}(Q_2 j)$ is a continuous function with

$$\lim_{Q_{2j}\to 0^+} f_{2j}(Q_{2j}) = -mk_0 - \frac{c_j w^2 (I_r - I_w)}{2} < 0$$

and

$$\lim_{Q_{2j}\to\infty} f_{2j}(Q_{2j}) = \infty > 0$$

$$(j=n,n+1,n+2,\cdots,z).$$

Furthermore, the first-order derivative of $f_{2j}(Q_{2j})$ is given by

$$f_{2j}{}'(Q_{2j}) = Q_{2j}[I_r c_j - m\alpha(\ln\overline{q})^2 \overline{q}^{Q_{2j}}] \qquad \text{(C2)}$$

$$(j=n,n+1,n+2,\cdots,z).$$

Now, in light of Equation (B2) given in Appendix B, we know that Equation (C2) is similar to Equation (B2). Therefore, by employing the same procedure as in Appendix B, we deduce that Theorem 2 holds true and thus complete the proof of Theorem 2.

Appendix D

Proof of Theorem 3. When $0<q<1$, let

$$B(Q_j) = 1-\overline{q}^{Q_j} + Q_j \overline{q}^{Q_j} \ln\overline{q}.$$

Then $B(Q_j)$ is a continuous function of Q_j with $B(0)=0$ and

$$\lim_{Q_j\to\infty} B(Q_j) = 1.$$

Furthermore, we have

$$B'(Q_j) = Q_j \overline{q}^{Q_j} (\ln\overline{q})^2 > 0 \quad \text{for} \quad Q_j > 0 \quad (j=1,2,\cdots,z).$$

This implies that $B(Q_j)$ is strictly increasing as Q_j increases, and thus

$$0 < B(Q_j) < 1$$

for $Q_j>0$ and $j=1,2,\cdots,z$. Therefore, we have

$$0 < B(Q_j) = 1-\overline{q}^{Q_j} + Q_j\overline{q}^{Q_j}\ln\overline{q} < 1$$

for $Q_j>0$ and $j=1,2,\cdots,z$. This evidently completes the proof of Theorem 3.

Appendix E

Proof of Theorem 4. We consider the following cases:

Case (i): $\alpha \leqq 0$

In view of Equation (B2) given in Appendix B, we have

$$f_{1j}{}'(Q_{1j}) > 0 \qquad \text{and} \qquad Q_{1j} > 0$$

for $j=1,2,\cdots,n$. We also note that $f_{1j}(Q_{1j})$ is increasing on $[0,\infty)$ and

$$f_{1j}(0) = -mK.$$

According to Theorem 3, we have

$$\begin{aligned} f_{1j}(Q_{1j0}) &= -m\alpha[1-\overline{q}^{Q_{1j0}} + Q_{1j0}\overline{q}^{Q_{1j0}}(\ln\overline{q})] \\ &> 0 = f_{1j}(Q_{1j}^*) \end{aligned}$$

and

$$\begin{aligned} f_{1j}(Q_{1j1}) &= mR - m\alpha[1-\overline{q}^{Q_{1j1}} + Q_{1j1}\overline{q}^{Q_{1j1}}(\ln\overline{q})] \\ &> 0 = f_{1j}(Q_{1j}^*). \end{aligned}$$

This implies that

$$Q_{1j}^* > Q_{1j0} \qquad \text{and} \qquad Q_{1j}^* \leqq Q_{1j1}$$

when $\alpha \leqq 0$. Furthermore, from Equations (15a) and (15b), we know that

$$0 < Q_{1j0} \leqq Q_{1j1}.$$

Therefore, we have

$$0 < Q_{1j}^* \leqq Q_{1j0} \leqq Q_{1j1}$$

for $j=1,2,\cdots,n$.

Case (ii): $\alpha > 0$

Substituting from Equation (15b) into Equation (B1) given in Appendix B for this case, we have

$$f_{1j}(Q_{1j1}) = m\left[R\left(1-B(Q_{1j1}) + \frac{c_r\theta\overline{q}}{q}B(Q_{1j1})\right)\right]$$

Thus, from Theorem 3, we know that

$$f_{1j}(Q_{1j1}) > 0 = f_{1j}(Q_{1j}^*).$$

This implies that

$$Q_{1j}^* \leqq Q_{1j1}$$

when $\alpha>0$. Furthermore, from Equation (B1) given in Appendix B and Equation (15a), we know that

$$f_{1j}(Q_{1j0}) = -m\alpha B(Q_{1j0}).$$

In this case, we have

$$f_{1j}(Q_{1j0}) < 0 = f_{1j}(Q_{1j}^*)$$

when $\alpha > 0$. This implies that

$$Q_{1j}^* > Q_{1j0}.$$

Therefore, we have

$$0 < Q_{1j0} < Q_{1j}^* < Q_{1j1} \qquad (j = 1, 2, \cdots, n).$$

By combining Case (i) and Case (ii), we have completed the proof of Theorem 4.

Appendix F

Proof of Theorem 5. We consider the following cases:

Case (i): $\alpha \leqq 0$

In light of Equation (C2) given in Appendix C, we have

$$f_{2j}{}'(Q_{2j}) > 0 \qquad \text{and} \qquad Q_{2j} > 0$$

for $j = n, n+1, n+2, \cdots, z$. We also note that $f_{2j}(Q_{2j})$ is increasing on $[0, \infty)$ and

$$f_{2j}(0) = -mK - \frac{c_j\, w^2(I_r - I_w)}{2}.$$

Thus, according to Theorem 3, we have

$$\begin{aligned} f_{2j}(Q_{2j0}) &= -m\alpha\left[1 - \overline{q}^{Q_{2j0}} + Q_{2j0}\overline{q}^{Q_{2j0}}(\ln \overline{q})\right] \\ &> 0 = f_{2j}(Q_{2j}^*) \end{aligned}$$

and

$$\begin{aligned} f_{2j}(Q_{2j1}) &= mR + \frac{c_j\, w^2(I_r - I_w)}{2} \\ &\quad - m\alpha\left[1 - \overline{q}^{Q_{2j1}} + Q_{2j1}\overline{q}^{Q_{2j1}}(\ln \overline{q})\right] \\ &> 0 = f_{2j}(Q_{2j}^*). \end{aligned}$$

This implies that

$$Q_{2j}^* > Q_{2j0} \qquad \text{and} \qquad Q_{2j}^* \leqq Q_{2j1}$$

when $\alpha \leqq 0$. Furthermore, from Equations (16a) and (16b), we know that

$$0 < Q_{2j0} \leqq Q_{2j1}.$$

Therefore, we have

$$0 < Q_{2j}^* \leqq Q_{2j0} \leqq Q_{2j1}$$

for $j = n, n+1, n+2, \cdots, z$.

Case (ii): $\alpha > 0$

Upon substituting from Equation (16b) into Equation (C1) given in Appendix C for this case, we have

$$f_{2j}(Q_{2j1}) = m\left[R\left(1 - B(Q_{2j1}) + \frac{c_r\theta\overline{q}}{q}B(Q_{2j1})\right)\right].$$

Therefore, from Theorem 3, we know that

$$f_{2j}(Q_{2j1}) > 0 = f_{2j}(Q_{2j}^*).$$

This shows that

$$Q_{2j}^* \leqq Q_{2j1}$$

when $\alpha > 0$. Furthermore, from Equation (C1) given in Appendix C and Equation (15b), we know that

$$f_{2j}(Q_{2j0}) = -m\alpha B(Q_{2j0}).$$

In this case, we have

$$f_{2j}(Q_{2j0}) < 0 = f_{2j}(Q_{2j}^*)$$

when $\alpha > 0$. This implies that

$$Q_{2j}^* > Q_{2j0}.$$

Consequently, we have

$$0 < Q_{2j0} < Q_{2j}^* < Q_{2j1} \qquad (j = n, n+1, n+2, \cdots, z).$$

Finally, by combining Case (i) and Case (ii), we complete the proof of Theorem 5.

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