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# The Differences in the Shape of Characteristic Curves of a Column Subjected to Eulers Load Obtained on the Basis of Bernoulli- Euler and Timoshenko Theories

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**Abstract:** In this paper the formulation of the boundary problem of the natural vibration frequency on the basis of Hamiltons principle with consideration of Timoshenko theory has been presented. The investigated column is loaded by external compressive load with constant line of action (Eulers load). Introduction of discreet elements such as springs on both ends of the column allows one to create the general form of boundary conditions. The proper set of their stiffness corresponds to different boundary conditions. In this study an influence of slenderness and external load as well as boundary condition on the shape of characteristic curves calculated with consideration of Bernoulli- Euler and Timoshenko theories has been shown. The presented results show the direction of proper choice of the beam theory in the studies on natural vibration frequency of slender supporting systems.

**Keywords:** Vibrations, Column, Timoshenko beam Theory, Instability, Euler load

## 1 Introduction

In the formulation of the boundary problem of the natural vibration of columns subjected to compressive load the theory of Bernoulli Euler [1,3,4,5,6,7,9,10,11] or Timoshenko can be used. In the literature also the Shear Beam Model [4,9] and Rayleigh Beam Model [4,9] can be found. When the theory of Timoshenko is taken into account the two models can be found in which the form of differential equations is different. Model 1 has been presented by Kolousek [6] while Model 2 by Nemat-Nasser [10]. Sato [16] in his study has presented the forms of differential equations of motion for Model 1 and 2 by means of Hamiltons principle. He also concluded that the results of numerical calculations are more accurate when Model 2 has been taken into account. This phenomenon occurs due to ignoring in the equations of Model 1 of $\gamma^2$($\gamma$ shear angle). With the high shear angle magnitude the results obtained on the basis of Model 1 can differ significantly from the reality. That is why in the theoretical investigations with consideration of high shear angle the application of Model 2 gives better results. Katsikadelis and Kounadis [5] have considered Timoshenko beam column subjected to compressive follower force [2]. In their investigations the Timoshenko theory has been used Model 1 and Model 2. The presented in [5] results of numerical simulations are focused on the flutter loading and flutter vibration frequency. The calculations have been done at different magnitude of the slenderness factor as well as moment of inertia of the concentrated mass localized on the loaded end of the column. Authors have proved that the magnitudes of flutter loading as well as flutter vibration frequency are greater for Model 1 than for Model 2. An increase of the slenderness factor magnitude of the system causes the reduction of the investigated parameters.

An important step in the investigations on the supporting systems is the creation of relation between external loading force and natural vibration frequency. Curves created on the basis of these results are called characteristic curves. The characteristics curves can vary due to different types of action of external load. In the figure 1 the curves calculated for divergence (figure.1a see. [12,14,15]), flutter (figure. 1b see. [2,13]) and divergence-pseudoflutter (figure. 1c see. [14]) systems respectively have been shown. Besides of characteristic curves presented in the figure 1, the curves corresponding to limiting case of the hybrid systems [8] can be found. In

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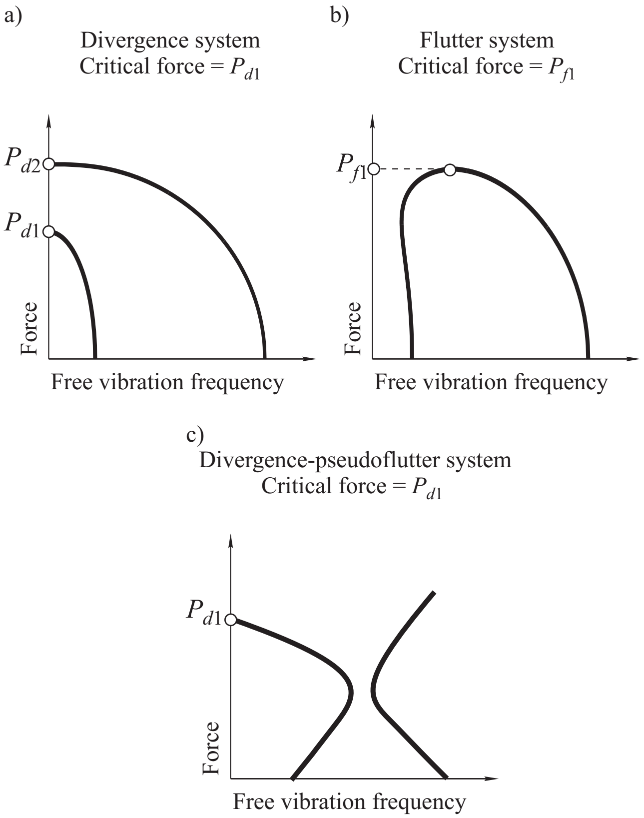


**Fig. 1:** Types of characteristic curves on the plane external load – free vibration frequency: a) divergence system, b) flutter system, c) divergence – pseudoflutter system

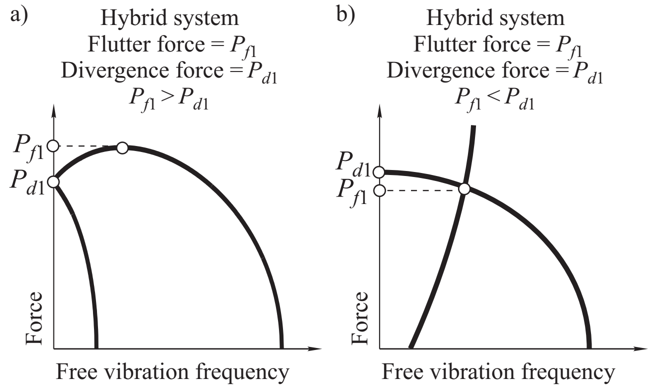


**Fig. 2:** Types of characteristic curves on the place external load – free vibration frequency of the hybrid system

the hybrid systems the type of instability (divergence or flutter) depends on structural and loading parameters. Implementation of limiting structural and loading parameters allows one to obtain the curve external load vibration frequency which is characterized by both divergence and flutter instability. In the case of hybrid systems the two types of characteristic curves can be presented. In the first one the divergence critical loading is smaller than the flutter one (figure. 2a see [13, 14]). The second type is characterized by smaller magnitude of flutter critical load than critical divergence load (figure 2b see [17]).

Abramovich in the work [1] has done research on the column subjected to axial compressive force. The author has presented the relation between frequency and loading parameters. The frequency parameter in the paper [1] has been defined as square of relation between natural vibration frequency and reference vibration frequency (in this case the reference vibration frequency is the one calculated for zero magnitude of external load). The loading parameter is a ratio of external load to critical force of the system. The frequency and loading parameters can change from 0 up to 1. Abramovich [1] has compared the results calculated on the basis of Bernoullie-Euler and Timoshenko theories. It has been concluded that the greatest difference in the obtained results on the basis of both theories can be found at loading parameter 0.5. The main purpose of this paper is to study an influence of different types of supports of the column on the difference between the shape of characteristic curves calculated on the basis of two theories Bernoullie  Euler and Timoshenko (Model 2).

## 2 Problem formulation

In this paper the column with installed on both ends discreet elements is presented. The discreet elements are as follows: two rotational springs (stiffness: $C_{R_0}$ ($x = 0$) and $C_{R_1}$ ($x =$ l)) and one translational one (stiffness $C_T$). The proper selection of stiffness allows one to achieve different types of supports (boundary conditions) of the investigated slender system presented in the figure 3. The column is composed of one rod with circular cross section area ($d$  diameter). The column is loaded by a force with constant line of action when the system leans out of static equilibrium. In the literature this type of loading is called Eulers force.

In the paper the formulation of natural vibrations has been done with consideration of Timoshenko theory. The boundary problem can be derived by means of Hamiltons principle (see [16]):

$$\delta \int_{t_1}^{t_2} (T - V)dt = 0 \tag{1}$$

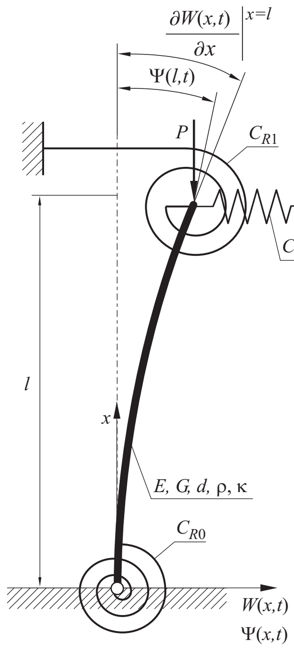


**Fig. 3:** Investigated system subjected to Euler's load

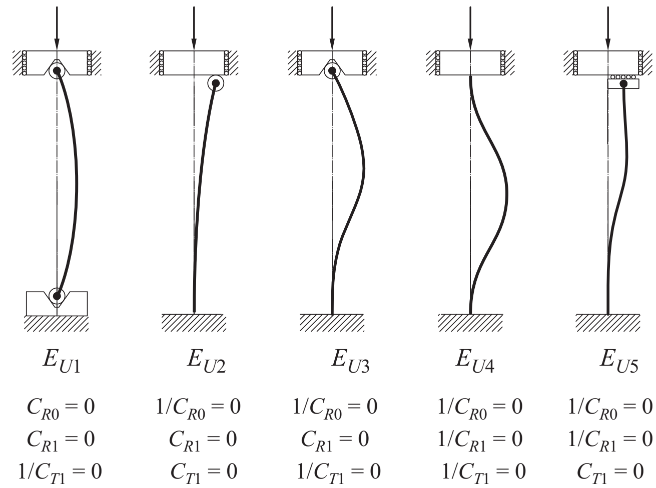


**Fig. 4:** Specific types of the considered slender system

The kinetic $T$ and potential $V$ energies of the considered column can be written in the form:

$$T=\frac{1}{2}(\rho A)\int_0^l\left[\frac{\partial W(x,t)}{\partial t}\right]^2dx+\frac{1}{2}(\rho J)\int_0^l\left[\frac{\partial \Psi(x,t)}{\partial t}\right]^2dx \tag{2}$$

$$\begin{aligned}V=&\tfrac{1}{2}(EJ)\int_0^l\left[\tfrac{\partial \Psi(x,t)}{\partial x}\right]^2dx-\tfrac{1}{2}P\int_0^l\left[\tfrac{\partial W(x,t)}{\partial x}\right]^2dx+\\&+\tfrac{1}{2}(AG\kappa)\int_0^l\left[\tfrac{\partial W(x,t)}{\partial x}-\Psi(x,t)\right]^2dx+\tfrac{1}{2}C_{R0}(\Psi(0,t))^2+\\&+\tfrac{1}{2}C_{R1}(\Psi(l,t))^2+C_{T1}(W(l,t))^2\end{aligned} \tag{3}$$

where: $W(x,t)$  deflection of the section, $\psi(x,t)$ rotation angle of the section, $E$ Young modulus, $G$ Kirchhoff modulus, $A$  cross-section area, $J$  axial geometrical moment of inertia of the column's section, $\kappa$ - the shear coefficient which depends on section's shape (circular cross-section  = 0.91), $\rho$- density of the material.

Five different types of supports of the slender system has been taken into account. The schemes of each type of support are presented in the figure 4 ($E_{U_i}$ notation). Individual types of supports are dependent on value of springs stiffness (comp. figure 4).

The differential equations of motion obtained on the basis of Hamiltons principle (1) are as follows (see [16]):

$$\begin{aligned}&(EJ)\tfrac{\partial^2\Psi(x,t)}{\partial x^2}+AG\kappa\left[\tfrac{\partial W(x,t)}{\partial x}-\Psi(x,t)\right]+\\&-(\rho J)\tfrac{\partial^2\Psi(x,t)}{\partial t^2}=0\end{aligned} \tag{4}$$

$$\begin{aligned}&AG\kappa\left[\tfrac{\partial^2 W(x,t)}{\partial x^2}-\tfrac{\partial \Psi(x,t)}{\partial x}\right]-P\tfrac{\partial^2 W(x,t)}{\partial x^2}+\\&-(\rho A)\tfrac{\partial^2 W(x,t)}{\partial t^2}=0\end{aligned} \tag{5}$$

There exists only one geometrical boundary condition of the column presented in the figure 3:

$$W(0,t)=0 \tag{6}$$

Introduction of (6) into Hamiltons principle leads to natural boundary conditions:

$$(EJ)\left.\frac{\partial \Psi(x,t)}{\partial x}\right|_{x=0}-C_{R0}\Psi(0,t)=0 \tag{7}$$

$$(EJ)\left.\frac{\partial \Psi(x,t)}{\partial x}\right|^{x=l}+C_{R1}\Psi(l,t)=0 \tag{8}$$

$$\begin{aligned}&(AG\kappa)\left(\left.\tfrac{\partial W(x,t)}{\partial x}\right|^{x=l}-\Psi(x,t)\right)-P\left.\tfrac{\partial W(x,t)}{\partial x}\right|^{x=l}+\\&+C_{T1}W(l,t)=0\end{aligned} \tag{9}$$

The boundary conditions of each configuration (figure 4 - configurations dependent on value of springs stiffness) are in the form:

- column $E_{U_1}$ ($C_{R_0}=0$; $C_{R_1}=0$; $1/C_{T_1}=0$)

$$W(0,t)=0 \tag{10a}$$

$$(EJ)\left.\frac{\partial\Psi(x,t)}{\partial x}\right|_{x=0}=0 \quad (10b)$$

$$W(l,t)=0 \quad (10c)$$

$$(EJ)\left.\frac{\partial\Psi(x,t)}{\partial x}\right|^{x=l}=0 \quad (10d)$$

- column $E_{U_2}$ ($1/C_{R_0}$ = 0; $C_{R_1}$ = 0; $C_{T_1}$ = 0)

$$W(0,t)=0 \quad (11a)$$

$$\Psi(0,t)=0 \quad (11b)$$

$$(EJ)\left.\frac{\partial\Psi(x,t)}{\partial x}\right|^{x=l}=0 \quad (11c)$$

$$(AG\kappa)\left(\left.\frac{\partial W(x,t)}{\partial x}\right|^{x=l}-\Psi(x,t)\right)-P\left.\frac{\partial W(x,t)}{\partial x}\right|^{x=l}=0 \quad (11d)$$

- column $E_{U_3}$ ($1/C_{R_0}$ = 0; $C_{R_1}$ = 0; $1/C_{T_1}$ = 0)

$$W(0,t)=0,\Psi(0,t)=0,W(l,t)=0 \quad (12a\text{-}c)$$

$$(EJ)\left.\frac{\partial\Psi(x,t)}{\partial x}\right|^{x=l}=0 \quad (12d)$$

- column $E_{U_4}$ ($1/C_{R_0}$ = 0; $1/C_{R_1}$ = 0; $1/C_{T_1}$ = 0)

$$W(0,t)=0,\Psi(0,t)=0,W(l,t)=0,\Psi(l,t)=0 \quad (13a\text{-}d)$$

- column $E_{U_5}$ ($1/C_{R_0}$ = 0; $1/C_{R_1}$ = 0; $C_{T_1}$ = 0)

$$W(0,t)=0,\Psi(0,t)=0,\Psi(l,t)=0 \quad (14a\text{-}c)$$

$$(AG\kappa)\left(\left.\frac{\partial W(x,t)}{\partial x}\right|^{x=l}-\Psi(x,t)\right)-P\left.\frac{\partial W(x,t)}{\partial x}\right|^{x=l}=0 \quad (14d)$$

The solution of the differential equations (4) and (5) can be presented as a harmonic functions:

$$W(x,t)=w(x)\cos(\omega t),\Psi(x,t)=\psi(x)\cos(\omega t) \quad (15a\text{-}b)$$

Introduction of solutions (15) and completion of mathematical operations allows one to write (4) and (5) in the form:

$$y^{IV}(\xi)+\frac{\Theta^2\left(-\Omega^2[\varphi+1]-\Theta^2\varphi\lambda\right)+\Omega^2\lambda}{\Theta^2\left(\lambda-\Theta^2\varphi\right)}y^{II}(\xi)+ +\frac{\Omega^2\left(\Theta^4\varphi-\Omega^2\right)}{\Theta^2\left(\lambda-\Theta^2\varphi\right)}y(\xi)=0 \quad (16)$$

$$\psi^{IV}(\xi)+\frac{\Theta^2\left(-\Omega^2[\varphi+1]-\Theta^2\varphi\lambda\right)+\Omega^2\lambda}{\Theta^2\left(\lambda-\Theta^2\varphi\right)}\psi^{II}(\xi)+ +\frac{\Omega^2\left(\Theta^4\varphi-\Omega^2\right)}{\Theta^2\left(\lambda-\Theta^2\varphi\right)}\psi(\xi)=0 \quad (17)$$

where:

$$\xi=\frac{x}{l},y(\xi)=\frac{w(x)}{l},\lambda=\frac{Pl^2}{(EJ)},\Theta^2=\frac{Al^2}{J},\varphi=\frac{\kappa G}{E}, \quad (18a\text{-}e)$$

$$\Omega^2=\frac{(\rho A)\,l^4\omega^2}{(EJ)} \quad (18f)$$

The differential equations (16) and (17) depend on one spatial variable $\zeta$ . That is why this condition must be met in every time period $t$ for $\zeta\in$ (0, 1). The solution of (16) and (17) must met the boundary conditions in which the equations (15) are introduced.

The solutions of differential equations (16) and (17) depend on relation between parameters:

$$\Gamma=\frac{\Theta^2\left(-\Omega^2[\varphi+1]-\Theta^2\varphi\lambda\right)+\Omega^2\lambda}{\Theta^2(\lambda-\Theta^2\varphi)} \quad (19a)$$

and

$$\Phi=\frac{\Omega^2\left(\Theta^4\varphi-\Omega^2\right)}{\Theta^2(\lambda-\Theta^2\varphi)} \quad (19b)$$

The solutions can be expressed as follows:

- solution A

if ($\Gamma>0$ and $\Gamma$R/2 $<(\Gamma^2/4+\Phi)^{0.5}$)
or
( $\Gamma<0$ and (/2$\Gamma$ + $(\Gamma^2/4+\Phi)^{0.5}$) ¿ 0):

$$y(\xi)=B_{A1}\cosh(\alpha_A\xi)+B_{A2}\sinh(\alpha_A\xi)+ +B_{A3}\cos(\beta_A\xi)+B_{A4}\sin(\beta_A\xi) \quad (20a)$$

$$\psi(\xi)=C_{A1}\cosh(\alpha_A\xi)+C_{A2}\sinh(\alpha_A\xi)+ +C_{A3}\cos(\beta_A\xi)+C_{A4}\sin(\beta_A\xi) \quad (20b)$$

where:

$$\alpha_A=\sqrt{-\frac{\Gamma}{2}+\sqrt{\frac{\Gamma^2}{4}+\Phi}} \quad (21a)$$

$$\beta_A=\sqrt{\frac{\Gamma}{2}+\sqrt{\frac{\Gamma^2}{4}+\Phi}} \quad (21b)$$

$$C_{A1}=B_{A2}\frac{\alpha_A^2\left(\varphi\Theta^2-\lambda\right)+\Omega^2}{\alpha_A\varphi\Theta^2} \quad (22a)$$

$$C_{A2} = B_{A1}\frac{\alpha_A^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\alpha_A\varphi\Theta^2} \tag{22b}$$

$$C_{A3} = B_{A4}\frac{\beta_A^2\left(\varphi\Theta^2 - \lambda\right) - \Omega^2}{\beta_A\varphi\Theta^2} \tag{22c}$$

$$C_{A4} = B_{A3}\frac{-\beta_A^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\beta_A\varphi\Theta^2} \tag{22d}$$

- solution B

if $(\Gamma < 0 \; and \; \Gamma/2 > (\Gamma^2/4 + \Phi)^{0.5})$:

$$y(\xi) = B_{B1}\cos(\beta_{B1}\xi) + B_{B2}\sin(\beta_{B1}\xi) + \\ + B_{B3}\cos(\beta_{B2}\xi) + B_{B4}\sin(\beta_{B2}\xi) \tag{23}$$

$$\psi(\xi) = C_{B1}\cos(\beta_{B1}\xi) + C_{B2}\sin(\beta_{B1}\xi) + \\ + C_{B3}\cos(\beta_{B2}\xi) + C_{B4}\sin(\beta_{B2}\xi) \tag{24}$$

where:

$$\beta_{B1} = \sqrt{\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \Phi}} \tag{25a}$$

$$\beta_{B2} = \sqrt{\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} + \Phi}} \tag{25b}$$

$$C_{B1} = B_{B2}\frac{\beta_{B1}^2\left(\varphi\Theta^2 - \lambda\right) - \Omega^2}{\beta_{B1}\varphi\Theta^2} \tag{26a}$$

$$C_{B2} = B_{B1}\frac{-\beta_{B1}^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\beta_{B1}\varphi\Theta^2} \tag{26b}$$

$$C_{B3} = B_{B4}\frac{\beta_{B2}^2\left(\varphi\Theta^2 - \lambda\right) - \Omega^2}{\beta_{B2}\varphi\Theta^2} \tag{26c}$$

$$C_{B4} = B_{B3}\frac{-\beta_{B2}^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\beta_{B2}\varphi\Theta^2} \tag{26d}$$

solution C

if ( $\Gamma < 0$ and ($\Gamma/2$ + ( $\Gamma^2/4\Phi)0^{.5}$) ¡ 0)

$$y(\xi) = B_{C1}\cosh(\alpha_{C1}\xi) + B_{C2}\sinh(\alpha_{C1}\xi) + \\ + B_{C3}\cosh(\alpha_{C2}\xi) + B_{C4}\sinh(\alpha_{C2}\xi) \tag{27}$$

$$\psi(\xi) = C_{C1}\cosh(\alpha_{C1}\xi) + C_{C2}\sinh(\alpha_{C1}\xi) + \\ + C_{C3}\cosh(\alpha_{C2}\xi) + C_{C4}\sinh(\alpha_{C2}\xi) \tag{28}$$

where:

$$\alpha_{C1} = \sqrt{-\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} + \Phi}} \tag{29a}$$

$$\alpha_{C2} = \sqrt{-\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \Phi}} \tag{29b}$$

$$C_{C1} = B_{C2}\frac{\alpha_{C1}^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\alpha_{C1}\varphi\Theta^2} \tag{30a}$$

$$C_{C2} = B_{C1}\frac{\alpha_{C1}^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\alpha_{C1}\varphi\Theta^2} \tag{30b}$$

$$C_{C3} = B_{C4}\frac{\alpha_{C2}^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\alpha_{C2}\varphi\Theta^2} \tag{30c}$$

$$C_{C4} = B_{C3}\frac{\alpha_{C2}^2\left(\varphi\Theta^2 - \lambda\right) + \Omega^2}{\alpha_{C2}\varphi\Theta^2} \tag{30d}$$

Introduction of solutions (according to the relation $\Gamma$ and $\Phi$) into boundary conditions leads to system of equations:

$$[a_{ij}] \; col\{B_{i1}, B_{i2}, B_{i3}, B_{i4}\} = 0 \tag{31}$$

where: i stands for chosen method of solution A, B or C. The determinant of the matrix of coefficients (31) equated to zero leads to transcendental equation on natural vibration frequency, obtained on the basis of Timoshenko theory.

$$|a_{ij}| = 0 \tag{32}$$

## 3 Results of numerical calculations

In this paper the results of numerical calculations on the differences between characteristic curves obtained on the basis of two theories Bernoulli Euler and Timoshenko are presented. The following non dimensional parameters have been used for the presentation purposes:

$$\Lambda_P = \frac{P}{P_{crB-E}}, \Lambda_{\omega i} = \frac{\omega_{B-Ei} - \omega_{Ti}}{\omega_{B-Ei}}100\% \tag{33a,b}$$

where: $\Lambda_P$ parameter associated with external load. The force $P_{crB-E}$ is a critical force designated on the basis of Bernoulli-Euler theory and $P$ is an external load magnitude. By means of $\Lambda_{\omega_i}$ the difference between vibration frequencies is presented. The frequencies shown in (33b) are calculated under consideration of proper theory (Bernoulli - Euler ($\omega_{B-E_i}$) and Timoshenko ($\omega_{T_i}$)). In this paper an investigations on the difference in the first three natural vibration frequencies have been done. In the (33) the each frequency is marked by $i$ index. Additionally the slenderness $\zeta$ parameter (quotient of total length to the diameter $\zeta = l/d$) has been used in the numerical calculations. The presented results of numerical calculation corresponds to the particular cases shown in the figure 4.

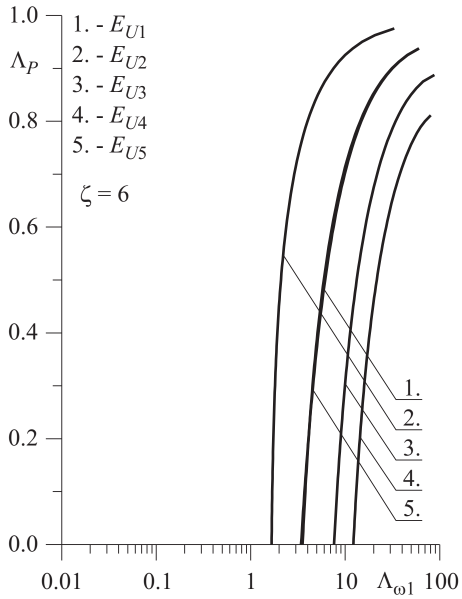


**Fig. 5:** Relation $\Lambda_P(\Lambda_{\omega_1}) : \zeta = 6$

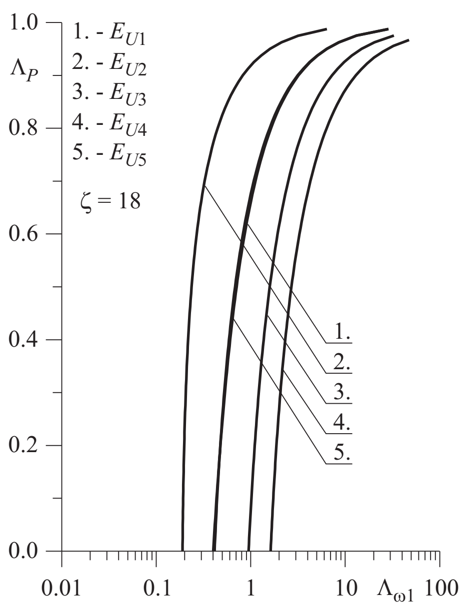


**Fig. 6:** Relation $\Lambda_P(\Lambda_{\omega_1}) : \zeta = 18$

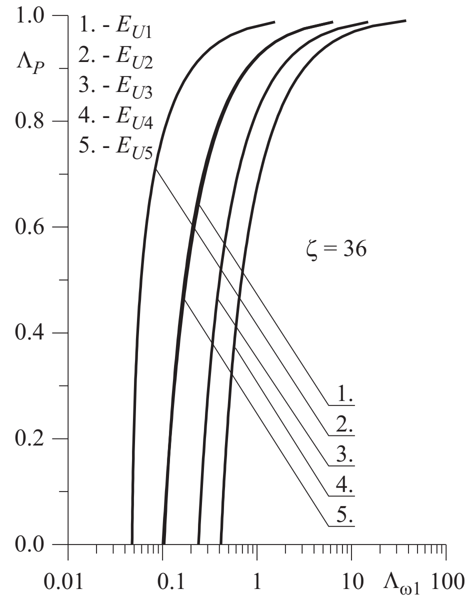


**Fig. 7:** Relation $\Lambda_P(\Lambda_{\omega_1}) : \zeta = 36$

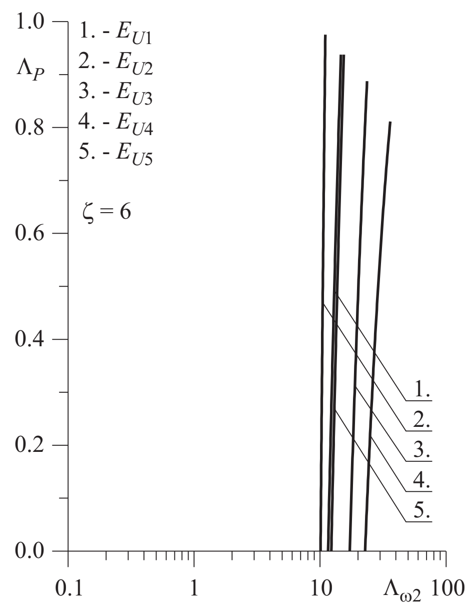


**Fig. 8:** Relation $\Lambda_P(\Lambda_{\omega_2}) : \zeta = 6$

In the figures 5 13 the curves $\Lambda_P(\Lambda_{\omega_i})$with consideration of different types of supports and $\zeta$ coefficient magnitude (figures 5,8,11 $\zeta = 6$, figures 6,9,12 $\zeta = 18$, figures 7,10,13 $\zeta = 36$) have been presented. On the basis of the analysis of the results of numerical calculations it can be concluded that difference in both theories (Bernoulli-Euler, Timoshenko) are getting greater at higher vibration frequencies. For instance when the configuration $E_{U_2}$ is taken into account at zero magnitude of external load and lowest slenderness parameter ($\zeta = 6$ ) the differences in the results are as follows: frequency 1 $\approx$ 1.8 % frequency 2 $\approx$ 10% frequency 3 $\approx$ 20%. The different relation can be found when an influence of external load on the investigated parameters has been taken into account. The greatest differences in the magnitudes of $\Lambda_{\omega_i}$ along the external load axis have been observed at first vibration frequency. in this case the magnitude of $\Lambda_{\omega_i}$ parameter is increasing

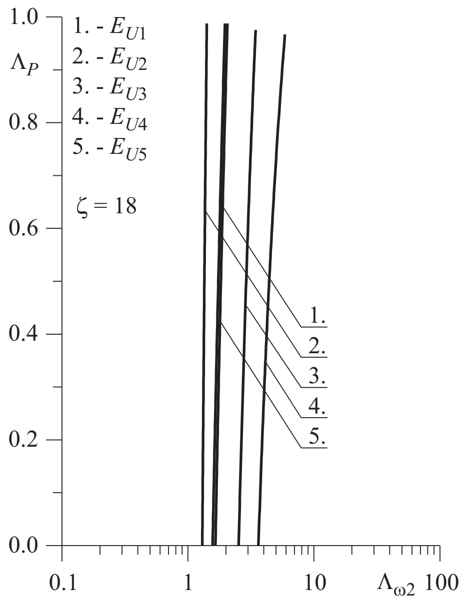


**Fig. 9:** Relation $\Lambda_P(\Lambda_{\omega_2}) : \zeta = 18$

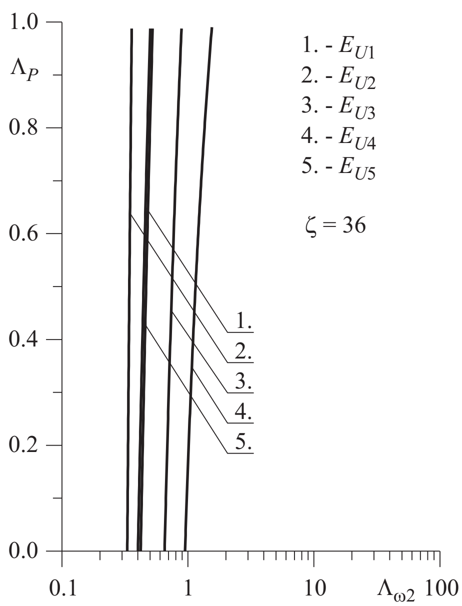


**Fig. 10:** Relation $\Lambda_P(\Lambda_{\omega_2}) : \zeta = 36$

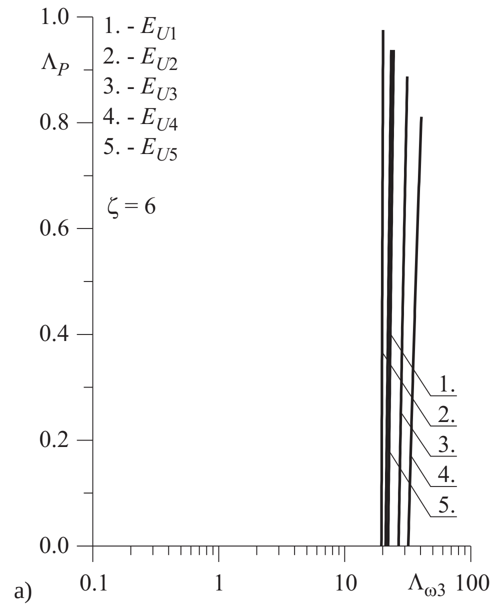


**Fig. 11:** Relation $\Lambda_P(\Lambda_{\omega_3}) : \zeta = 6$

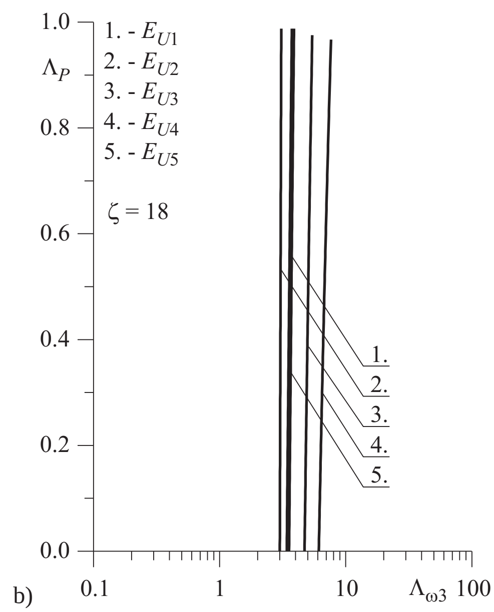


**Fig. 12:** Relation $\Lambda_P(\Lambda_{\omega_3}) : \zeta = 18$

according to increase of external load. At higher natural vibration frequencies (second and third) the change of $\Lambda_{\omega_i}$ has not been observed. An increase of slenderness parameter $\zeta$ causes the reduction of $\Lambda_{\omega_i}$.

Taking into account the presented results of numerical calculations in can be concluded that the type of support has an influence on the magnitudes of natural vibration frequencies obtained on the basis of theories of Bernoulli Euler and Timoshenko. The highest magnitudes of $\Lambda_{\omega_i}$) have been obtained at $E_{U_4}$ configuration (column in the $E_{U_4}$ configuration has the greatest magnitude of critical load) and the lowest ones at $E_{U_2}$. Presented curves on the plane $\Lambda_P(\Lambda_{\omega_i}$ are ending at different magnitudes of $\Lambda_P$ because calculations are limited to the critical force obtained on the basis of Timoshenko theory which is smaller than critical one obtained with Bernoulli-Euler theory.

c)

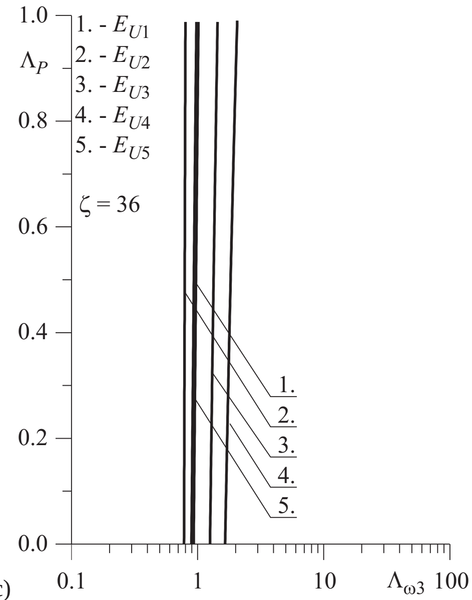


**Fig. 13:** Relation $\Lambda_P(\Lambda_{\omega_3}) : \zeta = 36$

## 4 Concluding remarks

In this paper the slender system (column) subjected to compressive conservative external load (defined by Euler) has been presented. In the problem formulation the discreet elements in the form of springs (one translational and two rotational) have been used. The boundary problem has been formulated on the basis of Hamiltons principle and Timoshenko theory. Implementation of discreet elements allows one to obtain boundary conditions for different configurations of the system (see figure 4) by means of proper set of their stiffness. In this study the difference in the vibration frequencies calculates on the basis of two theories have been compared. It has been shown that the type of support, slenderness and external load have an influence on the results of numerical calculations relating to vibration frequency. On the basis of the results presented in this paper the proper choice of beam theory can be done. The type of theory (Bernoulli-Euler simpler with easier implementation or more complicated Timoshenko) used in the studies can be chosen with consideration of systems parameters such as slenderness, external load or support type. The issue of research shown in this paper can be developed in the future. In the next studies an influence of different types of external load, mass and mass moment of inertia localized on both ends of the column on shape of characteristic curves should be done.

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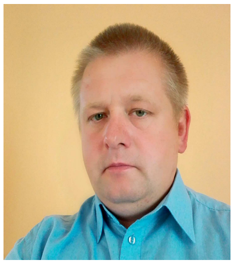
## References

[1] H. Abramovich, Natural frequencies of Timoshenko beams under compressive axial loads, Journal of Sound and Vibration, 157(1), 183-189 (1992).
[2] M. Beck, Die knicklast des einseiting eingespannten tangential geduruckten stabes, ZAMP, 3, 225-228 (1952).
[3] P. A. Djondjorov and V. M. Vassilev, On the dynamic stability of a cantilever under tangential follower force according to Timoshenko beam theory, Journal of Sound and Vibration, 311, 1431-1437 (2008).
[4] M. H. Kargarnovin and R. A. Jafari-Talookolaei, Aplication of the homotopy method for the analytic approach of the nonlinear free vibration analysis of the simple end beams using four engineering theories, Acta Mechanica, 212, 199-213 (2010).
[5] J. T. Katsikadelis and A. N. Kounadis, Flutter loads of a Timoshenko beam-column under a follower force governed by two variants of equations of motion, Acta Mechanica, 48, 209-217 (1983).
[6] V. Kolousek, Calcul des efforts dynamiques dans les assatures rigides, Vibrations des pouters, des pourtques at des arcs, Paris, Dunod, 86-89 (1959).
[7] A.N. Kounadis and J.T. Katsikadelis, Shear and rotary inertia effect on Becks columns, Journal of Sound and Vibration, 49(2), 171-178 (1976).
[8] H. Leipholz, Aspects of dynamic stability of structures, Journal of engineering Mechanics Division, 109-124 (1975).
[9] L. Meirovitch, Elements of Vibration Analysis, McGraw-Hill, New York, 1986.
[10] S. Nemat-Nasser, Instability of a cantilever under a follower force according to Timoshenko beam theory, Journal of Applied Mechanics, 34, 484-485 (1967).
[11] V. Stojanovic, P. Kozic, R. Pavlovic, G. Janevski, Natural frequencies of Timoshenko beams under compressive axial loads, Journal of Sound and Vibration, 157(1), 183-189 (1992).
[12] K. Sokol, Linear and Nonlinear Vibrations of a Column with an Internal Crack, Journal of Engineering Mechanics, 140(5), 2014.
[13] C. Sundararajan, Influence of an elastic end support on the vibration and stability of Becks column, International Journal of Mechanical Sciences, 18(5), 239-241 (1976).
[14] L. Tomski and S. Uzny, Free vibration and the stability of a geometrically non-linear column loaded by a follower force directed towards the positive pole, International Journal of Solids and Structures, 45(1), 87-112 (2008).
[15] L. Tomski and S. Uzny, The regions of flutter and divergence instability of a column subjected to Becks generalized load, taking into account the torsional flexibility of the loaded end of the column, Mechanics Research Communications, 38, 95-100 (2011).
[16] K. Sato, On the governing equations for vibration and stability of a Timoshenko beam: Hamiltons principle, Journal of Sound and Vibration, 145(2), 338-340 (1991).
[17] L. Tomski, S. Uzny, Free Vibrations and Stability of a New Slender System Subjected to a Conservative or Non-Conservative Load, Journal of Engineering Mechanics-ASCE, 139(8), 1133-1148 (2013).



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