Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/090203

On the Failure of Gambling Systems Under Unfavorable Odds

Jason Parker[1,*], *Subhash Bagui*[1,*], *and K. L. Mehra*[2]

[1]Department of Mathematics and Statistics, The University of West Florida, Pensacola, FL 32514, United States
[2]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada

Received: 27 Sep. 2019, Revised: 13 March 2020, Accepted: 17 March 2020
Published online: 1 May 2020

Abstract: The analysis of systems of gambling, in which the gambler attempts to overcome a disadvantage of unfavorable odds in a sequence of plays by judicious choice of bet sizes, has been a recurring theme in the development of probability theory. The impossibility of these and related systems has at times been used to formalize the notion of a random sequence of trials. A more modern approach is to prove theorems to the effect that such systems fail with probability tending to unity as the number of trials increases. That failure of systems is here made more precise with a strong convergence result first stated by Thorp in his book, *The Mathematics of Gambling* [1] . Thorp's proof outline [2] apparently relies on the invalid assumption that the amounts won or lost on different trials are independent. Building on previous analyses by Doob [3] and Feller [4], we obtain a corrected proof.

Keywords: Gambling, Roulette, Strong Law of Large Numbers, Almost Sure Convergence, Unlimited Capital, Finite Starting Capital.

1 Introduction

1.1 Background

Suppose a gambler places bets on each of a sequence of games of roulette. If the ball lands on his number, his initial wager is returned to him, plus an additional 35 times that amount. Otherwise, he loses the amount he wagered. Since there are 38 slots on the wheel (1 through 36, zero, and double zero), the gambler would need to receive 37:1 payout odds to make the game fair. Thus on any given trial, the gambler has a slight negative expectation. In particular, he loses $2/38$, or slightly more than 5 cents, for every dollar bet on average [5].

Gamblers are typically aware of this disadvantage. But many of them are convinced that by varying the size of their wager in accordance with a strategy or "system", the disadvantage on any single trial can be reversed over the course of a sequence of trials. Indeed, one frequently finds books recommending such systems in the gaming section of the any bookstore. Bearing such impressive names as *Martingale*, *LaBouchere*, *d'Alembert* [6], the common theme of these systems is that they are "progressive." That is, as long as the bettor is losing, he chases his losses by making increasingly large wagers so that when he finally has some good luck he recovers his losses, plus a small amount. This will happen eventually, it is reasoned, and then the process can be repeated.

In practice, when odds are unfavorable, such systems may not work out precisely according to expectations. Eventually, a string of bad luck may either bankrupt the gambler, or require him to place a bet that exceeds the maximum limit allowed by the casino. Since bet sizes increase roughly exponentially in progressive systems, a disastrous losing streak need not be very long [1].

Therefore, we shall consider only systems that have been modified to operate between the minimum and maximum bet limits, and therefore can no longer guarantee a small win with probability one [2]. Before proceeding, it is worth noting that even in the ideal case of unlimited funds and no cap on the bet size, a gambler following a progressive strategy faces the unlikely (probability zero) but real possibility of an unending sequence of exponentially increasing losses. This is not attractive given that the gambler wins only modest sums when the system works as advertised. In such an idealized situation, it is not clear that the theoretically positive expectation makes the bet "rational."

* Corresponding author e-mail: jasonparker5011@gmail.com, sbagui@uwf.edu

1.2 Plan of Paper

Having thus restricted ourselves to systems that can be implemented in practice, we will show in Section 3 that when minimum and maximum bet limits are imposed, any system for choosing the size of the current bet based on the results of previous wagers must fail in the following sense, stated by Thorp [1]:

– First, given any initial capital, the probability that the gambler has been ruined before the n^{th} trial tends to one as n increases. This result corresponds to the weak law of large numbers.
– Second, if the player has unlimited capital and continues to play forever, his average gain per unit wagered will converge almost surely to the expected value of a single unit bet, which is assumed to be negative. This result corresponds to the strong law of large numbers.

Surely, no system with such inevitably poor long-term results can be properly called a “winning” system. The most that can be said for it is that it may often result in a small win, all the while running the risk of a huge loss.

In Section 4, we extend these results to two slightly more advanced cases, in which the bettor gets to choose among multiple wager options as well as their size, and also in which the wager has more than two outcomes.

In Section 5, we discuss the similarities and differences between these results and the more standard Strong Law of Large Numbers theorems, drawing attention to the typical lack of independence between amounts won on different trials. We give examples to show that the ordinary sample mean of the amounts won need not converge in any of the usual senses: in probability or almost surely.

To prove these two basic results in Section 3, we will require two elementary lemmas. They are presented in Section 2.

2 Two Lemmas

Lemma 1. *If $\{X_n\}$ and $\{Y_n\}$ are sequences of random variables such that $X_n \xrightarrow{a.s.} 0$ as $n\to\infty$, and for some λ, $|Y_n|\le\lambda$ for all n, then we have that*

$$X_nY_n \xrightarrow{a.s.} 0, \quad as \quad n\to\infty. \tag{1}$$

Proof. By assumption, for every $\varepsilon>0$, $P(\{|X_n|>\varepsilon \quad i.o.\})=0$. Then for every $\varepsilon'>0$ we must have:

$$\begin{aligned} P(\{|X_nY_n|>\varepsilon' \ i.o.\}) &= P(\{|X_nY_n|>\varepsilon' \ i.o.\},\forall n|Y_n|\le\lambda) \\ &\quad + P(\{|X_nY_n|>\varepsilon' \ i.o.\},\exists n|Y_n|>\lambda) \\ &\le P(\{|X_n|>\varepsilon'/\lambda \ i.o.\}) \\ &= 0. \end{aligned} \tag{2}$$

The result follows. □

Lemma 2. *Suppose that there are M nondecreasing sequences $\left\{\{n_j\}:1\le j\le M;\quad \sum_{j=1}^{M} n_j=n\right\}$ of random variables, taking natural number values, which sum to n (and are thus indexed by n, although it is not shown explicitly), where we also suppose that for each $j\in\{1,2,\ldots,M\}$, we have $n_j\to\infty$ as $n\to\infty$. Further, suppose that there are M sequences of r.v.'s $\{X_{jn_j}:1\le j\le M\}$ such that $X_{jn_j}\xrightarrow{a.s.}0$ as $n_j\to\infty$ for each j, $1\le j\le M$, as $n\to\infty$. Then we have*

$$\sum_{j=1}^{M}\frac{n_j}{n}X_{jn_j}\xrightarrow{a.s.}0, \quad as \quad n\to\infty. \tag{3}$$

Proof. For each j, the sequence $\{Y_{jn}\}=\{\frac{n_j}{n}\}$ is bounded between 0 and 1 as $n\to\infty$. Applying Lemma 1, we get for each $j\in\{1,2,\ldots,M\}$

$$\frac{n_j}{n}X_{jn_j}\xrightarrow{a.s.}0, \quad \text{as} \quad n\to\infty. \tag{4}$$

Then the result follows from the linearity of almost sure convergence. □

3 Failure of Gambling Systems

Here we use the preceding lemmas together with properties of almost sure convergence to extend the result given in Feller [4] that a gambler restricted to a fixed bet size cannot obtain an advantage by selecting a sub-sequence of trials on which to bet. From this we will obtain the two results stated in Subsection 1.2 above. We will consider the case of unlimited capital first in Subsection 3.1, followed by the case of finite starting capital in Subsection 3.2.

3.1 Unlimited Capital

Feller [4, pp. 185-187] analyzed “systems” in which the bet size is fixed, but the gambler may decline to bet on certain trials of his choice. Feller's analysis uses the result of Doob [3] that sub-sequences of independent Bernoulli trials are themselves i.i.d Bernoulli sequences. Provided that the gambler wagers infinitely often, his long-term results are asymptotically no different than if he had wagered on all trials. By the SLLN for i.i.d. Bernoulli sequences, the fraction of bets won will tend to the success probability p. By assumption, the odds $\mathscr{O}$ paid on winning bets is such that the game is unfair: $p\mathscr{O}-q<0$. Thus the gambler will be a long-term loser almost surely.

With our two lemmas, the SLLN for Bernoulli sequences, and Doob's result on subsequences, we may draw conclusions about the long-run performance of a gambling system in which the bet size is varied from one trial to the next. First, we will consider that the gambler has unlimited access to capital, and thus continues playing forever.

Initially, we assume that the gambling system is such that each wager size occurs infinitely often with unit probability. Then we will show the restriction can be relaxed.

By Doob's result, the sub-sequences on which the player bets $j = 1,2,...,M$ units are each infinite, i.i.d. Bernoulli sequences. We are assuming that $n_j \to \infty$ as $n \to \infty$ for each j. Thus for each such sub-sequence, the SLLN applies. Let $S_{jn_j} = \sum_{i=1}^{n_j} I_{ji}$, where $I_{ji} = 1$ if the i^{th} bet of the j^{th} sequence (of j-unit bets) wins and is zero otherwise. Then by the SLLN, for each j, $1 \le j \le M$,

$$\frac{S_{jn_j}}{n_j} - p \xrightarrow{a.s.} 0, \quad \text{as} \quad n_j \to \infty \quad (\text{and thus as } n \to \infty). \tag{5}$$

From this we obtain:

$$\frac{W_{jn_j}}{n_j} - j[(\mathscr{O}+1)p-1] \xrightarrow{a.s.} 0, \tag{6}$$

where $W_{jn_j} = j[(\mathscr{O}+1)S_{n_j} - n_j]$ is the net gain from all bets of size j units during the first n trials. Rewriting the above convergence relation, we get

$$\frac{W_{jn_j}}{n_j} - j(p\mathscr{O}-q) \xrightarrow{a.s.} 0 \quad \text{for } j \in \{1,2,...,M\}, \tag{7}$$

which, by virtue of the second lemma, results in

$$\sum_{j=1}^{M} \frac{n_j}{n}\left[\frac{W_{jn_j}}{n_j} - j(p\mathscr{O}-q)\right] \xrightarrow{a.s.} 0, \quad \text{as} \quad n \to \infty. \tag{8}$$

This is the same as $\frac{W_n}{n} - \frac{A_n}{n}(p\mathscr{O}-q) \xrightarrow{a.s.} 0$, where $W_n = \sum_j W_{jn_j}$ is the total net gain after n trials, and $A_n = \sum_j jn_j$ is the total amount wagered (known as the *action*). Finally, the sequence $Y_n = \frac{n}{A_n}$ is a bounded sequence with $1/M \le Y_n \le 1$. Therefore, we can apply the first lemma again to obtain our first result:

$$\frac{W_n}{A_n} - (p\mathscr{O}-q) \xrightarrow{a.s.} 0, \quad \text{as} \quad n \to \infty. \tag{9}$$

Now we relax the requirement that all bet sizes are used infinitely often. Since we are assuming that the betting continues forever, at least one bet size must occur infinitely often. Assume that, for the system $\mathscr{S}$ in question, some bet size, say $j = M$, occurs only finitely often with nonzero probability. Then it is easy to construct a system $\mathscr{S}'$ with the same convergent behavior as $\mathscr{S}$, but in which wagers of size $j = M$ occur infinitely often. Starting with $\mathscr{S}$, the idea is simply to substitute size $j = M$ for one of the more persistent sizes at increasingly long intervals, in such a way that the ratio n_M/n is not affected as $n_M \to \infty$. This has no effect on the long-run tendency of W_n/A_n. Then, since the above result applies to the modified system $\mathscr{S}'$, it must apply to the original system $\mathscr{S}$ as well. Thus we have shown the following:

Theorem 1. *For the case of unlimited starting capital, the ratio of the net gain W_n to the total amount wagered A_n converges almost surely as follows:*

$$\frac{W_n}{A_n} \xrightarrow{a.s.} p\mathscr{O}-q, \quad as \quad n \to \infty. \tag{10}$$

From this we conclude that a gambler following any system of choosing bet sizes based on previous results will eventually become a loser and stay a loser forever, with probability one. And the ratio of his total losses to total action will tend to converge to the expectation on a single unit wager.

3.2 Finite Starting Capital

Since almost sure convergence implies convergence in probability, we immediately have that for any $\varepsilon > 0$,

$$P(W_n > A_n(\varepsilon + p\mathscr{O}-q)) \le P\left(\left|\frac{W_n}{A_n} - (p\mathscr{O}-q)\right| > \varepsilon\right) \to 0, \tag{11}$$

as $n \to \infty$. Since $p\mathscr{O}-q < 0$ by assumption, for small ε we have $\varepsilon + (p\mathscr{O}-q) < 0$ as well. Given any initial capital C, however large, for sufficiently large n we will have

$$A_n(\varepsilon + p\mathscr{O}-q) < -C. \tag{12}$$

Therefore, we have as our second result,

Theorem 2. *Given finite starting capital C, the net gain W_n satisfies:*

$$P(W_n > -C) \le P(W_n > A_n(\varepsilon + p\mathscr{O}-q)) \to 0. \tag{13}$$

Thus the probability that the gambler has any capital remaining after n trials tends to zero.

4 Two Extensions

The above result can be extended in two different directions.

4.1 Extension I

The first is to assume that, at the start of each new play, one may choose among L different binary wagers. Each wager has its own success probability p_i and odds $\mathscr{O}_i$ paid out on wins. Following the previous analysis, the net gain after n plays becomes

$$W_n = \sum_{j=1}^{M} \sum_{i=1}^{L} j\left[(\mathscr{O}_i + 1)S_{ijn_{ij}} - n_{ij}\right], \qquad (14)$$

where n_{ij} is the number of times a bet of size j was placed on wager option i during the first n plays. Thus $\sum_i \sum_j n_{ij} = n$.

Similarly, the total amount wagered A_n is the sum of all the amounts wagered on each of the L options: $A_n = \sum_{i=1}^{L} A_{in} = \sum_{i=1}^{L}\sum_{j=1}^{M} jn_{ij} = \sum_{j=1}^{M} jn_j$. Here n_j is simply the number of bets of size j placed in the first n plays, regardless of which option they were placed on.

Using the two lemmas as before, we obtain,

$$\frac{W_n}{A_n} - \frac{\sum_{i=1}^{L} A_{in}(p_i\mathscr{O}_i - q_i)}{A_n} \xrightarrow{a.s.} 0. \qquad (15)$$

This result agrees with our intuition that the average gain per unit bet should tend to converge to a weighted average of the expectations for each of the L wager options, where the weights are the total amounts wagered on each option.

4.2 Extension II

A second extension is to consider that, instead of multiple wager options to choose from at each play, there is a single option that is non-binary. Specifically, there are $K+1$ possible outcomes for each trial: $\Omega = \{\omega_0, \omega_1, \omega_2, ..., \omega_K\}$. If outcome ω_0 occurs, the amount wagered is lost. This happens with probability $p_0 = q$. For $k = 1, 2, ..., K$, if outcome ω_k occurs, the bettor wins $\mathscr{O}_k$ times the amount wagered. This happens with probability p_k, and we have that $q + \sum_{k=1}^{K} p_k = 1$.

An analysis entirely analagous to the ones above gives the result

$$\frac{W_n}{A_n} \xrightarrow{a.s.} -q + \sum_{k=1}^{K} p_k \mathscr{O}_k. \qquad (16)$$

The average gain per unit wagered converges almost surely to the expectation of a unit wager, which was to be expected.

5 Discussion

There are a couple of observations to be made. First of all, while the sequence of Bernoulli trials $\{X_k\}$ that the player wagers on is assumed to consist of independent trials, the actual sequence of gains and losses are not generally independent. Suppose that the amount wagered, Y_k on a given trial is determined by the outcomes of previous trials. Then the amount B_k won on the k^{th} trial is a random variable:

$$B_k = Y_k[(\mathscr{O} + 1)X_k - 1]. \qquad (17)$$

To see that the gains on different trials are not in general independent, consider this simplistic gambling system: Bet one unit on the first trial. If it is a success, bet the maximum of M units thereafter. If it is a failure, continue betting the minimum of one unit thereafter. Assuming the wager pays even money (that is, $\mathscr{O} = 1$), it can be shown that the covariance of B_1 and B_k is proportional to $p - q$:

$$\begin{aligned} Cov(B_1, B_k) &= (p-q)[pM - q - (p-q)(pM+q)] \\ &= 2pq(p-q)(M-1). \end{aligned} \qquad (18)$$

In Thorp's treatment of this problem, the B_k's are considered to be independent [2, p. 85]. This assumption appears to be a necessary part of the derivation, thus invalidating the proof.

A second observation is that the strong convergence result proved here concerns the ratio of two random variables, rather than an ordinary sample mean. The total amount wagered A_n is random because bet sizes depend on the outcomes of previous trials. Further, while the result could be rearranged into a format more recognizable for SLLN-type results, i.e.

$$\frac{W_n}{n} - \frac{A_n}{n}(p\mathscr{O} - q) \xrightarrow{a.s.} 0, \qquad (19)$$

which refers to the sample mean of the winnings per bet, it is not merely an immediate result of any of the standard laws of large numbers. For example, it is not possible to conclude that

$$\frac{W_n}{n} - \frac{E[A_n]}{n}(p\mathscr{O} - q) \xrightarrow{a.s.} 0, \qquad (20)$$

as we could if the gains on different trials were independent (though not identically distributed). In other words, we cannot conclude that the sample mean of the gain from a wager converges almost surely to average expected gain per trial. Although the restrictions on bet sizes guarantees that the individual wagers have finite mean and variance, the possible long-range dependence between bet sizes and previous outcomes prevents us from saying that the sample mean will converge in the above sense. Thus Kolmogorov's SLLN for independent r.v.'s with variance restrictions does not apply [7, p. 94].

To see this, consider once more the strategy in which all bets on trials $k = 2,3,4,\ldots$ are the same size: the minimum (one unit) if the first trial was a failure, or the maximum (M units) if it was a success. For all trials after the first, the expected size of a bet is thus $pM+q$, and therefore $E[A_n]/n \to pM+q$ as $n \to \infty$. However, the sample mean W_n/n converges either to $M(p\mathscr{O}-q)$ or $p\mathscr{O}-q$ according as the first trial was a success or a failure.

It is even easier to see that the most well known versions of the SLLN, in which the sample mean converges to a contant value, cannot apply here. A betting scheme which is completely deterministic can result in the non-convergence of the sample mean. For example, by simply alternating between betting the minimum, the maximum, and an intermediate value for periods of exponentially increasing length, the sample mean will continually bounce between three values.

Below we simulate a Bernoulli sequence where the success probability is $p = 1/3$. The odds paid out are only $\mathscr{O} = 1.5$, less than the 2:1 odds required to make the game fair. The bettor makes wagers of 1 unit on the first 100 trials, 100 units on the next 1000 trials, and 50 units on the next 10000 trials.

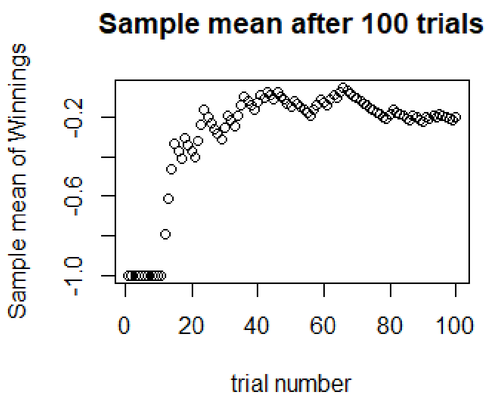

Fig. 1: Plot of sample mean for 100 Bernoulli trials with $p = 1/3$ and $\mathscr{O} = 1.5$
. Bet size is 1 unit.

In Figure 1, we plot the sample mean of the gain over 100 trials of unit bets. We see that the bettor starts off on a losing streak, but soon recovers. The sample mean approaches the expected value of -1/6.

During the next 1000 trials, the bettor wagers 100 units each trial. In Figure 2 we see that the sample mean now starts to approach the expectation of a single wager,

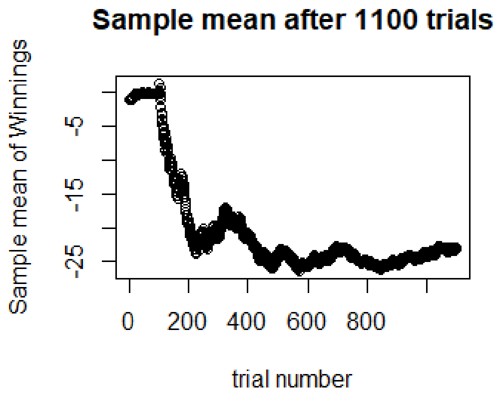

Fig. 2: Plot of sample mean for 1100 trials with $p = 1/3$ and $\mathscr{O} = 1.5$. Bet size increases to 100 units for the last 1000 trials.

$-100/6 \approx -16.67$. The fluctuation about that value is still considerable, since the number trials is not yet large. The influence of the first 100 small bets is negligible.

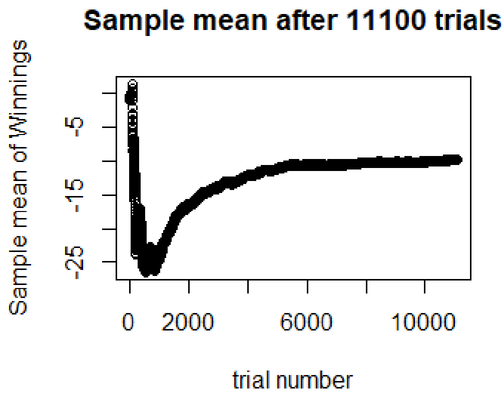

Fig. 3: Plot of sample mean for 11100 trials with $p = 1/3$ and $\mathscr{O} = 1.5$. Bet size decreases to 50 for last 10000 trials.

The next 10000 trials involve bets of 50 units each. In Figure 3, the sample mean starts to approach the theoretical expectation of $-50/6 \approx -8.33$ quite closely after around 6000 trials. From these three plots, we see that during each stage of constant bet size, the sample mean starts to converge before changing abruptly when the bet size suddenly changes. Thus there is no long-run convergence.

However, in Figure 4 we see that the sample average winnings per unit wagered, W_n/A_n, converges to the expected value of a unit bet, which is $-1/6$ in this simulation.

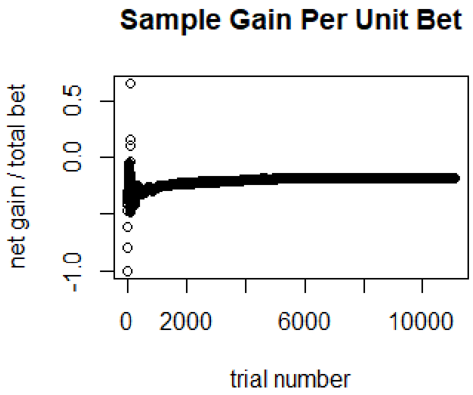

Fig. 4: Plot of sample gain per unit wagered for 11100 trials with $p = 1/3$ and $\mathscr{O} = 1.5$.

6 Conclusion

In this paper we have confirmed Thorp's results on the failure of gambling systems, offering a corrected proof that does not assume that the amounts won on different trials are independent. In particular, we have shown

– Given any initial capital, the probability that the gambler has been ruined before the n^{th} trial tends to one as n increases.
– Second, if the player has unlimited capital and continues to play forever, his average gain per unit wagered will converge almost surely to the expected value of a single unit bet, which is assumed to be negative.

Our approach for dealing with the potential dependence between amounts won followed that of Feller's analysis [4, pp. 185-187] of systems where the bet size is fixed, but the bettor has the option to wager or not on any trial (provided he wagers infinitely often). This analysis uses Doob's result [3] that a sub-sequence of a sequence of i.i.d. Bernoulli trials, in which a trial is selected or not based only on the outcomes of previous trials, is itself an i.i.d Bernoulli sequence. That result effectively allows us to consider the sequence of trials and wagers as M separate sequences running concurrently. Since each of these concurrent sequences is an i.i.d Bernoulli sequence on which bets of constant size are made, the ordinary SLLN for Bernoulli sequences provides the convergence behavior. Then the linearity property of strong convergence, together with two elementary lemmas, led to the conclusion that the ratio of the net gain to the total amount wagered converges almost surely to the expectation of a single unit wager.

We have extended this result to sequences of trials in which there are more than two outcomes, and also to sequences in which the bettor has a choice among several different wagers.

Finally, we noted that the strong convergence result, while very similar to the statement of the strong law of large numbers, does not appear to be a direct application of any of the more well-known versions of it. Of course, the SLLN plays a crucial role in the proof, and it is satisfying that the most elementary version of it (for Bernoulli sequences) is sufficient.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] E.O. Thorp, *The Mathematics of Gambling*, Gambling Times, Hollywood, California (1984).
[2] E.O. Thorp, *Elementary Probability*, Robert E. Krieger, Huntington, New York (1977).
[3] J.L. Doob, *Annals of Mathematics* **37** 363–367 (1936).
[4] W. Feller, *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, New York (1957).
[5] J. Scarne, *Scarne's Complete Guide to Gambling*, Simon & Schuster, New York, (1961)
[6] M. Jensen, *Secrets of Winning Roulette*, Cardoza, Las Vegas, Nevada, (1998)
[7] C.R. Rao, *Linear Statistical Inference and Its Applications*, John Wiley & Sons, New York (1965).
[8] S.N. Ethier, *Operations Research* **35** 814–819 (1987).