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Well-Posedness of General Time-Fractional Diffusion Equations Involving Atangana-Baleanu Derivative

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Abstract: In this paper, we study a general time-fractional diffusion equation involving the Atangana-Baleanu derivative of Caputo sense. First, we derive weak maximum-minimum principles to the associated fractional differential operators of the parabolic type, then we apply these principles to establish uniqueness and stability results to initial-boundary value problem and to obtain a norm estimate of the solution. For the existence of solution to the problem, we apply the eigenfunction expansion method to construct a formal solution, which under certain conditions proved to be a weak solution.

Keywords: Atangana-Baleanu derivative, fractional diffusion equations, maximum-minimum principle, weak solution.

1 Introduction, motivation and preliminaries

Fractional diffusion models have been implemented to model various problems in several fields, such as physical sciences [1,2,3,4,5], engineering [5,6], medicine [7,8,9] and biology [4]. Recently, many studies devoted to develop the theory of fractional diffusion equations. For instance, Luchko in [10] considered a class of time-fractional diffusion equation with Caputo fractional derivative. He established uniqueness and stability results by a maximum-principle technique. The existence of solutions was proved using the eigenfunction expansion method. Analogous results were obtained in [11] for the generalized time-fractional diffusion problem where the fractional derivative is of distributed order. The applicability of maximum principle to linear and nonlinear fractional diffusion equations with the Riemann-Liouville fractional derivative, multi-term and distributed order fractional derivatives of Riemann-Liouville type was first discussed and proved by Al-Refai and Luchko in [12,13,14]. In [15], maximum-minimum principles were used to analyze a class of fractional diffusion equations which generalizes the single and the multi-term time-fractional diffusion equations.

Fractional calculus is a branch of mathematical analysis that has wide applications in different fields in engineering and science [16,17,18,19,20]. In these fields, various analytical and numerical methods including their applications have been proposed in recent years [21,22,23,24,25]. Recently, two new types of non-local fractional derivatives with non singular kernels have been developed, the Atangana-Baleanu derivative [26] and, the Caputo-Fabrizio derivative [27]. Many researchers have studied these new types and their applications [25,28,29,30,31,32]. In [33] Al-Refai and Abdeljawad extended the results presented in [12] for a class of the time-fractional diffusion equations involving the Caputo fractional derivative with exponential kernel, and recently Al-Refai in [34] analyzed the solutions of a class of fractional differential equations involving Atangana-Baleanu fractional derivative of Caputo sense (ABC-fractional derivative).

In this paper, we extend the results obtained in [33] for the general time-fractional diffusion operator involving the ABC-fractional derivative. First, we derive new maximum-minimum principles for general time-fractional diffusion operators over an open bounded domain $\Omega \times (0,T], \Omega \subset \mathbb{R}^n$, then we use these principles to prove uniqueness and stability of the solutions to the initial-boundary value problems associated with the general time-fractional operators.

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Furthermore we derive some a priori norm estimates of these solutions. Finally, we propose a definition for the weak solution and prove the existence of solutions for this class of general time-fractional diffusion problems.

The paper is organized as follows. First, we present some fundamental definitions and results about ABC-fractional derivative. In Section 2, we derive new weak maximum-minimum principles for time-fractional diffusion operator of parabolic type. In Section 3, we analyze the solutions of linear time-fractional diffusion equations using the obtained maximum-minimum principles. The notion of the weak solution of the initial-boundary value problem for the general time-fractional diffusion equation is introduced in Section 4 and some existence results are given. Finally, in Section 5 we present some illustrated examples and concluding remarks.

Definition 1. *Let $u \in H^1(0,T), T>0$, the Atangana-Baleanu fractional derivative of Caputo sense of order $\alpha \in (0,1)$ is defined by*

$$({}_0^{ABC}D_t^{\alpha}u)(t) = \gamma_\alpha \frac{\mathscr{M}(\alpha)}{\alpha}\int_0^t u'(s)E_\alpha\left[-\gamma_\alpha(t-s)^\alpha\right]ds, \tag{1}$$

where

$$\gamma_\alpha = \frac{\alpha}{1-\alpha},$$

$\mathscr{M}(\alpha)$ is a normalization function satisfying

$$\mathscr{M}(\alpha) = (1-\alpha)+\frac{\alpha}{\Gamma(\alpha)}, \quad \mathscr{M}(0)=\mathscr{M}(1)=1,$$

and $E_\alpha(\cdot)$ is the Mittag-Leffler function.

Now we present the main properties of AB-fractional derivative of Caputo sense, then using these results we derive new maximum-minimum principles for some general linear fractional operators based on this derivative.

Lemma 1. *[34] Let $u \in H^1(0,T)$ attain its maximum at a point $t_0 \in [0,T]$. Then for $0<\alpha<1$ we have*

$$({}_0^{ABC}D_t^{\alpha}u)(t_0) \geq \gamma_\alpha \frac{\mathscr{M}(\alpha)}{\alpha}E_\alpha\left[-\gamma_\alpha t_0^\alpha\right](u(t_0)-u(0)) \geq 0. \tag{2}$$

Lemma 2. *[34] Let $\lambda \in \mathbb{R}$ and $0<\alpha<1$. The fractional initial value problem*

$$\begin{cases} ({}_0^{ABC}D_t^{\alpha}u)(t) = \lambda u(t)+g(t), \;\; 0<t\leq T, \\ u(0)=u_0, \end{cases} \tag{3}$$

has the unique solution in the fractional space $H^1(0,T)\cap\mathscr{C}[0,T]$, if and only if $\lambda u_0+g(0)=0$. The solution of the fractional initial value problem (3) is given in the form

$$u(t) = \frac{\mathscr{M}(\alpha)}{\mathscr{M}(\alpha)-\lambda(1-\alpha)}u_0E_\alpha[\mu_\alpha t^\alpha] + \frac{(1-\alpha)}{\mathscr{M}(\alpha)-\lambda(1-\alpha)}(r(t)*g'(t)+g(0)r(t)), \tag{4}$$

where

$$\mu_\alpha = \frac{\lambda\alpha}{\mathscr{M}(\alpha)-\lambda(1-\alpha)}, \quad r(t) = E_\alpha[\mu_\alpha t^\alpha]+\frac{\alpha}{1-\alpha\Gamma(\alpha)}\left(t^{\alpha-1}*E_\alpha[\mu_\alpha t^\alpha]\right).$$

Lemma 3. *[34] Let $p \in \mathscr{C}[0,T]$ where $p(t)>0$, $t\in[0,T]$, and $g(t)$ is piecewise continuous function. If $u \in H^1(0,T)$ is a solution of*

$$({}_0^{ABC}D_t^{\alpha}u)(t)+p(t)u(t)=g(t), \;\; t>0, \;\; 0<\alpha<1,$$

then

$$\| u \|_{\mathscr{C}[0,T]} = \max_{0\leq t\leq T} | u(t) | \leq \left\| \frac{g}{p} \right\|_{\mathscr{C}[0,T]}. \tag{5}$$

2 Weak maximum-minimum principles

In this section, we derive weak maximum and minimum principles to the following general fractional differential operator

$$P_\alpha(u) = \left({}^{ABC}{}_0D_t^\alpha\right)u - L(u) + q(x)u,\ \ 0<\alpha<1, t\in(0,T], x\in\Omega,$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open domain with smooth boundary $\partial\Omega, q(x)\geq c>0, x\in\bar{\Omega}$, and L is the uniformly elliptic operator defined by

$$L(u) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}.$$

The coefficients $a_{i,j}, b_i$, satisfy the following conditions

a)$a_{i,j}(\cdot,t)\in\mathscr{C}^1(\bar{\Omega}),\ b_i(\cdot,t)\in\mathscr{C}(\bar{\Omega}),\ \ \forall i,j=1,2,...,n,\ \ \forall t\in(0,T].$
b)$a_{ij}(x,t) = a_{ji}(x,t)\ \ \forall i,j=1,2,...,n,\ \ \forall(x,t)\in\Omega\times(0,T].$
c)There exist a positive constant $\mu>0$ which is independent of x and t such that

$$\sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq \mu\sum_{i=1}^{n}|\,\xi_i\,|^2\quad \forall(x,t)\in\Omega\times[0,T]\ \ and\ (\xi_1,...,\xi_n)\in\mathbb{R}^n.$$

Theorem 1.*[Weak maximum principle]*
Let $u\in C^2(\bar{\Omega})\cap H^1(0,T]$ satisfy $P_\alpha(u)\leq 0$ in $R=\Omega\times(0,T]$, then

$$\max_{(x,t)\in\bar{R}} u(x,t) \leq \max_{(x,t)\in\partial R}\{u(x,t),0\}. \tag{6}$$

Proof Assume by contradiction that the result (6) is not true. Since u is continuous, then u attains a positive maximum at a point $(x_0,t_0)\in R$ with $u(x_0,t_0)=M>0$. We have

$$\frac{\partial u}{\partial x_i}|_{(x_0,t_0)} = 0,\ \ \frac{\partial^2 u}{\partial x_i\partial x_j}\,|_{(x_0,t_0)}\leq 0,\ i,j=1,...,n,$$

and by virtue of the result in [36] we have

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}\right)_{|(x,t)=(x_0,t_0)} \leq 0, \tag{7}$$

and hence

$$L(u)\,|_{(x_0,t_0)} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}\right)_{|(x_0,t_0)} + \left(\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}\right)_{|(x_0,t_0)} \leq 0.$$

From Lemma 1, we have

$$\begin{aligned}\left({}^{ABC}{}_0D_t^\alpha(u(x_0,t))\right)(t_0) &\geq \frac{M(\alpha)}{1-\alpha}E_\alpha\left[-\gamma_\alpha t_0^\alpha\right](u(x_0,t_0)-u(x_0,0))\\ &= \frac{M(\alpha)}{1-\alpha}E_\alpha\left[-\gamma_\alpha t_0^\alpha\right](M-u(x_0,0))>0,\end{aligned}$$

which together with

$$q(x_0)u(x_0,t_0) = q(x_0)M\geq 0,$$

implies that

$$P_\alpha(u)\,|_{(x,t)=(x_0,t_0)}>0,$$

which is a contradiction to the assumption that $P_\alpha(u)\leq 0$.

Theorem 2(Weak minimum principle).
Let $u\in C^2(\bar{\Omega})\cap H^1(0,T]$ satisfy $P_\alpha(u)\geq 0$ in $R=\Omega\times(0,T]$, then

$$\min_{(x,t)\in\bar{R}} u(x,t) \geq \min_{(x,t)\in\partial R}\{u(x,t),0\}. \tag{8}$$

Proof By applying Theorem 1 to $-u$, we obtain the result (8).

3 Linear fractional diffusion problem

We consider the initial-boundary value problem

$$({}^{ABC}{}_{0}D_t^{\alpha})u = L(u) - q(x)u + g(x,t), \quad (x,t) \in R = \Omega \times (0,T], \tag{9}$$

$$u(x,0) = f(x), \quad x \in \bar{\Omega}, \tag{10}$$

$$u(x,t) = h(x,t), \quad (x,t) \in \partial\Omega \times [0,T]. \tag{11}$$

Where $g(x,t), f(x), h(x,t)$ are continues on $\bar{R}, \bar{\Omega}$ and $\partial\Omega \times [0,T]$, respectively.

Definition 2.*A classical solution to the problem (9-11) is a function $u \in \mathscr{C}(\bar{R}) \cap \mathscr{C}^2(\Omega) \cap H_t(0,T)$ that satisfy the equation (9) and the conditions (10-11), where*

$$H_t(0,T) = \{u(x,\cdot) \in \mathscr{C}^1[0,T], u(x,\cdot) \in H^1(0,T)\}.$$

In the following we present uniqueness and stability results and norm estimate of solution to the problem (9-11).

Theorem 3.*(Uniqueness)* *The fractional initial-boundary value problem (9-11) posses at most one classical solution .*

Proof Assume that the problem (9-11) has two solutions u_1, u_2, and let $w = u_1 - u_2$. Then it holds that

$$\begin{cases} ({}^{ABC}{}_{0}D_t^{\alpha})w = L(w) - q(x)w, \quad (x,t) \in R, \\ w(x,0) = 0, \quad x \in \bar{\Omega}, \\ w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T]. \end{cases}$$

Applying the weak maximum principle in Theorem 1 we have

$$w(x,t) \leq 0, \quad (x,t) \in \bar{R}.$$

The above statements hold true for $-w(x,t)$, and thus

$$-w(x,t) \leq 0, \quad (x,t) \in \bar{R}.$$

Hence $w(x,t) = 0$, $\forall (x,t) \in \bar{R}$, which complete the proof.

Theorem 4.*(Stability)* *Let $u_1(x,t)$ and $u_2(x,t)$ be two classical solutions to the initial-boundary value problem (9) that satisfy the boundary conditions (11) and the initial conditions*

$$u_1(x,0) = f_1(x), \;\; u_2(x,0) = f_2(x), \quad \forall x \in \bar{\Omega}.$$

Then

$$\| u_1 - u_2 \|_{\mathscr{C}(\bar{R})} = \max_{\bar{R}} |u(x,t)| \leq \| f_1 - f_2 \|_{\mathscr{C}(\bar{\Omega})} .$$

Proof Let $w(x,t) = u_1(x,t) - u_2(x,t)$. Then $w(x,t)$ satisfies

$$\begin{cases} {}^{ABC}{}_{0}D_t^{\alpha} w = L(w) - q(x)w, \quad (x,t) \in R, \\ w(x,0) = f_1(x) - f_2(x), \quad x \in \bar{\Omega} \\ w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T]. \end{cases}$$

By virtue of the weak maximum principle, we have

$$\max_{\bar{R}} w(x,t) \leq \max_{\partial R} \{f_1(x) - f_2(x), 0\}.$$

Applying analogue statements for $-w(x,t)$, we have

$$\max_{\bar{R}} \{-w(x,t)\} \leq \max_{\partial R} \{f_2(x) - f_1(x), 0\}.$$

Thus

$$\|w\|_{\mathscr{C}(\bar{R})} = \max_{\bar{R}} |w(x,t)| \leq \max_{\bar{\Omega}} |f_2(x) - f_1(x)| = \|f_1 - f_2\|_{\mathscr{C}(\bar{\Omega})}. \blacksquare$$

Theorem 5. *Let $u(x,t)$ be a classical solution to the time-fractional initial boundary value problem (9-11) with $g(x,0)=0$. Let $v(t)\in C[0,T]$ be such that*

$$|g(x,t)|\le v(t),\quad (x,t)\in\bar{R},\quad v(0)=0,$$

and $y(t)$ be the unique solution of

$$({}^{ABC}\,{}_0D_t^{\alpha}y)(t) = -q(x)y(t)+v(t),\ \ t>0,\ \ 0<\alpha<1, q(x)\ge c>0, \tag{12}$$

$$y(0) = 0. \tag{13}$$

Then it holds that

$$\| u\|_{\mathscr{C}(\bar{R})}\le\frac{1}{c}\| v\|_{\mathscr{C}([0,T])}+\max\left\{\| f\|_{\mathscr{C}(\bar{\Omega})},\| h\|_{\mathscr{C}(\partial\Omega\times[0,T])}+\frac{1}{c}\| v\|_{\mathscr{C}([0,T])},0\right\}. \tag{14}$$

Proof From Lemma 2 the initial-value problem (12) has a unique solution $y(t)$, and from the inequality (5) the solution $y(t)$ is bounded.
Let $w(x,t)=u(x,t)-y(t)$, then

$$\begin{aligned}P_\alpha(w(x,t)) &= -L(w(x,t))+q(x)w(x,t)+{}^{ABC}\,{}_0D_t^{\alpha}w(x,t)\\ &= -L(u(x,t))+q(x)u(x,t)+{}^{ABC}\,{}_0D_t^{\alpha}u-{}^{ABC}\,{}_0D_t^{\alpha}y(t)-q(x)y(t)\\ &= g(x,t)-v(t)\le 0.\end{aligned}$$

The initial and boundary conditions of $w(x,t)$ are

$$w(x,0)=f(x)-y(0)=f(x),\quad \forall x\in\bar{\Omega},$$
$$w(x,t)=h(x,t)-y(t),\ \ \forall (x,t)\in\partial\Omega\times[0,T].$$

Using the weak maximum principle, we have

$$w(x,t)\le\max_{(x,t)\in\partial R}\{f(x),h(x,t)-y(t),0\},$$

thus

$$u(x,t)\le y(t)+\max_{(x,t)\in\partial R}\{f(x),h(x,t)-y(t),0\}. \tag{15}$$

Let $w(x,t)=-u(x,t)-y(t)$, then $w(x,t)$ satisfies

$$\begin{aligned}P_\alpha(w(x,t)) &= -L(w(x,t))+q(x)w+{}^{ABC}\,{}_0D_t^{\alpha}w\\ &= L(u(x,t))-q(x)u(x,t)-{}^{ABC}\,{}_0D_t^{\alpha}u-{}^{ABC}\,{}_0D_t^{\alpha}y(t)-q(x)y(t)\\ &= -g(x,t)-v(t)\le 0,\end{aligned}$$

and it holds that

$$w(x,0)=-f(x)-y(0)=-f(x),\quad \forall x\in\bar{\Omega},$$
$$w(x,t)=-h(x,t)-y(t),\ \ \forall (x,t)\in\partial\Omega\times[0,T].$$

Applying the weak maximum principle, we have

$$w(x,t)\le\max_{(x,t)\in\partial R}\{-f(x),-h(x,t)-y(t),0\},$$

thus

$$-u(x,t)\le y(t)+\max_{(x,t)\in\partial R}\{-f(x),-h(x,t)-y(t),0\}. \tag{16}$$

Combining equations (15) and (16) we get

$$|u|\le\| y\|_{\mathscr{C}([0,T])}+\max\{\| f\|_{\mathscr{C}(\bar{\Omega})},\| h\|_{\mathscr{C}(\partial\Omega\times[0,T])}+\| y\|_{\mathscr{C}([0,T])},0\}.$$

By inequality (5), we have

$$\| y\|_{\mathscr{C}([0,T])}\le\max_{t\in[0,T]}\left\{\frac{|v(t)|}{|q(x)|}\right\}=\frac{1}{q(x)}\max_{t\in[0,T]}|v(t)|\le\frac{1}{c}\| v\|_{\mathscr{C}([0,T])}, \tag{17}$$

which implies that

$$\| u\|_{\mathscr{C}(\bar{R})}\le\frac{1}{c}\| v\|_{\mathscr{C}([0,T])}+\max\left\{\| f\|_{\mathscr{C}(\bar{\Omega})},\| h\|_{\mathscr{C}(\partial\Omega\times[0,T])}+\frac{1}{c}\| v\|_{\mathscr{C}([0,T])},0\right\},$$

and this completes the proof.■

Corollary 1. *Let u be a classical solution to the fractional initial value problem (9-11) with $g(x,t)=0$. Then it hold that*

$$\| u \|_{\mathscr{C}(\bar{R})} \leq \max\left\{ \| f \|_{\mathscr{C}(\bar{\Omega})}, \| h \|_{\mathscr{C}(\partial\Omega\times[0,T])}, 0 \right\}. \tag{18}$$

Proof Since $g(x,t)=0$, we can choose $v(t)=0$, and thus by virtue of the result in (14), we obtain the result in inequality (18).

4 Weak solution

In this section, we prove the existence of the unique solution to the initial-boundary value problem (9-11) under certain conditions. First, we define the concept of the weak solution in the sense of Vladimirov [35].

Definition 3. *Let $g_n \in \mathscr{C}(\bar{R}_T), f_n \in \mathscr{C}(\bar{\Omega})$ and $h_n \in \mathscr{C}(\partial\Omega \times [0,T]), n = 1,2,....$ and let $v_n \in \mathscr{C}([0,T])$ be such that $|g_n(x,t)| \leq v_n(t),\ \forall (x,t) \in \bar{R}$ and suppose that these sequences of functions satisfy the following conditions:*

1.

$$\| g_n - g \|_{\mathscr{C}(\bar{R}_T)} \longrightarrow 0 \ \ as \ \ n\to\infty, \ \ \| f_n - f \|_{\mathscr{C}(\bar{\Omega})} \longrightarrow 0 \ \ as \ \ n\to\infty,$$
$$\| h_n - h \|_{\mathscr{C}(\partial\Omega\times[0,T])} \longrightarrow 0 \ \ as \ \ n\to\infty, \ \ \| v_n - v \|_{\mathscr{C}([0,T])} \longrightarrow 0 \ \ as \ \ n\to\infty,$$

and let $y_n(t)$ be the unique solution of

$$({}^{ABC}{}_0D_t^{\alpha} y_n)(t) = -q(x)y_n(t) + v_n(t), \ \ t>0, \ \ 0<\alpha<1,$$
$$y_n(0) = 0,$$

for all $n = 1,2,...,$ such that $y_n(t)$ satisfy

$$\| y_n - y \|_{\mathscr{C}([0,T])} \longrightarrow 0 \ \ as \ \ n\to\infty.$$

2. For any $n = 1,2,...$ there exist a classical solution $u_n = u_n(x,t)$ to the initial-boundary value problem

$$\begin{cases} {}^{ABC}{}_0D_t^{\alpha} u_n = L(u_n) - q(x)u_n + g_n(x,t), & (x,t)\in R, \\ u_n(x,0) = f_n(x), & x\in\bar{\Omega}, \\ u_n(x,t) = h_n(x,t), & (x,t)\in\partial\Omega\times[0,T], \end{cases} \tag{19}$$

then the function $u \in \mathscr{C}(\bar{R})$ defined by

$$\| u_n - u \|_{\mathscr{C}(\bar{R})} \longrightarrow 0 \ \ as \ \ n\to\infty, \tag{20}$$

is called a weak solution to the initial-boundary value problem (9-11).

In the following we prove that there always exist a function $u \in \mathscr{C}(\bar{R})$ that satisfies the property (20). To do this, we show that the sequences $u_n, n = 1,2,...$ is uniformly convergent in $\bar{R}$. Applying the estimate (14) to the functions u_p and u_q that are the classical solutions to the initial - boundary value problems (19) we immediately obtain the inequality

$$\| u_p - u_q \|_{\bar{R}} \leq \frac{1}{c} \| v_p - v_q \|_{[0,T]} + \max\{ \| f_p - f_q \|_{\bar{\Omega}}, \| h_p - h_q \|_{\partial\Omega\times[0,T]} + \frac{1}{c} \| v_p - v_q \|_{[0,T]} \}. \tag{21}$$

This mean that $u_n, n = 1,2,...$ is Cauchy sequences in $\mathscr{C}(\bar{R}_T)$, and since it is Banach space then u_n converges to a function $u \in \mathscr{C}(\bar{R}_T)$.
Moreover, the estimates in (14) and (18) obtained in Theorem 5 and Corollary 1 for the classical solution of the problem (9)-(11) remain valid for the weak solution, too. That is,

$$\| u_n \|_{\mathscr{C}(\bar{R})} \leq \frac{1}{c} \| v_n \|_{\mathscr{C}([0,T])} + \max\left\{ \| f_n \|_{\mathscr{C}(\bar{\Omega})}, \| h_n \|_{\mathscr{C}(\partial\Omega\times[0,T])} + \frac{1}{c} \| v_n \|_{\mathscr{C}([0,T])}, 0 \right\}, \tag{22}$$

is valid for all $n = 1,2,....$ Since $\| u_n - u \|_{\mathscr{C}(\bar{R})} \longrightarrow 0 \ \ as \ \ n\to\infty$, then $\| u_n \|_{\bar{R}} \to \| u \|_{\bar{R}}$ as $n\to\infty$. Similarly, for f_n, h_n and v_n.
The estimate (18) for the weak solution is used to prove the following uniqueness theorem.

Theorem 6.(Uniqueness of the weak solution) *The problem (9-11) possesses at most one weak solution in the sense of Definition 3*

Proof Assume that the problem (9-11) has two weak solutions u_1, u_2, and let $w = u_1 - u_2$, then

$$\begin{cases} {}^{ABC}{}_0D_t^{\alpha} w = L(w) - q(x)w, & (x,t) \in R, \\ w(x,0) = 0, & x \in \bar{\Omega}, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T]. \end{cases} \tag{23}$$

Furthermore, w satisfies the estimation (18)

$$\| w \|_{\bar{R}} \leq \max\left\{ \| f \|_{\bar{\Omega}}, \| h \|_{\partial\Omega\times[0,T]}, 0 \right\} = 0. \tag{24}$$

thus $w(x,t) = 0$ in $\bar{R}$.
In the rest of this section we prove the existence of a weak solution to the initial-boundary-value problem (9-11) under some conditions. We restrict ourselves to the case where the operator L is in divergence form, i.e., we choose

$$b_j(x,t) = \sum_{i=1}^{n} \frac{\partial a_{ij}(x,t)}{\partial x_i}, \quad j = 1, \ldots, n,$$

then

$$\begin{aligned} L(u) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{i,j}(x,t)\frac{\partial u}{\partial x_j}\right) \\ &= \mathbf{div}(A(x,t)\cdot\nabla u). \end{aligned}$$

where $A(x,t) = [a_{ij}(x,t)]_{1\leq i,j\leq n}$, and each $a_{ij}(\cdot,t) \in \mathscr{C}^1(\bar{\Omega})$, $i,j = 1,\ldots,n$. We treat the initial-boundary value problem

$$\begin{cases} {}^{ABC}{}_0D_t^{\alpha} u = \mathbb{L}(u) + g(x,t), & (x,t) \in R, \\ u(x,0) = f(x), & x \in \bar{\Omega} \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases} \tag{25}$$

where

$$\mathbb{L}(u) = L(u) - q(x)u, \tag{26}$$

and we suppose that the conditions

$$a_{ij}(\cdot,t) \in \mathscr{C}^1(\bar{\Omega}), \ \ i,j = 1,\ldots,n, \ \ q \in \mathscr{C}(\bar{\Omega}), \ q(x) \geq c > 0, x \in \Omega, \tag{27}$$

hold true. Now we show the existence of a solution to the initial-boundary value problem (25) by using the eigenfunction expansion method. Let

$$u(x,t) = X(x)T(t), \quad (x,t) \in R,$$

and substitute in the homogenous equation of the problem (25), we get

$$\frac{{}^{ABC}{}_0D_t^{\alpha} T}{T(t)} = \frac{\mathbb{L}(X)}{X(x)} = -\lambda,$$

where λ being a constant which does not depend on the variables t and x. Then the eigenvalue problem associated with (25) is given by

$$\begin{cases} \mathbb{L}(X) = -L(X) + q(x)X(x) = \lambda X(x), & x \in \Omega, \\ X(x) = 0, & x \in \partial\Omega. \end{cases} \tag{28}$$

The conditions (27) implies that the operator $\mathbb{L}$ is a positive definite and self adjoint operator. According to the theory of the eigenvalue problems for self adjoint operators [35], the eigenvalue problem (28) has a counted number of the positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ with finite multiplicity. Any function $f \in \mathscr{M}_{\mathbb{L}}$ can be represented thought its Fourier series in the form

$$f(x) = \sum_{k=1}^{\infty} < f, X_k > X_k(x), \tag{29}$$

where $< f, X_k >$ denotes the standard scalar product in $L^2(\Omega)$ and $X_k \in \mathscr{M}_{\mathbb{L}}$ are the eigenfunctions that correspond to the eigenvalues λ_k, that is,

$$\mathbb{L}(X_k) = -L(X_k) + q(x)X_k(x) = \lambda_k X_k(x), k = 1,2,... \tag{30}$$

and

$$\mathscr{M}_{\mathbb{L}} = \left\{ f \in \mathscr{C}^2(\Omega), f|_{\partial\Omega} = 0 \ \ and \ \ \mathbb{L}(f) \in L^2(\Omega) \right\}.$$

Suppose that the function $g(\cdot,t) \in \mathscr{M}_{\mathbb{L}}$ for any $t \in (0,T]$, then g can be represented in the form of a uniformly convergent series

$$g(x,t) = \sum_{k=1}^{\infty} g_k(t)X_k(x) \ \ where \ \ g_k(t) =< g, X_k >= \int_{\Omega} g(x,t)X_k(x)dx. \tag{31}$$

We write the solution of the problem (25) in the form of a Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x), \quad T_k(t) =< u(x,t), X_k(x) >, \tag{32}$$

with the initial condition

$$T_k(0) = (u(x,0), X_k) =< f(x), X_k >= \int_{\Omega} f(x)X_k(x)dx = f_k. \tag{33}$$

Substituting the representation (32) into the equation of the IBVP (25), and using (30) we get the following uncoupled system of ordinary fractional differential equations with AB- fractional derivative of Caputo sense

$$\begin{cases} ({}^{ABC}{}_0D_t^{\alpha} T_k)(t) + \lambda_k T_k(t) = g_k(t), \quad t \in (0,T], \\ T_k(0) = f_k, \quad k = 1,2,... \end{cases} \tag{34}$$

and this initial-value problem has a unique solution for all $k = 1,2,...$

$$T_k(t) = \frac{1}{M(\alpha) - \lambda_k(1-\alpha)} \left(M(\alpha) f_k E_\alpha[\omega_k t^\alpha] + (1-\alpha)(r_k(t) * g_k'(t) + g_k(0)r_k(t)) \right), \tag{35}$$

where $\omega_k = \frac{-\lambda_k \alpha}{M(\alpha)+\lambda_k(1-\alpha)}$, and

$$r_k(t) = E_\alpha[\omega_k t^\alpha] + \frac{\alpha t^{\alpha-1}}{1-\alpha\Gamma(\alpha)} * E_\alpha[\omega_k t^\alpha].$$

Thus the formal solution of the initial-boundary value problem (25) can be written in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x), \tag{36}$$

where $T_k(t)$ is defined by (35).

Theorem 7. *The formal solution (36) of the initial-boundary value problem (25) is a weak solution.*

Proof: Let

$$u_n(x,t) = \sum_{k=1}^{n} T_k(t)X_k(x),$$

be a sequence of partial sums of the Fourier series (36), then $u_n, n = 1,2,...$ is a classical solution of the initial-boundary-value problem (25), and from the inequality (5) we have

$$\| T_k \|_{\mathscr{C}[0,T]} \leq M_k = \max_{t\in[0,T]} \frac{| g_k(t) |}{\lambda_k},$$

since $\lambda_k \to \infty$ as $k \to \infty$, then the sequence $\frac{1}{\lambda_k}, k = 1,2,...$ is bounded, and the Fourier series

$$\sum_{k=1}^{\infty} g_k(t)X_k(x),$$

converges. By the Weierstrass M-test we conclude that the sequence $u_n, n = 1,2...$ converges absolutely and uniformly to (36).

5 Illustrated examples

*Example 1.*Consider the initial-boundary value problem

$$ {}_{0}^{ABC}D_t^{\alpha} u = u_{xx} - u + t\sin(x), \quad (0,\pi)\times(0,T], \tag{37}$$

$$ u(x,0) = 0, \quad x \in [0,\pi] \tag{38}$$

$$ u(0,t) = u(\pi,t) = 0. \quad t \in [0,T] \tag{39}$$

We have, $|g(x,t)| = |t\sin(x)| \le t, 0 < x < \pi$, so we can choose $v(t) = t$ in Eq. (12). Then

$$\| u \|_{\bar{R}} \le 2T.$$

The eigenvalues and the eigenfunctions of the eigenvalue problem (28) associated with (37-39) are

$$\lambda_n = n^2+1, \; X_n = \sin(nx).$$

The Fourier series for the function $g(x,t) = t\sin(x)$ is

$$ g(x,t) = \sum_{k=1}^{\infty} g_n(t)\sin(nx), \; g_n(t) = \begin{cases} t & n=1 \\ 0 & n\neq 1. \end{cases} \tag{40}$$

Thus, the solution of the initial-value problem (37-39) is

$$\begin{aligned} u(x,t) &= T_1(t)\sin(x) \\ &= \frac{1-\alpha}{M(\alpha)-2(1-\alpha)}\left[\int_0^t E_\alpha[\omega_1 s^\alpha] + \frac{\alpha s^{\alpha-1}}{1-\alpha\Gamma(\alpha)} * E_\alpha[\omega_1 s^\alpha]ds\right]\sin(x), \end{aligned}$$

where $\omega_1 = \frac{-2\alpha}{M(\alpha)+2(1-\alpha)}$.

*Example 2.*Consider the initial-boundary value problem

$$ {}_{0}^{ABC}D_t^{\alpha} u = u_{xx} - u + 5e^t\sin(2x), \quad (0,\pi)\times(0,1], \tag{41}$$

$$ u(x,0) = \sin(2x), \quad x \in [0,\pi] \tag{42}$$

$$ u(0,t) = u(1,t) = 0. \quad t \in [0,T] \tag{43}$$

We have, $|g(x,t)| = |5e^t\sin(2x)| \le 5e^t, 0 < x < \pi$, so we can choose $v(t) = 5e^t$ in equation (12), we get

$$\| u \|_{\bar{R}} \le 5e.$$

The Fourier series for the functions $f(x) = \sin(2x)$ and $g(x,t) = 5e^t\sin(2x)$ are

$$ f(x) = \sum_{k=1}^{\infty} f_n\sin(nx), \; f_n = \begin{cases} 1 & n=2 \\ 0 & n\neq 2. \end{cases} \tag{44}$$

$$ g(x,t) = \sum_{k=1}^{\infty} g_n(t)\sin(nx), \; g_n(t) = \begin{cases} 5e^t & n=2 \\ 0 & n\neq 2. \end{cases} \tag{45}$$

Thus, the solution of the initial-value problem (41-43) is

$$u(x,t) = T_2(t)\sin(2x),$$

where

$$ T_2(t) = \frac{1}{M(\alpha)-5(1-\alpha)}\left(M(\alpha)E_\alpha[\omega_2 t^\alpha] + 5(1-\alpha)(r_2(t)*e^t + r_2(t))\right), \tag{46}$$

where $\omega_2 = \frac{-5\alpha}{M(\alpha)+5(1-\alpha)}$, and

$$r_2(t) = E_\alpha[\omega_2 t^\alpha] + \frac{\alpha t^{\alpha-1}}{1-\alpha\Gamma(\alpha)} * E_\alpha[\omega_2 t^\alpha].$$

6 Conclusion

In this paper, We derived weak maximum-minimum principles for general time-fractional diffusion operators involving the ABC-fractional derivative to prove the uniqueness and stability of the solution of the initial-boundary value problem associated with the general time-fractional operators and then we prove the existence of a weak solution. Future work will be extending our results to include a nonlinear source function to the equation and applying the method of upper and lower solutions to establish existence and uniqueness of nonlinear time-fractional diffusion problems. Other work could be considering different types of fractional derivatives on time and space.

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