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Stancu-Durrmeyer Type Operators Based on q-Integers

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Abstract: In this paper, we construct the Stancu-Durrmeyer-type modification of q-Bernstein operators by means of q-Jackson integral. Here, we establish moment estimates and some direct results which include basic convergence theorem, local approximation theorem and approximation for a Lipschitz type space. Also, we establish the Korovkin type A-statistical approximation theorem and rates of A-statistical convergence in terms of the modulus of continuity.
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1 Introduction

In 1968, Stancu [28] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \longrightarrow C[0,1]$, depending on a non negative parameter α as

$$P_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x), \tag{1}$$

where $p_{n,k}^{(\alpha)}(x)$ is the Polya distribution with density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)},$$

$x \in [0,1]$. In case $\alpha = 0$, the operators (1) reduce to the classical Bernstein polynomials. For these operators , Lupas and Lupas [21] considered a special case for $\alpha = \frac{1}{n}$ which reduces to

$$P_n^{(\frac{1}{n})}(f;x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right)(nx)_k(n-nx)_{n-k}, \tag{2}$$

where the rising factorial $(x)_n$ is given by $(x)_n = x(x+1)(x+2).....(x+n-1)$ with $(x)_0 = 1$. Recently, Miclăuş [22] established some approximation results for these operators and for the operators (2) also.

In 2009, Nowak [23] defined the q-analogue for the operators (1) for any function $f \in C[0,1], q > 0, \alpha \geq 0$ and each $n \in \mathbb{N}$ as

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), x \in [0,1], \tag{3}$$

where,

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{\nu=0}^{k-1}(x+\alpha[\nu]_q)\prod_{\mu=0}^{n-k-1}(1-q^{\mu}x+\alpha[\mu]_q)}{\prod_{\lambda=0}^{n-1}(1+\alpha[\lambda]_q)}$$

and investigated the Korovkin type approximation properties for these operators. For $\alpha = 0$, operators defined by (3) reduce to q-Bernstein polynomials and for $q \to 1-$, these operators reduce to Bernstein-Stancu operators. For $\alpha = 0$ and $q \to 1-$, these operators reduce to the classical Bernstein polynomials. Jiyang et al. [17] studied the rate of convergence and Voronovskaya type theorem for these operators defined by (3). After that Agratini [4] introduced some estimates for the rate of convergence for these operators by means of modulus of continuity and Lipschitz type maximal function and also gave a probabilistic approach.
For $f \in C[0,1],\ 0 < q < 1$, Erencin et al. [9] introduced the Kantorovich type generalization of these operators by means of the Riemann type q-integral and investigated

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some approximation properties and also established a local approximation theorem. Durrmeyer [8] introduced the integral modification of classical Bernstein polynomials which was studied in different forms by several researches (cf. [5,13,15,27] etc.). Gupta [12] introduced the q-analogue of Bernstein Durrmeyer operators and studied some approximation properties for these operators. Gupta and Wang [16] introduced q-Durrmeyer type operators and studied estimation of the rate of convergence in terms of modulus of continuity. Subsequently, Finta and Gupta [10] studied some local and global approximation theorems for the q-Durrmeyer operators. Similar kind of results have been obtained in some significant papers eg. [1], [2] and [3]. Motivated by these studies, we now propose the Durrmeyer type integral modification for the operators defined by (3) as

$$D_n^{\alpha}(f;q;x)=[n+1]_q\sum_{k=0}^{n}p_{n,k}^{q,\alpha}(x)\int_0^1 p_{n,k}^{q}(t)f(t)d_qt, \quad (4)$$

where

$$p_{n,k}^{q,\alpha}(x)=\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{\prod_{\nu=0}^{k-1}(x+\alpha[\nu]_q)\prod_{\mu=0}^{n-k-1}(1-q^{\mu}x+\alpha[\mu]_q)}{\prod_{\lambda=0}^{n-1}(1+\alpha[\lambda]_q)},$$

and

$$p_{n,k}^{q}(t)=\begin{bmatrix} n\\ k\end{bmatrix}_q t^k(1-qt)_q^{n-k}.$$

Clearly, when $\alpha=0$ we have $p_{n,k}^{q}(t)=q^{-k}p_{n,k}^{q,\alpha}(qt)$ In this paper for the operators (4), we obtain moments, basic convergence theorem, local approximation theorem, A-statistical convergence theorem and the rate of A-statistical convergence.

Now, we recall some basic definitions of q-calculus. For more details we refer to [18]. For a non-negative integer n, the $q-$ integer $[n]_q$ and $q-$ factorial $[n]_q!$ are defined as

$$[n]_q=\begin{cases}\dfrac{1-q^n}{1-q}, & q\neq 1,\\ n, & q=1.\end{cases}$$

and

$$[n]_q!=\begin{cases}[1]_q[2]_q[3]_q.......[n]_q, & n\geq 1,\\ 1, & n=0.\end{cases}$$

The $q-$binomial coefficient is defined as

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The $q-$beta function is defined as

$$B_q(k,n)=\int_0^1 t^{k-1}(1-qt)_q^{n-1}d_qt$$

or

$$B_q(k,n)=\frac{[k-1]_q![n-1]_q!}{[n+k-1]_q!}.$$

Now suppose that $0<a<b$, $0<q<1$ and f be a real valued function. Then the q-Jackson integral of f over the interval $[0,b]$ and over the generic interval $[a,b]$ are respectively defined as

$$\int_0^b f(x)d_qx=(1-q)b\sum_{j=0}^{\infty}f(bq^j)q^j \quad (5)$$

and

$$\int_a^b f(x)d_qx=\int_0^b f(x)d_qx-\int_0^a f(x)d_qx,$$

provided the sum in (5) converges absolutely.

2 Auxiliary Results

In what follows, $\|.\|$ will denote the uniform norm on $[0,1]$.

Lemma 1.[23] *For $B_n^{q,\alpha}(t^m,x)$, $m=0,1,2$, one has*

1. $B_n^{q,\alpha}(1;x)=1$
2. $B_n^{q,\alpha}(t;x)=x$
3. $B_n^{q,\alpha}(t^2;x)=\frac{1}{(1+\alpha)}\left(x(x+\alpha)+\frac{x(1-x)}{[n]_q}\right)$.

Lemma 2.*For $D_n^{\alpha}(t^m;q;x)$, $m=0,1,2$, we have*

1. $D_n^{\alpha}(1;q;x)=1$
2. $D_n^{\alpha}(t;q;x)=\frac{1}{[n+2]_q}(1+q[n]_qx)$
3. $D_n^{\alpha}(t^2;q;x)=\frac{1}{[n+2]_q[n+3]_q}\left\{(1+q)+q(1+2q)[n]_qx+\frac{q^3[n]_q^2}{1+\alpha}\left(x(x+\alpha)+\frac{x(1-x)}{[n]_q}\right)\right\}$.

Consequently,

1. $D_n^{\alpha}(t-x;q;x)=\frac{1}{[n+2]_q}+\frac{1}{[n+2]_q}(q[n]_q-[n+2]_q)x$
2. $D_n^{\alpha}((t-x)^2;q;x)=\frac{1+q}{[n+2]_q[n+3]_q}+\left\{\frac{1}{[n+2]_q[n+3]_q}\left(q(1+2q)[n]_q+\frac{q^3[n]_q([n]_q\alpha+1)}{(1+\alpha)}\right)-\frac{2}{[n+2]_q}\right\}x+\left\{\frac{q^3[n]_q([n]_q-1)}{[n+2]_q[n+3]_q(1+\alpha)}-\frac{2q[n]_q}{[n+2]_q}+1\right\}x^2$.

3 Main Results

First we will establish the basic convergence theorem.

Theorem 1.*Let $<q_n>$ and $<\alpha_n>$ be the sequences such that $0<q_n<1,\alpha_n\geq 0$ and $\frac{1}{[n]_{q_n}}\to 0,$ as $n\to\infty$. Then for any $f\in C[0,1]$, $D_n^{\alpha_n}(f;q_n;x)$ converges to f uniformly on $[0,1]$ iff $\lim_n q_n=1$ and $\lim_n\alpha_n=0$.*

Proof. First we assume that $q_n \to 1$ *and* $\alpha_n \to 0$, as $n \to \infty$. From Lemma 2, we observe that

$$D_n^{\alpha_n}(1;q_n;x) = 1,$$

$$D_n^{\alpha_n}(t;q_n;x) = \frac{1}{[n+2]_{q_n}}(1+q_n[n]_{q_n}x) \to x,$$

and
$D_n^{\alpha_n}(t^2;q_n;x) =$

$$\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\Bigg\{(1+q_n)+q_n(1+2q_n)[n]_{q_n}x + \frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(x(x+\alpha_n)+\frac{x(1-x)}{[n]_{q_n}}\right)\Bigg\} \to x^2,$$

uniformly on $[0,1]$, as $n \to \infty$. Hence, by Bohman-Korovkin theorem $D_n^{\alpha_n}(f;q_n;x)$ converges to f uniformly on $[0,1]$, as $n \to \infty$

Conversely, suppose that $D_n^{\alpha_n}(f;q_n;x)$ converges to f uniformly on $[0,1]$, as $n \to \infty$ then

$$D_n^{\alpha_n}(t;q_n;x) = \frac{1}{[n+2]_{q_n}}(1+q_n[n]_{q_n}x) \to x$$

uniformly on $[0,1]$, as $n \to \infty$, which implies that $q_n \to 1$, as $n \to \infty$. Next
$D_n^{\alpha_n}(t^2;q_n;x) =$

$$\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\Bigg\{(1+q_n)+q_n(1+2q_n)[n]_{q_n}x + \frac{{q_n}^3[n]_{q_n}^2}{1+\alpha_n}\left(x(x+\alpha_n)+\frac{x(1-x)}{[n]_{q_n}}\right)\Bigg\} \to x^2,$$

uniformly on $[0,1]$, as $n \to \infty$.
Hence,

$$\frac{1+q_n}{[n+3]_{q_n}[n+2]_{q_n}} \to 0,$$

$$\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\Bigg\{q_n(1+2q_n)[n]_{q_n} + \frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(\alpha_n+\frac{1}{[n]_{q_n}}\right)\Bigg\} \to 0 \tag{6}$$

and

$$\frac{q_n^3[n]_{q_n}^2}{(1+\alpha_n)[n+2]_{q_n}[n+3]_{q_n}}\left(1-\frac{1}{[n]_{q_n}}\right) \to 1, \text{ as } n \to \infty. \tag{7}$$

From (6) it follows that, $\alpha_n \to 0$ *as* $n \to \infty$.

This completes the proof.

Now, we will prove a local approximation theorem for the operators $D_n^{\alpha}(f;q;x)$.
Let

$$W^2 = \Big\{g \in C[0,1] : g'' \in C[0,1]\Big\}.$$

For any $\delta > 0$, Peetre's K-functional [26] is defined by

$$K_2(f;\delta) = \inf_{g \in W^2}\Big\{\|f-g\|+\delta\|g''\|\Big\}, \tag{8}$$

where $\|.\|$ is the uniform norm on $C[0,1]$. From [6], there exists an absolute constant $C > 0$, such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \tag{9}$$

where $\omega_2(f;\sqrt{\delta})$ is the second order modulus of smoothness for $f \in C[0,1]$, defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<|h|\le\sqrt{\delta}} \sup_{x,x+h,x+2h \in [0,1]} |f(x+2h)-2f(x+h)+f(x)|.$$

The usual modulus of continuity of $f \in C[0,1]$ is defined by

$$\omega(f;\sqrt{\delta}) = \sup_{0<|h|\le\sqrt{\delta}} \sup_{x,x+h \in [0,1]} |f(x+h)-f(x)|.$$

Let us define an auxiliary operator as

$$\widetilde{D}_n^{\alpha}(f;q;x) = D_n^{\alpha}(f,q,x)+f(x)-f\left(\frac{1}{[n+2]_q}(1+q[n]_qx)\right). \tag{10}$$

Lemma 3. $\|D_n^{\alpha}(f;q;.)\| \le \|f\|$.

Proof. From (4)

$$\begin{aligned}|D_n^{\alpha}(f;q;x)| &\le [n+1]_q\sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x)\int_0^1 p_{n,k}^q(t)|f(t)|d_qt \\ &\le \|f\|\,[n+1]_q\sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x)\int_0^1 p_{n,k}^q(t)d_qt \\ &= \|f\|,\end{aligned}$$

which implies that

$\|D_n^{\alpha}(f;q;.)\| \le \|f\|$.

This completes the proof.

Hence from (10), it follows that

$$\|\widetilde{D}_n^{\alpha}(f;q;.)\| \le 3\|f\|. \tag{11}$$

In what follows, let q_n be a sequence in $(0,1)$ such that $q_n \to 1$, *as* $n \to \infty$ and $\frac{1}{[n]_{q_n}} \to 0$. Further, let α_n be a sequence of non-negative real numbers such that $\alpha_n \to 0$, *as* $n \to \infty$.

Theorem 2. *Let $f\in C[0,1]$. Then for each $x\in[0,1]$, we have* $|D_n^{\alpha_n}(f;q_n;x)-f(x)|$

$$\leq 2C\omega_2\left(f;\frac{\delta_{n,q_n}^{\alpha_n}(x)}{\sqrt{2}}\right)+\omega\left(f,\left|\frac{1}{[n+2]_{q_n}}+\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)x\right|\right),$$

where,

$$\delta_{n,q_n}^{\alpha_n}(x)=\left\{D_n^{\alpha_n}((t-x)^2;q_n;x)+\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-x\right)^2\right\}$$

Proof. From (10) we have $\widetilde{D}_n^{\alpha_n}(1;q_n;x)=1$ and $\widetilde{D}_n^{\alpha_n}(t;q_n;x)=x$. Hence $\widetilde{D}_n^{\alpha_n}(t-x;q_n;x)=0$.

For $g\in W^2$ and $x\in[0,1]$, by Taylor's theorem, we have

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Then,
$\widetilde{D}_n^{\alpha_n}(g(t);q_n;x)-g(x)=$

$$\begin{aligned}&\widetilde{D}_n^{\alpha_n}\left((t-x)g'(x);q_n;x\right)+\widetilde{D}_n^{\alpha_n}\left(\int_x^t(t-u)g''(u)du;q_n;x\right)\\
&=g'(x)\widetilde{D}_n^{\alpha_n}(t-x;q_n;x)+\widetilde{D}_n^{\alpha_n}\left(\int_x^t(t-u)g''(u)du;q_n;x\right)\\
&=\widetilde{D}_n^{\alpha_n}\left(\int_x^t(t-u)g''(u)du;q_n;x\right)\\
&=D_n^{\alpha_n}\left(\int_x^t(t-u)g''(u)du;q_n;x\right)-\int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}}\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-u\right)g''(u)du,\end{aligned}$$

which implies that
$|\widetilde{D}_n^{\alpha_n}(g(t);q_n;x)-g(x)|$

$$\begin{aligned}&\leq D_n^{\alpha_n}\left(\left|\int_x^t|t-u||g''(u)|du\right|;q_n;x\right)+\left|\int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}}\left|\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-u\right|g''(u)du\right|\\
&\leq\|g''\|D_n^{\alpha_n}\left(\left|\int_x^t|t-u|du\right|;q_n;x\right)+\|g''\|\int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}}\left|\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-u\right|du\\
&=I_1+I_2,\ \text{(say)}.\end{aligned}$$

Now,

$$\begin{aligned}I_1&=\|g''\|D_n^{\alpha_n}\left(\left|\int_x^t|t-u|du\right|;q_n;x\right)\\
&=\|g''\|D_n^{\alpha_n}\left(\frac{(t-x)^2}{2}:x\right),\end{aligned}$$

and

$$\begin{aligned}I_2&=\|g''\|\int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}}\left|\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-u\right|du\\
&=\frac{\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-x\right)^2}{2}\|g''\|.\end{aligned}$$

So, we have

$$\begin{aligned}|\widetilde{D}_n^{\alpha_n}(g(t);q_n;x)-g(x)|&\leq\frac{\|g''\|}{2}\left\{D_n^{\alpha_n}((t-x)^2;q_n;x)+\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-x\right)^2\right\}\\
&=\frac{\|g''\|}{2}(\delta_{n,q_n}^{\alpha_n}(x))^2,\ \text{(say)}.\end{aligned}\tag{12}$$

Now,
$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right|$

$$\begin{aligned}&=\left|\widetilde{D}_n^{\alpha_n}(f;q_n;x)-f(x)+f\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}\right)-f(x)\right|\\
&\leq\left|\widetilde{D}_n^{\alpha_n}(f-g;q_n;x)\right|+\left|\widetilde{D}_n^{\alpha_n}(g;q_n;x)-g(x)\right|\\
&\quad+\left|f(x)-g(x)\right|+\left|f\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}\right)-f(x)\right|.\end{aligned}$$

Using equations (11) and (12), we get
$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right|$

$$\leq 4\|f-g\|+4\frac{\|g''\|}{2}(\delta_{n,q_n}^{\alpha_n}(x))^2+\omega\left(f,\left|\frac{1}{[n+2]_{q_n}}+\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)x\right|\right).$$

Taking infimum on the right hand side of the above inequality over all $g\in W^2$, in view of the inequality (8),we obtain
$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right|$

$$\leq 4K_2\left(f,\frac{(\delta_{n,q_n}^{\alpha_n}(x))^2}{2}\right)+\omega\left(f,\left|\frac{1}{[n+2]_{q_n}}+\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)x\right|\right).$$

Using the relation between K-functional and second modulus of smoothness from equation (9), we get

$$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right|$$

$$\leq C\omega_2\left(f,\frac{\delta_{n,q_n}^{\alpha_n}(x)}{\sqrt{2}}\right) +\omega\left(f,\left|\frac{1}{[n+2]_{q_n}}+\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)x\right|\right).$$

This completes the proof.

For $0<r\leq 1$, we consider the following Lipschitz-type space [25]:

$$Lip_M(r)=\left\{f\in C[0,1]:|f(t)-f(x)|\leq M\frac{|t-x|^r}{(t+x)^{\frac{r}{2}}};\right. \left. M>0, t\in[0,1]\text{ and }x\in(0,1]\right\}.$$

In our next theorem, we estimate the error in the approximation for a function in $Lip_M(r)$.

Theorem 3. *For $f\in Lip_M(r)$, $0<r\leq 1$, $x\in(0,1]$ and $n\in\mathbb{N}$, we have*

$$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right|\leq M\left(\frac{\mu_{n,2}^{q_n,\alpha_n}(x)}{x}\right)^{\frac{r}{2}},$$

where $\mu_{n,2}^{q_n,\alpha_n}(x)=D_n^{\alpha_n}((t-x)^2;q_n;x)$.

Proof. We may write

$$|D_n^{\alpha_n}(f,q_n,x)-f(x)| \leq [n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\int_0^1 p_{n,k}^{q_n}(t)|f(t)-f(x)|d_{q_n}t.$$

Applying Hölder's inequality in the integral form for $p=\frac{2}{r}$ and $q_n=\frac{2}{2-r}$

$$|D_n^{\alpha_n}(f,q_n,x)-f(x)|$$

$$\leq [n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\left(\int_0^1 p_{n,k}^{q_n}(t)|f(t)-f(x)|^{\frac{2}{r}}d_{q_n}t\right)^{\frac{r}{2}} \left(\int_0^1 p_{n,k}^{q_n}d_{q_n}t\right)^{\frac{2-r}{2}}$$

$$=[n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\left(\int_0^1 p_{n,k}^{q_n}(t)|(f(t)-f(x))|^{\frac{2}{r}}d_{q_n}t\right)^{\frac{r}{2}} \left(\frac{1}{[n+1]_{q_n}}\right)^{\frac{2-r}{r}}$$

Again, applying Hölder's inequality for the sum for $p=\frac{2}{r}$ and $q_n=\frac{2}{2-r}$

$$|D_n^{\alpha_n}(f,q_n,x)-f(x)|$$

$$\leq\left([n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\int_0^1 p_{n,k}^{q_n}(t)|f(t)-f(x)|^{\frac{2}{r}}d_{q_n}t\right)^{\frac{r}{2}} \left(\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\right)^{\frac{2-r}{2}}$$

$$\leq\left\{[n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n},\alpha_n(x)\int_0^1 p_{n,k}^{q_n}(t)\left(M\frac{|t-x|^r}{(t+x)^{\frac{r}{2}}}\right)^{\frac{2}{r}}d_{q_n}t\right\}^{\frac{r}{2}}.1$$

$$\leq\frac{M}{(x)^{\frac{r}{2}}}\left([n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\int_0^1 p_{n,k}^{q_n}(t)|t-x|^2d_{q_n}t\right)^{\frac{r}{2}}$$

$$\leq\frac{M}{(x)^{\frac{r}{2}}}\left(D_n^{\alpha_n}((t-x)^2,q_n,x)\right)^{\frac{r}{2}}$$

$$=M\left(\frac{\mu_{n,2}^{q_n,\alpha_n}(x)}{x}\right)^{\frac{r}{2}}.$$

This completes the proof.

Lipschitz type maximal function of order β [20], is defined as

$$\widetilde{\omega}_\beta(f,x)=\sup_{t\neq x,t\in[0,1]}\frac{|f(t)-f(x)|}{|t-x|^\beta},\ x\in[0,1]\text{ and }\beta\in(0,1].$$

Now, we will obtain a local direct estimate for the operator $D_n^{\alpha_n}(f,q_n,x)$ in terms of the $\widetilde{\omega}_\beta(f,x)$.

Theorem 4. *Let $f\in C[0,1], 0<\beta\leq 1$. Then $\forall\, x\in[0,1]$, we have*

$$|D_n^{\alpha_n}(f,q_n,x)-f(x)|\leq\widetilde{\omega_\beta}(f,x)\left(\mu_{n,2}^{q_n,\alpha_n}(x)\right)^{\frac{\beta}{2}},$$

where $\mu_{n,2}^{q_n,\alpha_n}(x)$ is as defined in equation (12).

Proof. Applying Hölder's inequality twice first for the integration and then for the summation with $p=\frac{2}{\beta}$ and $q_n=\frac{2}{2-\beta}$, we have

$$|D_n^{\alpha_n}(f,q_n,x)-f(x)|$$

$$\leq\left([n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\int_0^1 p_{n,k}^{q_n}(t)|f(t)-f(x)|^{\frac{2}{\beta}}d_{q_n}t\right)^{\frac{\beta}{2}}.$$

Using the definition of the Lipschitz-type maximal function, we obtain

$$|D_n^{\alpha_n}(f,q_n,x)-f(x)|$$

$$\leq\widetilde{\omega_\beta}(f,x)\left([n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\int_0^1 p_{n,k}^{q_n}(t)|t-x|^2d_{q_n}t\right)^{\frac{\beta}{2}}$$

$$=\widetilde{\omega_\beta}(f,x)\left(D_n^{\alpha_n}((t-x)^2;q_n;x)\right)^{\frac{\beta}{2}}$$

$$=\widetilde{\omega_\beta}(f,x)\left(\mu_{n,2}^{q_n,\alpha_n}(x)\right)^{\frac{\beta}{2}}.$$

This completes the proof.

4 A-Statistical Convergence

Let $A=(a_{jn})$ be a non-negative infinite summability matrix. For a given sequence $x=<x_n>$, the A-transform of x denoted by $Ax=(Ax)_j$ is defined as

$$(Ax)_j=\sum_{n=1}^{\infty}a_{jn}x_n$$

provided the series converges for each j. A is said to be regular if $\lim_j (Ax)_j=L$ whenever $\lim_n x_n=L$. Now the A-density of $K, K\subseteq\mathbb{N}$(the set of the natural numbers), denoted by $\delta_A(K)$, is defined as

$\delta_A(K)=\lim_j\sum_{n=1}^{\infty}a_{jn}\chi_K(n)$, provided the limit exists, where $\chi_K(n)$ is the characteristic function of K.

Then $x=<x_n>$ is said to be A-statistically convergent to L i.e. $st_A-\lim_n x_n=L$ if for every $\varepsilon>0$,

$\lim_j\sum_{n:|x_n-L|\geq\varepsilon}a_{jn}=0$ or equivalently $\delta_A\{n\in K:|x_n-L|\geq\varepsilon\}=0$.

If we replace A by C_1 then A is a Cesaro matrix of order one and A-statistical convergence is reduced to the statistical convergence. Similarly, if $A=I$, the identity matrix then A-statistical convergence reduces to ordinary convergence. Kolk [19] proved that statistical convergence is stronger than ordinary convergence. In this direction, the significant contributions have been made by (cf. [7], [11], [14], [24] etc.).

Theorem 5.*[11] If the sequence of positive linear operators $L_n:C[a,b]\to C[a,b]$ satisfies the conditions $st-\lim_n\|L_n(e_i;q;.)-e_i\|=0$ where $e_i(t)=t^i, i=0,1,2,$ then for any $f\in C[a,b]$, we have $st-\lim_n\|L_n(f;q;.)-f\|=0.$*

The result given above also works for A-statistical convergence. Now we will establish the following A-statistical approximation theorem for the operator $D_n^{\alpha}(f,q,x)$.

Theorem 6.*Let $A=(a_{jn})$be a non-negative infinite regular summability matrix and $q=<q_n>, 0<q_n<1$ and $\alpha=<\alpha_n>$ be the sequences satisfying the following conditions:*

$$st_A-\lim_n q_n=1,\ \ st_A-\lim_n q_n^n=a, a<1$$

$$st_A-\lim_n\alpha_n=0 \text{ and } st_A-\lim_n\frac{1}{[n]_{q_n}}=0, \tag{13}$$

then for $f\in C[0,1]$, we have $st_A-\lim_n\|D_n^{\alpha_n}(f,q_n,.)-f\|=0.$

*Proof.*From Theorem 5, it is enough to prove that $st_A-\lim_n\|D_n^{\alpha_n}(e_i,q_n,.)-e_i\|_{C[0,1]}=0$ for $i=0,1,2.$

In view of Lemma 2, we have $D_n^{\alpha_n}(e_0,q_n,x)=1$, hence $st-\lim_n\|D_n^{\alpha_n}(e_0,q_n,.)-e_0\|=0.$

Now,

$$\begin{aligned}\|D_n^{\alpha_n}(e_1,q_n,.)-e_1\| &= \sup_{[0,1]}\left|\frac{1}{[n+2]_{q_n}}(1+q_n[n]_{q_n}x)-x\right|\\ &\leq\left|\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right|+\frac{1}{[n+2]_{q_n}}.\end{aligned} \tag{14}$$

For any given $\varepsilon>0$, let us define the following sets

$$U=\left\{n:\|D_n^{\alpha_n}(e_1,q_n,.)-e_1\|\geq\varepsilon\right\},$$

$$U_1=\left\{n:\left|\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right|\geq\frac{\varepsilon}{2}\right\},$$

and

$$U_2=\left\{n:\frac{1}{[n+2]_{q_n}}\geq\frac{\varepsilon}{2}\right\}.$$

From (14) it is easy to see that $U\subseteq U_1\bigcup U_2$, so we have

$$\sum_{n\in U}a_{jn}\leq\sum_{n\in U_1}a_{jn}+\sum_{n\in U_2}a_{jn}. \tag{15}$$

From equation (13), we obtain

$$st_A-\lim_n\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)=0$$

and

$$st_A-\lim_n\left(\frac{1}{[n+2]_{q_n}}\right)=0.$$

Hence by taking limit on both sides of (15), as $j\to\infty$, we get

$$st_A-\lim_n\|D_n^{\alpha_n}(e_1,q_n,.)-e_1\|=0. \tag{16}$$

Similarly, using Lemma 2 , we have

$$\begin{aligned}&|D_n^{\alpha_n}(e_2,q_n,.)-e_2\|\\ &=\sup_{[0,1]}\left|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\left\{(1+q_n)+q_n(1+2q_n)[n]_{q_n}x\right.\right.\\ &\quad\left.\left.+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(x(x+\alpha_n)+\frac{x(1-x)}{[n]_{q_n}}\right)\right\}-x^2\right|\\ &\leq\frac{1+q_n}{[n+2]_{q_n}[n+3]_{q_n}}+\left|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\left(q_n(1+2q_n)[n]_{q_n}\right.\right.\\ &\quad\left.\left.+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(\alpha_n+\frac{1}{[n]_{q_n}}\right)\right)\right|\\ &\quad+\left|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(1-\frac{1}{[n]_{q_n}}\right)-1\right|.\end{aligned} \tag{17}$$

For $\varepsilon>0$, let us define the following sets:

$$U=\left\{n:\|D_n^{\alpha_n}(e_2,q_n,.)-e_2\|\geq\varepsilon\right\},$$

$U_1=\left\{n:\frac{1+q_n}{[n+2]_{q_n}[n+3]_{q_n}}\geq\frac{\varepsilon}{3}\right\},$

$U_2=\left\{n:\left|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\left(q_n(1+2q_n)[n]_{q_n}\right.\right.\right.$
$\left.\left.\left.+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(\alpha_n+\frac{1}{[n]_{q_n}}\right)\right|\geq\frac{\varepsilon}{3}\right\},$

and

$$U_3=\left\{n:\left|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(1-\frac{1}{[n]_{q_n}}\right)-1\right|\geq\frac{\varepsilon}{3}\right\}.$$

From (17) it follows that $U\subseteq U_1\bigcup U_2\bigcup U_3$, hence

$$\sum_{n\in U}a_{jn}\leq\sum_{n\in U_1}a_{jn}+\sum_{n\in U_2}a_{jn}+\sum_{n\in U_3}a_{jn}. \tag{18}$$

Now, using (13) we find

$$st-\lim_n\frac{1+q_n}{[n+2]_{q_k}[n+3]_{q_k}}=0,$$

$st-\lim_n\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\left\{q_n(1+2q_n)[n]_{q_n}\right.$
$\left.+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(\alpha_n+\frac{1}{[n]_{q_n}}\right)\right\}=0,$

and $st-\lim_n\left\{\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(1-\frac{1}{[n]_{q_n}}\right)-1\right\}=0.$
Hence, by taking limits on both sides of (18), as $j\to\infty$, we get

$$st_A-\lim_n\|D_n^{\alpha_n}(e_2,q_n,.)-e_2\|=0. \tag{19}$$

This completes the proof.

5 Rate of A-statistical convergence

Let $f\in C[0,1]$. Then for any $x,t\in[0,1]$, we have $|f(t)-f(x)|\leq\omega(f,|t-x|)$, which implies that

$$|f(t)-f(x)|\leq(1+\delta^{-2}(t-x)^2)\omega(f,\delta),\ \ \delta>0. \tag{20}$$

Let $A=(a_{jn})$ be a non negative infinite regular summability matrix and $<b_j>$ be a positive non increasing sequence. If for every $\varepsilon>0,\lim_j\frac{1}{b_j}\sum_{n:|x_n-L|\geq\varepsilon}a_{jn}=0$, then we say that the sequence $x=<x_n>$, converges A-statistically to number L with the rate of $o(b_j)$ and this is denoted by $x_n-L=st_A-o(b_n)$ as $n\to\infty$. If for every $\varepsilon>0$, $\sup_j\frac{1}{j}\sum_{n:|x_n|\geq\varepsilon}a_{jn}<\infty$, then x is called A-statistically bounded with the rate $O(b_n)$, as $n\to\infty$.

If we consider the concept of A-statistical convergence in the measure from measure theory, then we have the following definitions: The sequence $x=<x_n>$ is said to be A-statistically convergent to L with the rate of $o_\eta(b_n)$, denoted by $x_n-L=st_A-o_\eta(b_n)$, as $n\to\infty$ if for every $\varepsilon>0,\lim_j\sum_{n:|x_n-L|\geq\varepsilon b_n}a_{jn}=0$.
The sequence $x=<x_n>$ is called A-statistically bounded with the rate of $O_\eta(b_n)$, if there exists a positive number M such that $\lim_j\sum_{n:|x_n|\geq Mb_n}a_{jn}=0$. We denote it by writing $x_n=st_A-O_\eta(b_n)$, as $n\to\infty$.
In our next theorem we give the rate of A-statistical convergence for the operator $D_n^{\alpha}(f;q;x)$ in terms of modulus of continuity.

Theorem 7.*Let $A=(a_{jn})$ be a non negative regular summability matrix and for each $x\in[0,1]$, $<b_n(x)>$ be a positive non-increasing sequence and let $q=<q_n>,0<q_n<1$ and $\alpha=<\alpha_n>$ be sequences satisfying equation (13) and $\omega(f;\mu_{n,2}^{q_n,\alpha_n})=st_A-o(b_n(x))$ with $\mu_{n,2}^{q_n,\alpha_n}(x)=D_n^{\alpha_n}((t-x)^2;q_n;x)$, then for any function $f\in C[0,1]$ and $x\in[0,1]$, we have $D_n^{\alpha_n}(f;q_n;x)-f(x)=st_A-o(b_n(x))$.*

*Proof.*By monotonicity and linearity of the operator $D_n^{\alpha_n}(f;q_n;x)$, we have $|D_n^{\alpha_n}(f;q_n;x)-f(x)|$

$$\leq D_n^{\alpha_n}(|f(t)-f(x)|;q_n;x)$$
$$\leq\left(1+\delta^{-2}D_n^{\alpha_n}((t-x)^2;q_n;x)\right)\omega(f;\delta),\text{for any }\delta>0.$$

Taking δ as $\sqrt{\mu_{n,2}^{q_n,\alpha_n}(x)}$, we get

$$|D_n^{\alpha_n}(f;q_n;x)-f(x)|\leq2\omega(f;\sqrt{\mu_{n,2}^{q_n,\alpha_n}}). \tag{21}$$

For $\varepsilon>0$, let us define the following sets:

$U=\left\{n:|D_n^{\alpha_n}(f;q_n;x)-f(x)|\geq\varepsilon\right\}$ and
$U_1=\left\{n:2\omega\left(f,\sqrt{\mu_{n,2}^{q_n,\alpha_n}}(x)\right)\geq\varepsilon\right\}.$

From (21), we have $\frac{1}{b_n(x)}\sum_{n\in U}a_{jn}\leq\frac{1}{b_n(x)}\sum_{n\in U_1}a_{jn}.$

Taking limits on above inequality as $j\to\infty$ and using $\omega(f;\mu_{n,2}^{q_n,\alpha_n})=st_A-o(b_n(x))$, we obtain the required result.
This completes the proof.

Theorem 8.*Let $A=(a_{jn})$, $<b_n(x)>$, $q=<q_n>$, and $\alpha=<\alpha_n>$ be all same as in Theorem 7. Assume that the operators $D_n^{\alpha_n}(f;q_n;x)$ satisfy the condition $\omega(f;\mu_{n,2}^{q_n,\alpha_n}(x))=st_A-o_\eta(b_n(x))$ with $\mu_{n,2}^{q_n,\alpha_n}(x)=D_n^{\alpha_n}((t-x)^2;q_n;x)$. Then for all $f\in C[0,1]$, we have $D_n^{\alpha}(f;q;x)-f(x)=st_A-o_\eta(b_n(x))$.*

Similar results hold when little "o_η" is replaced by the big "O_η".

In the following theorem we establish the rate of A-statistical convergence for the operator $D_n^{\alpha}(f;q_n;x)$ in terms of the second modulus of continuity in the space W^2.

Theorem 9. *Let $A=(a_{jn})$ be a non negative regular summability matrix and let $q=<q_n>, 0<q_n<1$ and $\alpha=<\alpha_n>$ be sequences satisfying equation (13). For each $f\in C[0,1]$, we have*

$$\|D_n^{\alpha_n}(f;q_n;.)-f\|\leq C\omega_2(f;\sqrt{\delta_n^{q_n,\alpha_n}}),$$

where

$$\delta_n^{q_n,\alpha_n}=\|D_n^{\alpha_n}((e_1-.);q_n;.)\|+\left\|D_n^{\alpha_n}((e_1-.)^2;q_n;.)\right\|.$$

Proof. For $g\in W^2$, applying Taylor's expansion , we have

$$D_n^{\alpha_n}(g;q_n;x)-g(x)=g'(x)D_n^{\alpha_n}(e_1-x;q_n;x)+\frac{1}{2}g''(\xi)D_n^{\alpha_n}((e_1-x)^2;q_n;x),$$

where ξ lies between t and x. We may write,

$$\begin{aligned}\|D_n^{\alpha_n}(g;q_n;.)-g\| &\leq \|g'\|\|D_n^{\alpha_n}((e_1-.);q_n;.)\|\\ &\quad+\frac{1}{2}\|g''\|\left\|D_n^{\alpha_n}((e_1-.)^2;q_n;.)\right\|\\ &\leq \delta_{n,x}^{q_n,\alpha_n}\|g\|_{W^2}, \text{ (say)}.\end{aligned}$$

For $f\in C[0,1]$ and $g\in W^2$, we have

$$\begin{aligned}\|D_n^{\alpha_n}(f;q_n;.)-f\| &\leq \|D_n^{\alpha_n}(f;q_n;.)-D_n^{\alpha_n}(g;q_n;.)\|\\ &\quad+\|D_n^{\alpha_n}(g;q_n;.)-g\|+\|g-f\|\\ &\leq 2\|g-f\|+\|D_n^{\alpha_n}(g;q_n;.)-g\|\\ &\leq 2\|g-f\|+\delta_n^{q_n,\alpha_n}\|g\|_{W^2}\\ &\leq 2\left(\|g-f\|+\delta_n^{q_n,\alpha_n}\|g\|_{W^2}\right).\end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g\in W^2$ and using equation (8), we get

$$\begin{aligned}\|D_n^{\alpha_n}(f;q_n;.)-f\| &\leq 2K(f;\delta_n^{q_n,\alpha_n})\\ &\leq C\omega_2(f;\sqrt{\delta_n^{q_n,\alpha_n}}).\end{aligned}$$

Using equations (16) and (19), we get $st_A-\lim_n\delta_n^{q_n,\alpha_n}=0$, hence $st_A-\lim_n\omega_2(f;\sqrt{\delta_n^{q_n,\alpha_n}})=0$, which gives the required result. This completes the proof.

Conclusion: The degree of approximation of the Stancu-Durrmeyer-type modification of q-Bernstein operators is studied by means of the modulus of continuity, Lipschitz type maximal function and Lipschitz type space. Also, the Korovkin type A-statistical approximation theorem and rates of A-statistical convergence in terms of the modulus of continuity are established.

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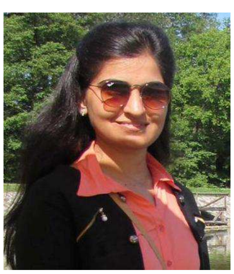

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