Applied Mathematics & Information Sciences
An International Journal

Some results on the digamma function

Biljana Jolevska-Tuneska[1] *and Ilija Jolevski*[2]

[1] Faculty of Electrical Engineering and Informational Technologies, Karpos II bb, Skopje, Republic of Macedonia
[2] Technical Faculty Bitola, University of St. Kliment Ohridski, Bitola, Republic of Macedonia

Received: 19 Jul. 2011; Revised 11 Oct. 2012; Accepted 23 Oct. 2012
Published online: 1 Jan. 2012

Abstract: The digamma function is defined for $x > 0$ as a locally summable function on the real line by

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\, dt\,.$$

In this paper we use the neutrix calculus to extend the definition for digamma function for the negative integers. Also we consider the derivatives of the digamma function for negative integers.

Keywords: Digamma Function, Gamma Function, Delta Function, Neutrices

1. Introduction

The gamma function $\Gamma(x)$ was introduced by Leonard Euler as a generalization of the factorial function on the set of all real numbers. It is defined for $x > 0$ by:(see [3] and [13])

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt\,.$$

Due to the difficulties in dealing with $\Gamma'(x)$ in particulary because it is a large function that increases very rapidly, the logarithmic derivative of $\Gamma(x)$ is studied instead. This function is known as the digamma function (or psi function) $\psi(x)$ (see [5,2]) and is defined for positive real numbers x by:

$$\psi(x) = \frac{d\,[\ln \Gamma(x)]}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}\,.$$

The infinite family of approximations of the Digammma function were recently considered by I. Muqattash and M.Yahdi (see [1]).

In this paper we consider the values of the digamma function for negative integers. Also we consider the derivatives of the digamma function for negative integers. To define the digamma function for this values we use neutrix calculus. The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as Hadamard finite part.

Using the concepts of the neutrix and the neutrix limit due to van der Corput [4], Fisher gave the general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been exploited in context of distributions, see [7,6].

We also note that recently Ng and van Dam applied the neutrix calculus, in conjunction with the Hadamar integral, developed by van der Corput, to the quantum field theories, in particular to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, and obtained finite renormalization in the loop calculations, see [16,17].

2. Neutrix calcucus

In the following, we let N be the neutrix, see van der Corput [4], having domain $N' = \{\epsilon : 0 < \epsilon < \infty\}$ and negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \quad \ln^r \epsilon : \quad \lambda < 0, \quad r = 1, 2, \ldots$$

and all functions $f(\epsilon)$ which tend to zero in the normal sense as ϵ tends to zero.

* Corresponding author: e-mail: ilija.jolevski@uklo.edu.mk

If $f(\epsilon)$ is a real (or complex) valued function defined on N' and if it is possible to find a constant c such that $f(\epsilon) - c$ is in N, then c is called the neutrix limit of $f(\epsilon)$ as $\epsilon \to 0$, and we write $\mathrm{N}-\lim_{\epsilon\to 0} f(\epsilon) = c$. Note that taking the neutrix limit of a function $f(\epsilon)$ is equivalent to taking the usual limit of Hadamar's finite part of $f(\epsilon)$, and if a function $f(\epsilon)$ tends to c in the normal sense as ϵ tends to zero, also it converges to c in the neutrix sense. The reader may find the general definition of a neutrix limit with some examples in [9].

In the following we apply Fishers's principle to define the digamma function for negative integers.

It was proved in [9] that

$$\Gamma(x) = \mathrm{N}-\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{x-1}e^{-t}\,dt$$

for $x \neq 0, -1, -2, \ldots$, where Γ denotes the Gamma function. This suggested that $\Gamma(-m)$ be defined by:

$$\Gamma(-m) = \mathrm{N}-\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{-m-1}e^{-t}\,dt \tag{1}$$

for $m = 0, 1, 2, \ldots$.

It was also proved that

$$\Gamma(0) = \int_0^\infty e^{-t}\ln t\,dt = \Gamma'(1) = -\gamma,$$

where γ denotes Euler's constant, and

$$\Gamma(-m) = \int_1^\infty t^{-m-1}e^{-t}\,dt + \\ + \int_0^1 t^{-m-1}\Big[e^{-t} - \sum_{i=0}^{m} \frac{(-t)^i}{i!}\Big]dt + \\ - \sum_{i=0}^{m-1} \frac{(-1)^i}{i!(m-i)} \tag{2}$$

for $m = 0, 1, 2, \ldots$.

Fisher and Kuribayashi [9] proved the existence of $\Gamma^{(r)}(0)$ and then defined $\Gamma^{(r)}(0)$ by the equation:

$$\Gamma^{(r)}(0) = \mathrm{N}-\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{-1}\ln^r te^{-t}\,dt \\ = \int_1^\infty t^{-1}\ln^r te^{-t}\,dt + \int_0^1 t^{-1}\ln^r t[e^{-t} - 1]\,dt,$$

for $r = 0, 1, 2, \ldots$ This suggested that $\Gamma^{(r)}(-m)$ be defined by:

$$\Gamma^{(r)}(-m) = \mathrm{N}-\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{-m-1}\ln^r te^{-t}\,dt \\ = \int_1^\infty t^{-m-1}\ln^r te^{-t}\,dt + \\ + \int_0^1 t^{-m-1}\ln^r t\left[e^{-t} - \sum_{i=0}^{m}\frac{(-t)^i}{i!}\right]dt + \\ + \sum_{i=0}^{m-1}\frac{(-1)^i}{i!} r!(m-i)^{-r-1},$$

for $r = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$

In [10] and [8] Fisher et al. defined the incomplete gamma function $\gamma(-m, x)$ for $m = 0, 1, 2, \ldots$ Some convolutions and neutrix convolution of this function and other functions were considered in [11] and [12]. Recently, Ozcag et al. in [15] defined the incomplete beta function and its derivatives.

3. Main result

The digamma function (see [14]) has its integral representation

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\,dt,$$

and the integral is convergent for $x > 0$. Also, for $x > 0$, this can be written as:

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - t^{x-1}}{1 - t}\,dt. \tag{3}$$

It can be easily proved from equation (3) that:

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{4}$$

for $x > 0$ and this equation can be used to define $\psi(x)$ for negative, non-integer values of x. Thus if $-1 < x < 0$ then

$$\psi(x) = \psi(x+1) - \frac{1}{x}.$$

So, we have that if $-n < x < -n+1$, $n = 1, 2, \ldots$, then:

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - t^{x-1+n}}{1 - t}\,dt - \sum_{k=1}^{n} \frac{1}{x+k-1}. \tag{5}$$

It follows that if $-n < x < -n+1$, $n = 1, 2, \ldots$, and $x > 0$, then:

$$-\gamma + \int_\epsilon^1 \frac{1 - t^{x-1}}{1 - t}\,dt = -\gamma + \int_\epsilon^1 \frac{1 - t^{x-1+n}}{1 - t}\,dt - \\ - \int_\epsilon^1 \frac{t^{x-1}(1 - t^n)}{1 - t}\,dt = \\ = -\gamma + \int_\epsilon^1 \frac{1 - t^{x-1+n}}{1 - t}\,dt - \sum_{k=1}^{n}\frac{1}{x+k-1} + \\ + \sum_{k=1}^{n}\frac{\epsilon^{x+k-1}}{x+k-1},$$

and it follows that

$$\mathrm{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{1 - t^{x-1}}{1 - t}\,dt = \\ = \lim_{\epsilon\to 0}\left[\int_\epsilon^1 \frac{1 - t^{x-1+n}}{1 - t}\,dt - \sum_{k=1}^{n}\frac{1}{x+k-1}\right] + \\ + \mathrm{N}-\lim_{\epsilon\to 0} \sum_{k=1}^{n} \frac{\epsilon^{x+k-1}}{x+k-1} = \\ = \psi(x) + \gamma$$

on using equation (5). We have therefore shown that

$$\psi(x) = -\gamma + \text{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{1-t^{x-1}}{1-t}\,dt \tag{6}$$

for $x \neq 0, -1, -2, \ldots$. This suggest the following definition.

Definition 1 *The digamma function $\psi(x)$ is defined by*

$$\psi(-n) = -\gamma + \text{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{1-t^{-n-1}}{1-t}\,dt \tag{7}$$

for $n = 0, 1, 2, \ldots$, provided that the neutrix limit exists.

Now we have the following theorem.

Theorem 1. *The digamma function $\psi(x)$ have values for negative integers, and*

$$\psi(-n) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \tag{8}$$

for $n = 0, 1, 2, \ldots$

Proof. We will now prove the existence of $\psi(0)$. We have

$$\int_\epsilon^1 \frac{1-t^{-1}}{1-t}\,dt = -\int_\epsilon^1 \frac{dt}{t} = -\ln\epsilon\,,$$

and it follows that $\text{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{1-t^{-1}}{1-t}\,dt = 0$. $\psi(0)$ therefore exists and $\psi(0) = -\gamma$.

Next we have:

$$\int_\epsilon^1 \frac{1-t^{-n-1}}{1-t}\,dt = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k\epsilon^k} + \ln\epsilon\,,$$

and

$$\text{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{1-t^{-n-1}}{1-t}\,dt = \sum_{k=1}^{n} \frac{1}{k}.$$

Finally, we have that $\psi(-n)$ exists and $\psi(-n) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}$, proving the theorem.

Further it follows from (3) that for $x > 0$ we have

$$\psi'(x) = -\int_0^1 \frac{t^{x-1}\ln t}{1-t}\,dt\,.$$

Let now $-n < x < -n+1$, $n = 1, 2, \ldots$ Then

$$-\int_\epsilon^1 \frac{t^{x-1}\ln t}{1-t}\,dt = \sum_{i=0}^{\infty} \int_\epsilon^1 t^{x-1+i}\ln t\,dt$$
$$= \sum_{i=0}^{\infty} \left[\frac{\epsilon^{x+i}\ln\epsilon}{x+i} + \frac{\epsilon^{x+i}-1}{(x+i)^2}\right],$$

and it follows from above that when $-n < x < -n+1$, $n = 1, 2, \ldots$ we have:

$$\psi'(x) = \text{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{t^{x-1}\ln t}{1-t}\,dt\,.$$

This suggests the following definition:

Definition 2 *The derivative of the digamma function $\psi(x)$ is defined by*

$$\psi'(-n) = \text{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{t^{-n-1}\ln t}{1-t}\,dt \tag{9}$$

for $n = 0, 1, 2, \ldots$, provided that the neutrix limit exists.

Now we have the following theorem.

Theorem 2. *The derivative of the digamma function $\psi(x)$ have values for negative integers, and*

$$\psi'(-n) = \sum_{k=1}^{\infty} \frac{1}{(k-n)^2}. \tag{10}$$

for $n = 0, 1, 2, \ldots$

Proof. Using the equation

$$-\int_\epsilon^1 \frac{t^{-n-1}\ln t}{1-t}\,dt = \sum_{i=0}^{\infty} \int_\epsilon^1 t^{-n-1+i}\ln t\,dt$$
$$= \sum_{i=0}^{\infty} \left[\frac{\epsilon^{-n+i}\ln\epsilon}{-n+i} + \frac{\epsilon^{-n+i}-1}{(-n+i)^2}\right],$$

and taking the neutrix limit we have that

$$\text{N}-\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{t^{-n-1}\ln t}{1-t}\,dt = \sum_{k=1}^{\infty} \frac{1}{(k-n)^2},$$

proving the theorem.

Acknowledgement

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] I. Muqattash and M.Yahdi, Infinite family of approximations of the Digamma function, Mathematical and computer modeling, Volume 43, 11-12 (2006) pp 1329-1336.
[2] Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York: Dover. pp. 258-259.(1972)
[3] A.Erdelyi, W. Magnus, F.Oberhettinger and F.G. Tricomi, Higher Transcendental Functions, Vol I, McGraw-Hill, New york, London, Toronto (1953).
[4] J.G. van der Corput, Introduction to the neutrix calculus, J. Analyse Math., **7**(1959), 291-398.
[5] J. Jensen and T. Gronwall, An Elementary Exposition of the Theory of the Gamma Function, Ann. Math. Soc. 17 (3) 124-166 (1916).
[6] A, Kilicman and B. Fisher, Some Results on the Composition of Singular Distributions, Volume 5, No. 3 (Sep. 2011), pp:623-634.

[7] B. Fisher, Neutrices and distributions, Forth Internat. Conf.on Complex Analysis and Applications, Varna, Bulgaria (1987), 169-175.
[8] B. Fisher, On defining the incomplete gamma function $\gamma(-m, x_-)$, Integral Trans. Spec. Func., **1**5-6(2004), 467-476.
[9] B. Fisher and Y. Kuribayashi, Neutrices and the gamma function, The Journal of the faculty of education, Tottori University, **1**-2 36(1987), 1-7.
[10] B. Fisher, B. Jolevska-Tuneska and A. Kilisman, On defining the incomplete gamma function, Integral Trans. Spec. Func., **1**4-4(2004), 293-299.
[11] Fisher B., Jolevska-Tuneska B. and Takaci A., On the incomplete gamma function and the neutrix convolution, Mathematica Bohemica, 4 (2003) 367-377.
[12] Fisher B., Jolevska-Tuneska B. and Takaci A., On convolutions and neutrix convolutions involving the incomplete gamma function, Integral Transforms and Special Functions, Vol. 15/5 (2004) 404-415.
[13] Gel'fand, I.M. and Shilov, G.E. Generalized functions, Vol. I, Academic Press Chap. 1, 1964.
[14] L. Medina and V. Moll, The integrals in Gradshteyn and Ryzhik.Part 10: The digamma function, SCIENTIA Series A: Mathematical Sciences, Vol. 17 (2009), 4566.
[15] Ozcag E., Ege I., Gurcay H. and Jolevska-Tuneska B., On partial derivatives of the Incomplete Beta Function, Applied Mathematical Letters, Volume 21-7, (2008), 675-681.
[16] Y. Jack Ng and H. van Dam, Neutrix calculus and finite quantum field theory, J. Phys. A: Math. Gen. 38 (2005) 317323.
[17] Y. Jack Ng and H. van Dam, An application of neutrix calculus to quantum field theory, International Journal of Modern Physics A, Volume 21-2(2006) pp. 297-312.

Prof. B.Jolevska-Tuneska is a mathematician presently employed as Associated Professor at Faculty of Electrical engineering and Informational Technologies, University Ss. Cyril and Methodious in Skopje, Republic of Macedonia. She obtained her PhD from Novi Sad University (Serbia) in 2003. She is an active researcher with 15 years teaching experience in her home country. She is supervisor on two PhD students at University Ss. Cyril and Methodious in Skopje. She is working actively with lots of graduate students. She has published more than 40 (forty) research articles in reputed international journals of mathematical and engineering sciences.

Prof. Ilija Jolevski is presently employed as Associated Professor at Technical Faculty Bitola, University of St. Kliment Ohridski, Bitola, Republic of Macedonia. He obtained his PhD from Faculty of Electrical engineering and Informational Technologies, University Ss. Cyril and Methodious, Skopje in 2006. He is an active researcher with 5 year industry and 10 years teaching experience. Recently he has been visiting researcher at Freie University Berlin, Germany in their Department of Mathematics and Computer Science. He is working actively with number of graduate students. He has published more than 30 (thirty) research articles in reputed international conferences.